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ORBITS OF SMOOTH FUNCTIONS ON 2-TORUS AND THEIR HOMOTOPY TYPES

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Let $f\colon T^2 \to \mathbb{R}$ be a Morse function on 2-torus T^2 such that its Kronrod-Reeb graph $\Gamma(f)$ has exactly one cycle, i.e. it is homotopy equivalent to S^1. Under some additional conditions we describe a homotopy type of the orbit of f with respect to the action of the group of diffeomorphism of T^2.

This result holds for a larger class of smooth functions $f\colon T^2 \to \mathbb{R}$ having the following property: for every critical point z of f the germ of f at z is smoothly equivalent to a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

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Пусть $f\colon T^2 \to \mathbb{R}$ — функция Морса на 2-торе T^2 такая, что ее граф Кронрода-Риба имеет в точности один цикл, т.е. этот граф гомотопически эквивалентен окружности. При некоторых дополнительных условиях на f мы описываем гомотопический тип орбиты функции f относительно стандартного правого действия группы диффеоморфизмов T^2.

Полученный результат верен для более широкого класса функций, которые в своих критических точках эквивалентны однородным многочленам без кратных множителей.

1. Introduction. Let M be a smooth compact oriented surface. For a closed (possibly empty) subset $X \subset M$ denote by $\mathcal{D}(M,X)$ the group of diffeomorphisms of M fixed on X. This group naturally acts from the right on the space of smooth functions $C^\infty(M)$ by the following rule: if $h \in \mathcal{D}(M,X)$ and $f \in C^\infty(M)$ then the result of the action of h on f is the composition map

$$f \circ h\colon M \xrightarrow{\ h\ } M \xrightarrow{\ f\ } \mathbb{R}. \tag{1}$$

Such an action is the main object of study in singularities theory. For $f \in C^\infty(M)$ let

$$\mathcal{S}(f,X) = \{f \in \mathcal{D}(M,X) \mid f \circ h = f\}, \quad \mathcal{O}(f,X) = \{f \circ h \mid h \in \mathcal{D}(M,X)\}$$

be respectively the *stabilizer* and the *orbit* of f under the action (1).

Endow $\mathcal{D}(M,X)$, $C^\infty(M)$ and their subspaces $\mathcal{S}(f,X)$ and $\mathcal{O}(f,X)$ with the corresponding Whitney C^∞-topologies. Let also $\mathcal{S}_{\mathrm{id}}(f,X)$ be the path component of the identity map id_M in $\mathcal{S}(f,X)$, $\mathcal{D}_{\mathrm{id}}(M,X)$ the path component of id_M in $\mathcal{D}(M,X)$, and $\mathcal{O}_f(f,X)$ be

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the path component of f in $\mathcal{O}(f,X)$. If $X = \varnothing$ then we omit it from notation and write $\mathcal{D}(M) = \mathcal{D}(M,\varnothing)$, $\mathcal{S}(f) = \mathcal{S}(f,\varnothing)$, $\mathcal{O}(f) = \mathcal{O}(f,\varnothing)$, and so on.

Denote also

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M), \quad \mathcal{S}'(f,X) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M,X). \tag{2}$$

Thus $\mathcal{S}'(f,X)$ consists of diffeomorphisms h preserving f, fixed on X and isotopic to id_M relatively X, though the isotopy between h and id_M is not required to be f-preserving.

Remark 1. We assume that for $f \in \mathcal{F}(M)$ all the homotopy groups of $\mathcal{O}(f,X)$ have f as a base point, and all the homotopy groups of the groups of diffeomorphisms and the corresponding stabilizers of f are based at id_M. For instance, $\pi_k(\mathcal{O}(f,X))$ always means $\pi_k\big(\mathcal{O}(f,X), f\big)$. Notice that the latter group is also isomorphic to $\pi_k\big(\mathcal{O}_f(f,X), f\big)$.

Since $\mathcal{D}(M,X)$ and $\mathcal{S}(f,X)$ are topological groups, it follows that the homotopy sets $\pi_0\mathcal{D}(M,X)$, $\pi_0\mathcal{S}(f,X)$, and $\pi_1\big(\mathcal{D}(M,X),\mathcal{S}(f,X)\big)$ have natural groups structures such that

$$\pi_0\mathcal{D}(M,X) \;\cong\; \mathcal{D}(M,X)/\mathcal{D}_{\mathrm{id}}(M,X), \quad \pi_0\mathcal{S}(f,X) \;\cong\; \mathcal{S}(f,X)/\mathcal{S}_{\mathrm{id}}(f,X).$$

In the following part of exact sequence of homotopy groups of the pair $\big(\mathcal{D}(M,X),\mathcal{S}(f,X)\big)$

$$\cdots \to \pi_1\mathcal{D}(M,X) \xrightarrow{\;q\;} \pi_1\big(\mathcal{D}(M,X),\mathcal{S}(f,X)\big) \xrightarrow{\;\partial\;} \pi_0\mathcal{S}(f,X) \xrightarrow{\;i\;} \pi_0\mathcal{D}(M,X) \tag{3}$$

all maps are homomorphisms.

Moreover, $q\big(\pi_1\mathcal{D}(M,X)\big)$ is contained in the center of $\pi_1\big(\mathcal{D}(M,X),\mathcal{S}(f,X)\big)$.

Recall that two smooth germs $f,g\colon (\mathbb{R}^2,0) \longrightarrow (\mathbb{R},0)$ *smoothly equivalent* if there exist germs of diffeomorphisms $h\colon (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ and $\phi\colon (\mathbb{R},0) \to (\mathbb{R},0)$ such that $\phi\circ g = f\circ h$.

Definition 1. Denote by $\mathcal{F}(M)$ the subset in $C^\infty(M)$ which consists of functions f having the following two properties:

- f takes a constant value at each connected components of ∂M, and all critical points of f are contained in the interior of M;
- for each critical point z of f the germ of f at z is smoothly equivalent to a *homogeneous polynomial* $f_z\colon \mathbb{R}^2 \to \mathbb{R}$ *without multiple factors.*

Suppose a smooth germ $f\colon (\mathbb{R}^2,0) \longrightarrow (\mathbb{R},0)$ has a critical point $0 \in \mathbb{R}^2$. This point is called *non-degenerate* if f is smoothly equivalent to a homogeneous polynomial of the form $\pm x^2 \pm y^2$.

Evidently, a homogeneous polynomial $f\colon \mathbb{R}^2 \to \mathbb{R}$ has critical points only if $\deg f \geq 2$, and in this case the origin is always a critical point of f. If f has no multiple factors, then the origin 0 is a unique critical point. Moreover, 0 is non-degenerate if $\deg f = 2$, and degenerate for $\deg f \geq 3$, see [22, §7] for discussions.

Denote by $\mathrm{Morse}(M)$ the subset of $C^\infty(M)$ consisting of the Morse functions, that is functions having only *non-degenerate* critical points. It is well known that $\mathrm{Morse}(M)$ is open and everywhere dense in $C^\infty(M)$. Since $\pm x^2 \pm y^2$ has no multiple factors, we get the following inclusion $\mathrm{Morse}(M) \;\subset\; \mathcal{F}(M)$.

Let $f \in \mathcal{F}(M)$ and $c \in \mathbb{R}$. A connected component C of the level set $f^{-1}(c)$ is *critical* if C contains at least one critical point of f. Otherwise C is called *regular*. Consider a partition of M into connected component of level sets of f. It is well known that the corresponding quotient space $\Gamma(f)$ has a structure of a finite one-dimensional CW-complex and is called *Kronrod-Reeb graph* or simply KR-graph of the function f. In particular, the vertices of $\Gamma(f)$ are critical components of level sets of f, see Figure 1 below.

It is usually said that this graph was introduced by G. Reeb in [32], however in was used before by A. S. Kronrod in [11] for studying functions on surfaces. Applications of $\Gamma(f)$ to the Morse functions on surfaces are given e.g. in [1, 17, 12, 34, 35, 30].

The interest to the homotopical properties of the spaces Morse(M), $\mathcal{F}(M)$, and in particular to their subspaces $\mathcal{S}(f)$ and $\mathcal{O}(f)$ is motivated by the works of A. T. Fomenko ([5], [6]) on classification of Hamiltonian systems of two degrees of freedom, see also A. T. Fomenko, S. V. Matveev and V. V. Sharko ([31]), and A. V. Bolsinov and A. T. Fomenko ([1]).

The groups of cobordisms of Morse functions on surfaces were computed by K. Ikegami and O. Saeki ([9]) for orientable case and by B. Kalmar ([10]) for non-orientable surfaces.

Connected components of Morse(M) were described by S. Matveev ([12]), H. Zieschang, V. Sharko ([33]), and by S. Maksymenko ([19], [18], [23]) for circle valued Morse maps, E. Kudryavtseva ([12], [13]), and E. Kudryavtseva and D. Permyakov ([16]) for height functions and distinct classes of Morse functions with fixed sets of critical points, and by Yu. Burman ([2], [3]) for functions on non-compact surfaces without critical points.

Furthermore, the homotopy types of connected components of Morse(M) for orientable M with $\chi(M) < 0$ were described in E. Kudryavtseva ([14], [15]).

The homotopy types of stabilizers $\mathcal{S}(f)$ and higher homotopy groups of $\mathcal{O}_f(f)$ for $f \in$ Morse(M) were computed by S. Maksymenko ([21]) and then extended to the spaces $\mathcal{F}(M)$ in [25], [26], see Theorem 1 below. In particular, $\mathcal{O}_f(f)$ is aspherical for all surfaces distinct from S^2 and $\mathbb{R}P^2$, and so its homotopy type is determined by $\pi_1\mathcal{O}(f)$. It was also shown that for generic Morse functions connected components of $\mathcal{O}(f)$ are homotopy equivalent to some p-dimensional torus $T^p = S^1 \times \cdots \times S^1$ (p times), or to $T^p \times \mathbb{R}P^3$.

In the mentioned above papers [14], [15] E. Kudryavtseva also proved that if M is orientable with $\chi(M) < 0$, then for the non-generic Morse map $f\colon M \to \mathbb{R}$ the connected components of $\mathcal{O}(f)$ are homotopy equivalent to the quotient T^p/G of the p-torus by free action of some finite group G.

In [24] the first named author showed that if M is an arbitrary (possibly non-orientable) surface with $\chi(M) < 0$, then $\pi_1\mathcal{O}(f) \cong \prod_{i=1}^{k} \pi_1\mathcal{O}(f|_{M_i}, \partial M_i)$, where M_i is a certain subsurface of M diffeomorphic either to a 2-disk or a cylinder or a Möbius band, $f|_{M_i} \in \mathcal{F}(M_i)$ is the restriction of f to M_i, and $M_i \cap M_j = \varnothing$ for $i \neq j$. Moreover, in a recent preprint [27] he also described the algebraic structure of $\pi_1\mathcal{O}(f)$ for all orientable surfaces distinct from T^2 and S^2.

Thus the remained problem is to describe the homotopy types of $\mathcal{O}(f)$ for functions on T^2, S^2 and on all non-orientable surfaces.

The first step in this direction one can find in in [28]. They proved that if the KR-graph $\Gamma(f)$ of $f \in \mathcal{F}(T^2)$ is a tree, then under some additional assumptions $\pi_1\mathcal{O}(f) \cong \mathbb{Z}^2 \times \prod_{i=1}^{k} \pi_1\mathcal{O}(f|_{M_i}, \partial M_i)$, where each $M_i \subset M$ is a 2-disk and $M_i \cap M_j = \varnothing$ for $i \neq j$.

The aim of the present paper is to describe another class of functions $f \in \mathcal{F}(T^2)$ on 2-torus for which $\Gamma(f)$ contains a unique cycle, and $\pi_1\mathcal{O}(f)$ is isomorphic to $\mathbb{Z} \times \pi_1\mathcal{O}(f|_N)$, where $N \subset M$ is a cylinder, see Theorem 2 and Corollary 1 below.

The following theorem summarizes the information about $\pi_1\mathcal{O}(f, X)$ which will be used in the present paper.

Theorem 1 ([21, 25, 26])**.** *Let $f \in \mathcal{F}(M)$ and X be a finite (possibly empty) union of* *regular components of certain level sets of the function f. Then the following statements hold true.*

(1) $\mathcal{O}_f(f, X) = \mathcal{O}_f(f, X \cup \partial M)$, *and so* $\pi_k\mathcal{O}(f, X) \;\cong\; \pi_k\mathcal{O}(f, X \cup \partial M)$, $k \geq 1$.

(2) The following map $p\colon \mathcal{D}(M, X) \longrightarrow \mathcal{O}(f, X)$, $p(h) = f \circ h$ *is a Serre fibration with fiber* $\mathcal{S}(f, X)$, *i.e. it has the homotopy lifting property for CW-complexes. This implies that*

(a) $p(\mathcal{D}_{\mathrm{id}}(M, X)) = \mathcal{O}_f(f, X)$;

(b) *the restriction map*

$$p|_{\mathcal{D}_{\mathrm{id}}(M,X)}\colon \mathcal{D}_{\mathrm{id}}(M, X) \longrightarrow \mathcal{O}_f(f, X) \tag{4}$$

is also a Serre fibration with fiber $\mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M, X)$, *see* (2).

(c) *for each* $k \geq 0$ *we have an isomorphism* $j_k\colon \pi_k\big(\mathcal{D}(M, X), \mathcal{S}(f, X)\big) \longrightarrow \pi_k\mathcal{O}(f, X)$ *defined by* $j_k[\omega] = [f \circ \omega]$ *for a continuous map* $\omega\colon (I^k, \partial I^k, 0) \to \big(\mathcal{D}(M), \mathcal{S}(f), \mathrm{id}_M\big)$, *and making commutative the following diagram*

$$\cdots \longrightarrow \pi_k\mathcal{D}(M, X) \xrightarrow{\;q\;} \pi_k\big(\mathcal{D}(M, X), \mathcal{S}(f, X)\big) \xrightarrow{\;\partial\;} \pi_{k-1}\mathcal{S}(f, X) \longrightarrow \cdots$$

$$\pi_k\mathcal{D}(M, X) \xrightarrow{\;p\;} \pi_k\mathcal{O}(f, X), \qquad \pi_k\big(\mathcal{D}(M, X), \mathcal{S}(f, X)\big) \xrightarrow[\cong]{\;j_k\;} \pi_k\mathcal{O}(f, X), \qquad \pi_k\mathcal{O}(f, X) \xrightarrow{\;\partial\circ j_k^{-1}\;} \pi_{k-1}\mathcal{S}(f, X)$$

see for example [8, § 4.1, Theorem 4.1].

(3) Suppose either f has a critical point which is not a non-degenerate local extremum or M is a non-orientable surface. Then $\mathcal{S}_{\mathrm{id}}(f)$ *is contractible,* $\pi_n\mathcal{O}(f) = \pi_n M$ *for* $n \geq 3$, $\pi_2\mathcal{O}(f) = 0$, *and for* $\pi_1\mathcal{O}(f)$ *we have the following short exact sequence of fibration p*

$$1 \longrightarrow \pi_1\mathcal{D}(M) \xrightarrow{\;p\;} \pi_1\mathcal{O}(f) \xrightarrow{\;\partial\circ j_1^{-1}\;} \pi_0\mathcal{S}'(f) \longrightarrow 1. \tag{5}$$

Moreover, $p\big(\pi_1\mathcal{D}(M)\big)$ *is contained in the center of* $\pi_1\mathcal{O}(f)$.

(4) Suppose either $\chi(M) < 0$ *or* $X \neq \varnothing$. *Then* $\mathcal{D}_{\mathrm{id}}(M, X)$ *and* $\mathcal{S}_{\mathrm{id}}(f, X)$ *are contractible, whence from the exact sequence of homotopy groups of the fibration* (4) *we get* $\pi_k\mathcal{O}(f, X) = 0$ *for* $k \geq 2$, *and that the boundary map* $\partial \circ j_1^{-1}\colon \pi_1\mathcal{O}(f, X) \longrightarrow \pi_0\mathcal{S}'(f, X)$ *is an isomorphism.*

Suppose M is orientable and differs from the sphere S^2 and the torus T^2. Let $X = \partial M$. Then M and X satisfy assumptions of (4) of Theorem 1. Therefore from (1) of that theorem we get the following isomorphism

$$\pi_1\mathcal{O}(f) \overset{(1)}{\cong} \pi_1\mathcal{O}(f, \partial M) \overset{(4)}{\cong} \pi_0\mathcal{S}'(f, \partial M).$$

In fact papers [24], [27], [14], [15] study $\pi_0\mathcal{S}'(f, \partial M)$ instead of $\pi_1\mathcal{O}(f)$.

However when M is a sphere or a 2-torus the situation is more complicated, as $\pi_1\mathcal{D}(S^2) \cong \mathbb{Z}_2$, $\pi_1\mathcal{D}(T^2) \cong \mathbb{Z}^2$, and from the short exact sequence (5) we get only that $\pi_1\mathcal{O}(f)$ is an extension of $\pi_0\mathcal{S}'(f)$ with $\pi_1\mathcal{D}(M)$.

2. Main result. Suppose $M = T^2$. Then we prove that for each $f \in \mathcal{F}(T^2)$ its KR-graph $\Gamma(f)$ is either a tree or has exactly one simple cycle. Moreover, $\pi_1\mathcal{D}_{\mathrm{id}}(T^2) \cong \mathbb{Z}^2$, see [4, 7], and therefore sequence (5) can be rewritten as follows

$$1 \longrightarrow \mathbb{Z}^2 \xrightarrow{\;p\;} \pi_1\mathcal{O}_f(f) \xrightarrow{\;\partial\;} \pi_0\mathcal{S}'(f) \longrightarrow 1. \tag{6}$$

In [28] the authors studied the case where $\Gamma(f)$ is a tree and proved that under certain "triviality of $\mathcal{S}'(f)$-action" assumptions on f sequence (6) splits and we get an isomorphism $\pi_1\mathcal{O}_f(f) \cong \pi_0\mathcal{S}'(f) \times \mathbb{Z}^2$.

In the present paper we consider the situation when $\Gamma(f)$ has exactly one simple cycle Υ and under another "triviality of $\mathcal{S}'(f)$-action" assumption describe the homotopy type of $\mathcal{O}_f(f)$ in terms of $\pi_0\mathcal{S}'(f,C)$ for some regular component of some level-set of f, see Definition 2 and Theorem 2.

First we mention the following two simple lemmas which are left for the reader.

Lemma 1. *Let $f \in \mathcal{F}(T^2)$. Then the following conditions are equivalent:*

(i) $\Gamma(f)$ *is a tree;*

(ii) *every point $z \in \Gamma(f)$ separates $\Gamma(f)$;*

(iii) *every connected component of every level set of f separates T^2.*

Lemma 2. *Assume that $\Gamma(f)$ has exactly one simple cycle Υ. Let also $z \in \Gamma(f)$ be any point belonging to some open edge of $\Gamma(f)$ and C be the corresponding regular component of certain level set $f^{-1}(c)$ of f. Then the following conditions are equivalent:*

(a) $z \in \Upsilon$*;*

(b) *z does not separate $\Gamma(f)$;*

(c) *C does not separate T^2.*

Thus if $\Gamma(f)$ is not a tree, then there exists a connected component C of some level set of f that does not separate T^2, and this curve corresponds to some point z on an open edge of cycle Υ.

For simplicity we will fix once and for all such $f \in \mathcal{F}(T^2)$ and C, and use the following notation

$$\begin{aligned} &\mathcal{D}^{\mathrm{id}} := \mathcal{D}_{\mathrm{id}}(T^2), && \mathcal{O} := \mathcal{O}_f(f), && \mathcal{S} := \mathcal{S}'(f), && \mathcal{S}^{\mathrm{id}} := \mathcal{S}_{\mathrm{id}}(T^2), \\ &\mathcal{D}^{\mathrm{id}}_C := \mathcal{D}_{\mathrm{id}}(T^2, C), && \mathcal{O}_C := \mathcal{O}_f(f, C), && \mathcal{S}_C := \mathcal{S}'(f, C), && \mathcal{S}^{\mathrm{id}}_C := \mathcal{S}_{\mathrm{id}}(f, C). \end{aligned}$$

Let also $h \in \mathcal{S}$, so $f \circ h = f$ and h is isotopic to id_{T^2}. Then $h(f^{-1}(c)) = f^{-1}(c)$, and therefore h interchanges connected components of $f^{-1}(c)$. In particular, $h(C)$ is also a connected component of $f^{-1}(c)$. However, in general, $h(C)$ does not coincide with C, see Figure 1.

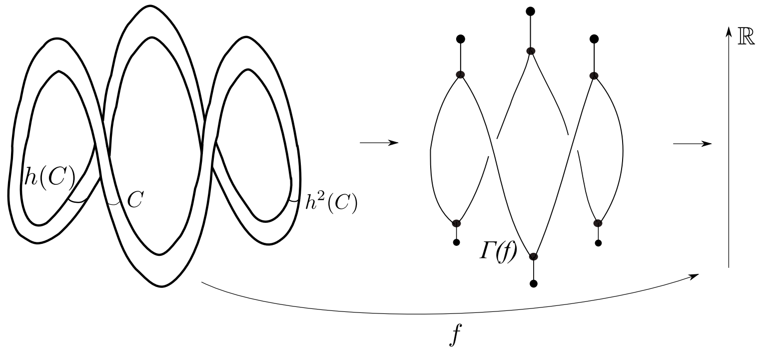
Definition 2. We will say that $\mathcal{S}$ *trivially acts on* C if $h(C) = C$ for all $h \in \mathcal{S}$.

Remark 2. Emphasize that the above definition only requires that $h(C) = C$ for all diffeomorphisms h that preserve f and are *isotopic to* id_{T^2}. We do not put any assumptions on diffeomorphisms that are not isotopic to id_{T^2}.

Remark 3. It is easy to see that if C_1 is another non-separating regular component of some level-set $f^{-1}(c_1)$, then $\mathcal{S}$ trivially acts on C if and only if $\mathcal{S}$ trivially acts on C_1.

The aim of the present paper is to establish the following theorem which will be proved in Section 4.

Fig. 1:



Theorem 2. *Let $f \in \mathcal{F}(T^2)$ be such that $\Gamma(f)$ has exactly one cycle, and C a regular connected component of certain level set $f^{-1}(c)$ of f which does not separate T^2. Suppose $\mathcal{S}$ trivially acts on C. Then there is a homotopy equivalence $\mathcal{O} \simeq \mathcal{O}_C \times S^1$. In particular, by (4) of Theorem 1 we have the following isomorphisms*

$$\pi_1\mathcal{O} \ \cong \ \pi_1\mathcal{O}_C \times \mathbb{Z} \ \overset{j\times \mathrm{id}_{\mathbb{Z}}}{\cong} \ \pi_0\mathcal{S}_C \times \mathbb{Z}.$$

Let C be the same as in Theorem 2, $x = f(C)$. For $\varepsilon > 0$ let U_ε be the connected component of $f^{-1}[x-\varepsilon, x+\varepsilon]$ containing C. If ε is sufficiently small, then U_ε contains no critical points of f and is diffeomorphic with a cylinder.

Corollary 1. *Suppose U_ε contains no critical points of f and let $N = \overline{T^2 \setminus U_{\varepsilon/2}}$ be "complementary" cylinder, see Figure 2 below. Then $\pi_0\mathcal{S}_C \cong \pi_0\mathcal{S}'(f|_N, \partial N)$, whence we have the following isomorphisms: $\pi_1\mathcal{O} \cong \pi_1\mathcal{O}(f|_N, \partial N) \times \mathbb{Z} \cong \pi_0\mathcal{S}'(f|_N, \partial N) \times \mathbb{Z}$.*

Proof. It follows from [26, Corollary 6.1] that the natural inclusions

$$\mathcal{S}'(f|_N, N \cap U_\varepsilon) \subset \mathcal{S}'(f|_N, \partial N), \mathcal{S}'(f, U_\varepsilon) \ \subset \ \mathcal{S}'(f, C) \ \equiv \ \mathcal{S}_C$$

are homotopy equivalences. On the other hand, the restriction map $r\colon \mathcal{S}'(f, U_\varepsilon) \to \mathcal{S}'(f|_N, N \cap U_\varepsilon)$, $r(h) = h|_N$ is evidently a homeomorphism. □

Added in proof. In paper [29], which was published faster than the present one, the authors described $\pi_1\mathcal{O}(f)$ for functions f on T^2 for which $\mathcal{S}'(f)$ *non-trivially* acts on C and so the total number of distinct images of C with respect to $\mathcal{S}'(f)$ is an integer $n \geq 2$. It is proved that then $\pi_1\mathcal{O}(f) \cong \pi_1\mathcal{O}(f|_N, \partial N) \wr_n \mathbb{Z}$, where $N \subset M$ is a sub-cylinder such that $f|_N \in \mathcal{F}(N)$, and $A \wr_n Z$ is the semidirect product of $A^n \rtimes \mathbb{Z}$ corresponding to a *non-effective* action of $\mathbb{Z}$ on A^n by cyclical shifts of coordinates.

Evidently, the case $n = 1$ corresponds to the triviality of $\mathcal{S}'(f)$-actions in the sense of Definition 2. Notice that $A \wr_1 Z = A \times \mathbb{Z}$, and so the obtained result holds for $n = 1$ due to Corollary 1. However the proofs in [29] essentially use non-triviality of $\mathcal{S}'(f)$-actions and so they are independent from Theorem 2.

The problems for further investigations of $\mathcal{O}(f)$ is to study the general case of $f \in \mathcal{F}(T^2)$ whose KR-graph is a tree, describe the structure of $\mathcal{O}(f)$ for functions on S^2 and on all non-orientable surfaces, extend the obtained results to actions of symplectic groups of diffeomorphisms of the corresponding surfaces, and relate the information about the homotopy types of $\mathcal{O}(f)$ with invariants of 3-manifolds.

3. Preliminaries.

3.1. Algebraic lemma.

Lemma 3. *Let L, M, Q, S be four groups. Suppose there exists a short exact sequence*

$$1 \to L \times M \xrightarrow{q} T \xrightarrow{\partial} S \to 1 \tag{7}$$

and a homomorphism $\varphi \colon T \to L \times 1$ such that

- $\varphi \circ q \colon L \times 1 \longrightarrow L \times 1$ *is the identity map and*
- $q(1 \times M) \subset \ker(\varphi)$.

Then we have the following exact sequence

$$1 \longrightarrow 1 \times M \xrightarrow{q} \ker(\varphi) \xrightarrow{\partial} S \longrightarrow 1.$$

Proof. It suffices to prove that

1) $\partial(\ker\varphi) = S$,
2) $\varphi(1 \times M) = \ker\varphi \cap \ker\partial$.

1) Let $s \in S$. We have to find $b \in \ker\varphi$ such that $\partial(b) = s$. Since $\partial(T) = S$, there exists $t \in T$ such that $\partial(t) = s$. Put $\hat{t} = q(\varphi(t))$ and $b = t\,\hat{t}^{-1}$. Then $\varphi(\hat{t}) = \varphi \circ q \circ \varphi(t) = \varphi(t)$, $\partial(\hat{t}) = \partial \circ q \circ \varphi(t) = 1$. Hence $b = t\hat{t}^{-1} \in \ker\varphi$ and $\partial(b) = \partial(t)\,\partial(\hat{t})^{-1} = \partial(t) = s$.

2) Let $a \in \ker\varphi \cap \ker\partial$. We should find $m \in M$ such that $q(1, m) = a$.

As $a \in \ker\partial = q(L \times M)$, so there exist $(l, m) \in L \times M$ such that $q(l, m) = a$. But $q(1, m) \in q(1 \times M) \subset \ker\varphi$, whence

$$(1, 1) = \varphi(a) = \varphi(q(l, m)) = \varphi(q(l, 1)) \cdot \varphi(q(1, m)) = \varphi(q(l, 1)) = (l, 1).$$

Hence $l = 1$, and so $a = q(1, m) \in q(1 \times M)$. □

3.2. Isotopies of T^2 fixed on a curve. We need the following general lemma claiming that if a diffeomorphism h of T^2 is fixed on a non-separating simple closed curve C and is isotopic to id_{T^2}, then an isotopy between h and id_{T^2} can be made fixed on C.

Lemma 4. *Let $C \subset T^2$ be a not null-homotopic smooth simple closed curve. Then*

$$\mathcal{D}_{\mathrm{id}}(T^2, C) = \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, C). \tag{8}$$

Proof. The inclusion $\mathcal{D}_{\mathrm{id}}(T^2, C) \subset \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, C)$ is evident. Therefore we have to establish the inverse one.

Let $h \in \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, C)$, so h is fixed on C and is isotopic to id_{T^2}. We have to prove that $h \in \mathcal{D}_{\mathrm{id}}(T^2, C)$, i.e. it is isotopic to id_{T^2} via an isotopy fixed on C.

Let $C_1 \subset T^2$ be a simple closed curve isotopic to C and disjoint from C, and $\tau \colon T^2 \to T^2$ a Dehn twist along C_1 fixed on C. Cut the torus T^2 along C and denote the resulting cylinder by Q.

Notice that the restrictions $h|_Q, \tau|_Q \colon Q \to Q$ are fixed on ∂Q. It is well-known that the isotopy class $\tau|_Q$ generates the group $\pi_0\mathcal{D}(Q, \partial Q) \cong \mathbb{Z}$. Hence there exists $n \in \mathbb{Z}$ such that $h|_Q$ is isotopic to $\tau^n|_Q$ relatively to ∂Q. This isotopy induces an isotopy between h and τ^n fixed on C.

By the assumption, h is isotopic to id_{T^2}, while τ^n is isotopic to id_{T^2} only for $n = 0$. Hence h is isotopic to $\tau^0 = \mathrm{id}_{T^2}$ via an isotopy fixed of C. □

3.3. Smooth shifts along trajectories of a flow. Let $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a smooth flow on a manifold M. Then for every smooth function $\alpha\colon M \to \mathbb{R}$ one can define the following map $\mathbf{F}_\alpha\colon T^2 \to \mathbb{R}$ by the formula

$$\mathbf{F}_\alpha(z) = \mathbf{F}(z, \alpha(z)), \ z \in M. \tag{9}$$

Lemma 5 (Claim 4.14.1, [21]). *If $\mathbf{F}_\alpha$ is a diffeomorphism then for each $t \in [0,1]$ the map*

$$\mathbf{F}_{t\alpha}\colon M \to M, \ \mathbf{F}_{t\alpha}(z) = \mathbf{F}(z, t\alpha(z))$$

is a diffeomorphism as well. In particular, $\{\mathbf{F}_{t\alpha}\}_{t\in I}$ is an isotopy between $\mathrm{id}_M = \mathbf{F}_0$ and $\mathbf{F}_\alpha$.

3.4. Some constructions associated with f. In the sequel we will regard the circle S^1 and the torus T^2 as the corresponding factor-groups $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2/\mathbb{Z}^2$. Let $e = (0,0) \in T^2$ be the unit of T^2. We always assume that e is a base point for all homotopy groups related with T^2 and its subsets. For $\varepsilon \in (0, 0.5)$ let $J_\varepsilon = (-\varepsilon, \varepsilon) \subset S^1$ be an open ε-neighborhood of $0 \in S^1$.

Let $f \in \mathcal{F}(T^2)$ be a function such that its KR-graph $\Gamma(f)$ has only one cycle, and let C be a regular connected component of certain level set of f not separating T^2. For this situation we define several constructions "adopted" with f.

Special coordinates. Since C is non-separating and is a regular component of $f^{-1}(c)$, one may assume (by a proper choice of coordinates on T^2) that the following two conditions hold:

(a) $C = 0 \times S^1 \ \subset \ \mathbb{R}^2/\mathbb{Z}^2 \equiv T^2$;

(b) there exists $\varepsilon > 0$ such that for all $t \in J_\varepsilon = (-\varepsilon, \varepsilon)$ the curve $t \times S^1$ is a regular connected component of some level set of f.

It is convenient to regard C as a *meridian* of T^2. Let $C' = S^1 \times 0$ be the corresponding *parallel*. Then $C' \cap C = e$, see Figure 2. Consider also the following loops $\lambda, \mu\colon I \to T^2$ defined by

$$\lambda(t) = (t, 0), \ \mu(t) = (0, t). \tag{10}$$

They represent the homotopy classes of C' and C in $\pi_1 T^2$ respectively.

Let us also mention that C is a *subgroup* of the group T^2. Therefore $\pi_1(T^2, C)$ has a natural group structure.

Let $k\colon C \hookrightarrow T^2$ be the inclusion map. Then the corresponding homomorphism $k\colon \pi_1 C \to \pi_1 T^2$ is injective. Since C is also connected, i.e. $\pi_0 C = \{1\}$, we get the following short exact sequence of homotopy groups of the pair (T^2, C)

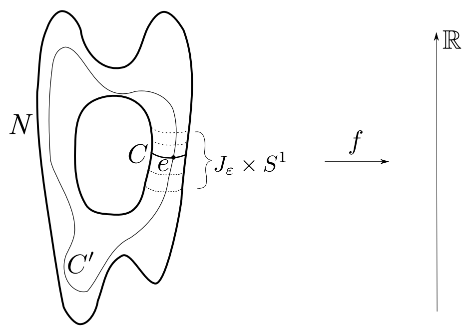
$$1 \longrightarrow \pi_1 C \xrightarrow{\ k\ } \pi_1 T^2 \xrightarrow{\ r\ } \pi_1(T^2, C) \longrightarrow 1. \tag{11}$$

As $\pi_1 C \cong \mathbb{Z}$ and $\pi_1 T^2 \cong \mathbb{Z}^2$, it follows that $\pi_1(T^2, C) \cong \mathbb{Z}$ and this group is generated by the image $r[\lambda]$ of the homotopy class of the parallel λ. In particular, there exists a section

$$s\colon \pi_1(T^2, C) \longrightarrow \pi_1 T^2 \tag{12}$$

such that $r \circ s[\lambda] = [\lambda]$, so $r \circ s$ is the identity map of $\pi_1(T^2, C)$.

Fig. 2:



The inclusion $\xi\colon T^2 \subset \mathcal{D}^{\mathrm{id}}$. Notice that T^2 is a connected Lie group. Therefore it acts on itself by smooth left translations. This yields the following embedding $\xi\colon T^2 \hookrightarrow \mathcal{D}^{\mathrm{id}}$: if $(a,b) \in T^2$, then $\xi(a,b)\colon T^2 \to T^2$ is a diffeomorphism given by the formula

$$\xi(a,b)(x,y) = \big(x + a \bmod 1, \quad y + b \bmod 1\big). \tag{13}$$

It is well known that ξ is a homotopy equivalence, e.g. [7].

Notice also that ξ yields the following map

$$\xi\colon C\big((I,\partial I),(T^2,e)\big) \longrightarrow C\big((I,\partial I),(\mathcal{D}^{\mathrm{id}},\mathrm{id}_{T^2})\big) \tag{14}$$

between the *spaces of loops* defined as follows: if $\omega\colon (I,\partial I) \longrightarrow (\mathcal{D}^{\mathrm{id}},\mathrm{id}_{T^2})$ is a continuous map, then $\xi(\omega) = \xi\circ\omega\colon\; I \longrightarrow \mathcal{D}^{\mathrm{id}}$. It is well known that this map is continuous with respect to compact open topologies. Moreover the corresponding map between the path components is just the homomorphism of fundamental groups

$$\xi\colon \pi_1 T^2 \longrightarrow \pi_1\mathcal{D}^{\mathrm{id}}. \tag{15}$$

Since ξ is a homotopy equivalence, homomorphism (15) is in fact an isomorphism.

To simplify notation we denoted all these maps with the same letter ξ. However this will never lead to confusion.

Isotopies L and M. Let

$$\mathbf{L} = \xi(\lambda), \quad \mathbf{M} = \xi(\mu) \tag{16}$$

be the images of the loops λ and μ in $\mathcal{D}^{\mathrm{id}}$ under the map given by (14). Evidently, they can be regarded as isotopies $\mathbf{L},\mathbf{M}\colon T^2\times[0,1] \to T^2$ defined by

$$\mathbf{L}(x,y,t) = (x + t \bmod 1,\ y), \quad \mathbf{M}(x,y,t) = (x,\ y+t \bmod 1), \tag{17}$$

for $x \in C'$, $y \in C$, and $t \in [0,1]$. Geometrically, $\mathbf{L}$ is a "*rotation*" of the torus along its parallels and $\mathbf{M}$ is a rotation along its meridians.

Denote by $\mathcal{L}$ and $\mathcal{M}$ the subgroups of $\pi_1\mathcal{D}^{\mathrm{id}}$ generated by loops $\mathbf{L}$ and $\mathbf{M}$ respectively. Since $\xi\colon T^2 \to \mathcal{D}^{\mathrm{id}}$ is a homotopy equivalence, and the loops λ and μ freely generate $\pi_1 T^2$, it follows that $\mathcal{L}$ and $\mathcal{M}$ are commuting free cyclic groups, and so we get an isomorphism $\pi_1\mathcal{D}^{\mathrm{id}} \cong \mathcal{L}\times\mathcal{M}$.

Notice that $\mathbf{L}$ and $\mathbf{M}$ can be also regarded as *flows* $\mathbf{L}, \mathbf{M}\colon T^2 \times \mathbb{R} \to T^2$ defined by the same formulas (17) for $(x, y, t) \in T^2 \times \mathbb{R}$. All orbits of the *flows* $\mathbf{L}$ and $\mathbf{M}$ are periodic of period 1. We denote these flows by the same letters as the corresponding *loops* (16), however this will never lead to confusion.

A flow F. As T^2 is an orientable surface, there exists a flow $\mathbf{F}\colon T^2 \times \mathbb{R} \to T^2$ having the following properties, see e.g. [21, Lemma 5.1]:

1) a point $z \in T^2$ is fixed for $\mathbf{F}$ if and only if z is a critical point of f;
2) f is constant along orbits of $\mathbf{F}$, that is $f(z) = f(\mathbf{F}(z, t))$ for all $z \in T^2$ and $t \in \mathbb{R}$.

It follows that every critical point of f and every regular components of every level set of f is an orbit of $\mathbf{F}$.

In particular, each curve $t \times S^1$ for $t \in J_\varepsilon$ is an orbit of $\mathbf{F}$. On the other hand, this curve is also an orbit of the flow $\mathbf{M}$. Therefore, we can always choose $\mathbf{F}$ so that

$$\mathbf{M}(x, y, t) = \mathbf{F}(x, y, t), \quad (x, y, t) \in J_\varepsilon \times S^1 \times \mathbb{R}. \tag{18}$$

Lemma 6 (Lemma 5.1, [21])**.** *Suppose a flow* $\mathbf{F}\colon T^2 \times \mathbb{R} \to T^2$ *satisfies the above conditions* 1) *and* 2) *and let* $h \in \mathcal{S}(f)$. *Then* $h \in \mathcal{S}_{\mathrm{id}}(f)$ *if and only if there exists a* C^∞ *function* $\alpha\colon T^2 \to \mathbb{R}$ *such that* $h = \mathbf{F}_\alpha$, *see equation* (9). *Such a function is unique and the family of maps* $\{\mathbf{F}_{t\alpha}\}_{t \in I}$ *constitute an isotopy between* id_M *and* h. □

4. Proof of Theorem 2. Let $f \in \mathcal{F}(T^2)$ be such that its KR-graph $\Gamma(f)$ has only one cycle, and let C be a non-separating regular connected component of certain level set of f. Assume also that $\mathcal{S}$ trivially acts of C. We have to prove that there exists a homotopy equivalence $\mathcal{O}_C \times S^1 \simeq \mathcal{O}$.

By (3) and (4) of Theorem 1 the orbits $\mathcal{O}$ and $\mathcal{O}_C$ are aspherical, as well as S^1, i.e. their homotopy groups π_k vanish for $k \geq 2$. Therefore, by Whitehead's Theorem ([8, § 4.1, Theorem 4.5]), it suffices to show that there exists an isomorphism $\pi_1\mathcal{O}_C \times \pi_1 S^1 \cong \pi_1\mathcal{O}$. Such an isomorphism will induce a required homotopy equivalence.

Moreover, due to (2) of Theorem 1 we have isomorphisms: $\pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \cong \pi_1\mathcal{O}_C$, $\pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \cong \pi_1\mathcal{O}$. Therefore it remains to find the following isomorphism

$$\pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \times \pi_1 S^1 \cong \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}). \tag{19}$$

Notice that every smooth function $f\colon T^2 \to \mathbb{R}$ has critical points being non-local extremes, since otherwise T^2 would be diffeomorphic to a 2-sphere S^2. Therefore by (3) and (4) of Theorem 2 the spaces $\mathcal{S}^{\mathrm{id}}$, $\mathcal{S}^{\mathrm{id}}_C$, and $\mathcal{D}^{\mathrm{id}}_C$ are contractible. Moreover, as noted above, $\mathcal{D}^{\mathrm{id}}$ is homotopy equivalent to T^2.

Let $i\colon (\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \subset (\mathcal{D}^{\mathrm{id}}, \mathcal{S})$ be the inclusion map. It yields a morphism between the exact sequences of homotopy groups of these pairs. The non-trivial part of this morphism is contained in the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1 & \longrightarrow & \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) & \xrightarrow{\partial_C} & \pi_0\mathcal{S}_C & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \\
1 & \longrightarrow & \pi_1\mathcal{D}^{\mathrm{id}} & \xrightarrow{q} & \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) & \xrightarrow{\partial} & \pi_0\mathcal{S} & \longrightarrow & 1.
\end{array} \tag{20}$$

The proof of Theorem 2 is based on propositions 1 and 2 below.

Proposition 1. *Under the assumptions of Theorem 2 there exists an epimorphism*

$$\varphi\colon \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \longrightarrow \mathcal{L}$$

such that

1) *φ is a left inverse for q, that is $\varphi \circ q = \mathrm{id}_{\mathcal{L}}$;*
2) *$q(\mathcal{M}) \subset \ker\varphi$;*
3) *$i_1\bigl(\pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C)\bigr) \subset \ker\varphi$.*

Corollary 2. a) *The map $\theta\colon \ker\varphi \times \mathcal{L} \longrightarrow \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$ defined by $\theta(\omega, l) = \omega \cdot q(l)$ for $(\omega, l) \in \ker\varphi \times \mathcal{L}$, is a groups isomorphism.*

b) *The following sequence is exact $1 \longrightarrow \mathcal{M} \xrightarrow{\ q\ } \ker\varphi \xrightarrow{\ \partial\ } \pi_0\mathcal{S} \longrightarrow 1$.*

Proof. Statement a) follows from 1) of Proposition 1 and for the fact that $q(\mathcal{L})$ is contained in the center of $\pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$, see (3) of Theorem 2. Statement b) is a direct consequence of statements 1) and 2) of Proposition 1 and Lemma 3 applied to the lower exact sequence of equation (20). We leave the details for the reader. □

Due to 2) and 3) of Proposition 1 and 3) of Corollary 2 we see that diagram (20) is reduced to the following one

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1 & \longrightarrow & \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) & \xrightarrow[\cong]{\partial_C} & \pi_0\mathcal{S}_C & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \\
1 & \longrightarrow & \mathcal{M} & \xrightarrow{q} & \ker\varphi & \xrightarrow{\partial} & \pi_0\mathcal{S} & \longrightarrow & 1.
\end{array} \tag{21}$$

Thus to complete Theorem 2 it suffices to prove that the middle vertical arrow, i_1, in equation (21) is an isomorphism between $\pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C)$ and $\ker\varphi$. As $\mathcal{L} \cong \mathbb{Z} \cong \pi_1 S^1$ we get the required isomorphism

$$\pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \times \mathbb{Z} \ \cong\ \ker\varphi \times \mathcal{L} \ \xrightarrow{\ \theta\ }\ \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}).$$

To show that $i_1\colon \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \to \ker\varphi$ is an isomorphism notice that ∂_C is also an isomorphism by (4) of Theorem 1. Therefore from the latter diagram (21) we get the following one

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker i_0 & \longrightarrow & \pi_0\mathcal{S}_C & \xrightarrow{i_0} & \pi_0\mathcal{S} & & \\
 & & \downarrow & & \downarrow{\scriptstyle i_1\circ\partial_C^{-1}} & & \| & & \\
1 & \longrightarrow & q(\mathcal{M}) & \longrightarrow & \ker\varphi & \xrightarrow{\partial} & \pi_0\mathcal{S} & \longrightarrow & 1.
\end{array}$$

Proposition 2. *Homomorphism $i_0\colon \pi_0\mathcal{S}_C \longrightarrow \pi_0\mathcal{S}$ is surjective, and the induced map $i_1 \circ \partial_C^{-1}\colon \ker i_0 \longrightarrow q(\mathcal{M})$ is an isomorphism.*

In other words, Proposition 2 claims that we have the following morphism between short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker i_0 & \longrightarrow & \pi_0\mathcal{S}_C & \xrightarrow{i_0} & \pi_0\mathcal{S} & \longrightarrow & 1 \\
 & & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle i_1\circ\partial_C^{-1}} & & \| & & \\
1 & \longrightarrow & q(\mathcal{M}) & \longrightarrow & \ker\varphi & \xrightarrow{\partial} & \pi_0\mathcal{S} & \longrightarrow & 1.
\end{array}$$

Since left and right vertical arrows are isomorphisms, it follow from five lemma ([8, § 2.1]), that $i_1 \circ \partial_C^{-1}$ is an isomorphism as well. This completes the proof of Theorem 2 modulo Propositions 1 and 2. The next two sections are devoted to the proof of those propositions.

Remark 4. It easily follows from statement a) of Corollary 2 that the map $\kappa : \mathcal{O}_C \times S^1 \to \mathcal{O}$ defined by

$$\kappa(g,t) = g \circ \mathbf{L}_t \tag{22}$$

is a homotopy equivalence. We leave the details for the reader.

5. Proof of Proposition 1. The existence of φ is guaranteed by statement (e) of the following lemma.

Lemma 7. *Suppose $\mathcal{S}$ trivially acts of C. Then there is a commutative diagram*

$$\begin{array}{ccccc} & & \pi_1\mathcal{D}^{\mathrm{id}} & \xleftarrow[\cong]{\ \xi\ } & \pi_1 T^2 \\ & & {\scriptstyle q}\downarrow & & {\scriptstyle s}\uparrow\ \downarrow{\scriptstyle r} \\ \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) & \xrightarrow{\ i_1\ } & \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) & \xrightarrow{\ \zeta\ } & \pi_1(T^2, C) \cong \mathbb{Z} \end{array} \tag{23}$$

in which

(a) *$r \circ s$ is the identity isomorphism of $\pi_1(T^2, C)$;*

(b) *$\xi \circ s$ is an isomorphism of $\pi_1(T^2, C)$ onto $\mathcal{L}$;*

(c) *ζ is surjective;*

(d) *$q(\mathcal{M}) \subset \ker\zeta$;*

(e) *$i_1(\pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C)) \ \subset\ \ker\zeta$;*

(f) *the composition $\varphi = \xi \circ s \circ \zeta \colon \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \longrightarrow \mathcal{L}$ satisfies the statement of Proposition 1.*

Proof. We need only to define the map ζ, since q appears in equation (3), and r, s, and ξ are described in §3.4.

Let $\omega\colon (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$ be a continuous map. In particular, $\omega(1)$ belongs to $\mathcal{S}$. By the assumption, $\mathcal{S}$ trivially acts on C, whence $\omega(1)(C) = C$. Consider the following path $\omega_e\colon I \to T^2$, $\omega_e(t) = \omega(t)(e)$. Then $\omega_e(0) = e$ and $\omega_e(1) \in C$. Therefore $\omega_e \in C\big((I, \partial I, 0), (T^2, C, e)\big)$. We put by definition $\zeta(\omega) = \omega_e$. The map $\zeta\colon \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \to \pi_1(T^2, C)$ in equation (23) is the induced mapping of the corresponding sets of homotopy classes. It is easy to verify that ζ is in fact a group homomorphism.

Commutativity of diagram (7)**.** Notice that the groups in the right rectangle of equation (7) are just the sets of path components of the corresponding spaces from the following diagram

$$\begin{array}{ccc} C\big((I, \partial I), (\mathcal{D}^{\mathrm{id}}, \mathrm{id}_{T^2})\big) & \xleftarrow{\ \xi\ } & C\big((I, \partial I), (T^2, e)\big) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ C\big((I, \partial I, 0), (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})\big) & \xrightarrow{\ \zeta\ } & C\big((I, \partial I, 0), (T^2, C, e)\big). \end{array} \tag{24}$$

Notice that the maps q and r here are just natural inclusions. It suffices to prove commutativity of diagram (24).

Let $\omega = (\alpha, \beta)\colon (I, \partial I) \longrightarrow (T^2, e)$ be a representative of some loop in $\pi_1 T^2$, where α and β are coordinate functions of ω. Then $q \circ \xi(\omega)(t)(x, y) = \bigl(x + \alpha(t) \bmod 1,\ y + \beta(t) \bmod 1\bigr)$. whence $\zeta \circ q \circ \xi(\omega)(t) = q \circ \xi(\omega)(t)(0, 0) = \bigl(\alpha(t), \beta(t)\bigr) = \omega(t)$.

But $r(\omega)(t) = \omega(t)$ as well, whence $\zeta \circ q \circ \xi(\omega) = r(\omega)$. Thus diagram (7) is commutative.

Property (a) is already established, see remark just after equation (12).

Property (b). Since $\xi \circ s(r[\lambda]) = \xi[\lambda] = \mathbf{L}$ and $\mathcal{L}$ is freely generated by $\mathbf{L}$, it follows that $\xi \circ s$ isomorphically maps $\pi_1(T^2, C)$ onto $\mathcal{L}$.

Property (c). By commutativity of equation (23) we have that $r[\lambda] = \zeta \circ q \circ \xi[\lambda]$. But $r[\lambda]$ generates $\pi_1(T^2, \lambda)$, whence ζ is surjective.

Property (d). As $\mathbf{M}$ generates $\mathcal{M}$, it suffices to show that $\zeta \circ q(\mathbf{M}) = 0$. We have that $[\mu] \in k(\pi_1 C)$. Therefore it follows from the exactness of sequence (11) that $r[\mu] = 0 \in \pi_1(T^2, C)$. Hence $\zeta \circ q[\mathbf{M}] = \zeta \circ q \circ \xi[\mu] = r[\mu] = 0$. Thus $q(\mathcal{M}) \subset \ker \varphi$.

Property (e). Let $\alpha \in \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C)$ and $\omega\colon (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C, \mathrm{id}_{T^2})$ be a path representing α. Then $i_1(\alpha)$ is represented by the homotopy class of the map $i \circ \omega\colon (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$. By the assumption $\omega(t) \in \mathcal{D}^{\mathrm{id}}_C$, i.e. it is fixed on C for all $t \in I$. In particular, since $e \in C$, we get that $\zeta(\omega)(t) = \omega_e(t) = \omega(t)(e) = e$. Thus $\zeta(\omega)\colon I \to T^2$ is a constant map, and so it represents a unit element of $\pi_1(T^2, C')$. Therefore $i_1(\alpha) \in \ker \zeta$.

Property (f). By (b) and (c), $\xi \circ s$ is an isomorphism, and ζ is surjective. Hence $\varphi = \xi \circ s \circ \zeta$ is surjective as well, and $\ker \varphi = \ker \zeta$. Therefore statements 2) and 3) of Proposition 1 follow from (d) and (e) respectively.

To prove 1) notice that

$$\varphi \circ q(\mathbf{L}) = (\xi \circ s \circ \zeta) \circ q\bigl(\xi[\lambda]\bigr) = \xi \circ s \circ (\zeta \circ q \circ \xi)[\lambda] = \xi \circ s \circ r[\lambda] = \xi[\lambda] = \mathbf{L}.$$

Hence $\varphi \circ q = \mathrm{id}_{\mathcal{L}}$. The proofs of Lemma 7 and Proposition 1 are completed. □

6. Proof of Proposition 2.

6.1. Image of i_0. Surjectivity of i_0 is guaranteed by the following lemma.

Lemma 8. *Suppose $\mathcal{S}$ trivially acts on C, i.e. $h(C) = C$ for all $h \in \mathcal{S}$. Then the induced homomorphism $i_0\colon \pi_0 \mathcal{S}_C \to \pi_0 \mathcal{S}$ is surjective.*

Proof. We prove that each $h \in \mathcal{S} \equiv \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2)$ is isotopic in $\mathcal{S}$ to a diffeomorphism g belonging to $\mathcal{S}_C \equiv \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, C)$.

Indeed, by the assumption, $h(C) = C$. Hence h is isotopic in $\mathcal{S}$ to a diffeomorphism g fixed on C, i.e. $g \in \mathcal{S} \cap \mathcal{D}(T^2, C)$. But due to Lemma 4

$$\mathcal{S} \cap \mathcal{D}(T^2, C) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, C) \overset{(8)}{=\!=\!=} \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, C) = \mathcal{S}_C,$$

whence $g \in \mathcal{S}_C$. Thus i_0 is an epimorphism. □

6.2. Kernel of i_0. Notice that the kernel of $i_0\colon \pi_0 \mathcal{S}_C \to \pi_0 \mathcal{S}$ consists of isotopy classes of diffeomorphisms in $\mathcal{S}_C$ isotopic to id_{T^2} by f-preserving isotopy, however such an isotopy need not be fixed on C. In other words, if we denote $\mathcal{K} := \mathcal{S}^{\mathrm{id}} \cap \mathcal{D}^{\mathrm{id}}_C = \mathcal{S}_{\mathrm{id}}(f) \cap \mathcal{D}(T^2, C)$, then

$$\ker i_0 = \pi_0 \mathcal{K}. \tag{25}$$

Also notice that $\mathcal{S}^{\mathrm{id}}_C$ is the identity path component of $\mathcal{K}$, whence $\ker i_0 = \pi_0 \mathcal{K} = \mathcal{K}/\mathcal{S}^{\mathrm{id}}_C$.

For the proof of Proposition 2 we will first establish in Lemma 9 that $\ker i_0 \cong \mathbb{Z}$ and then show in Lemma 10 that $i_1 \circ \partial_C^{-1}$ yields an isomorphism of $\ker i_0$ onto $\mathcal{M}$.

Since $\mathcal{K} := \mathcal{S}^{\mathrm{id}} \cap \mathcal{D}^{\mathrm{id}}_C \subset \mathcal{S}^{\mathrm{id}}$, it follows from Lemma 6 that for every $h \in \mathcal{K}$ there exists a unique smooth function $\delta \in C^\infty(T^2)$ such that $h = \mathbf{F}_\delta$.

Lemma 9. *Suppose the flow* $\mathbf{F}$ *satisfies relation* (18). *Then for each* $h = \mathbf{F}_\delta \in \mathcal{K}$ *the function* δ *is constant on* C *and its value on* C *is integer. Define a map* $\eta\colon \mathcal{K} \longrightarrow \mathbb{Z}$ *by* $\eta(h) = \delta|_C$, *for* $h = \mathbf{F}_\delta \in \mathcal{K}$. *Then* η *is a surjective homomorphism with* $\ker\eta = \mathcal{S}^{\mathrm{id}}_C$. *In particular* η *yields an isomorphism* $\ker i_0 = \pi_0\mathcal{K} = \mathcal{K}/\mathcal{S}^{\mathrm{id}}_C \cong \mathbb{Z}$.

Proof. We regard $\mathbf{M}$ as a flow on T^2. Notice that C is a closed trajectory of both flows $\mathbf{F}$ and $\mathbf{M}$ and its period with respect to $\mathbf{M}$ is equal to 1. Therefore relation (18) implies that the period of C with respect to $\mathbf{F}$ also equals 1.

Since $h \in \mathcal{K} = \mathcal{S}^{\mathrm{id}} \cap \mathcal{D}^{\mathrm{id}}_C$ is fixed on C, that is $y = h(y) = \mathbf{F}(y, \delta(y))$ for all $y \in C$, it follows that the value of $\delta(y)$ is a multiple of the period $\theta = 1$. Thus for every $y \in C$ there exists $n_y \in \mathbb{Z}$ such that $\delta(y) = n_y$. But δ is continuous, whence the mapping $y \mapsto n_y = \delta(y)$ is a continuous function $C \to \mathbb{Z}$. Therefore this function is constant, i.e. $\delta|_C = n$ for some $n \in \mathbb{Z}$.

We should prove that η has the desired properties.

Step 1. *η is a homomorphism.* Let $h_i = \mathbf{F}_{\delta_i} \in \mathcal{K}$ for $i \in \{0, 1\}$ such that $\delta_i|_C = n_i$ for certain $n_i \in \mathbb{Z}$. Define the function $\delta = \delta_1 \circ h_0 + \delta_0$. Since h_0 is fixed on C, we have that for each $z \in C$

$$\delta(z) \;=\; \delta_1 \circ h_0(z) + \delta_0(z) = \delta_1(z) + \delta_0(z) = n_1 + n_0.$$

Moreover, by [20, equation (8)], $h_1 \circ h_0 = \mathbf{F}_\delta$, whence

$$\eta(h_1 \circ h_0) = \delta|_C = n_1 + n_0 = \delta_1|_C + \delta_0|_C = \eta(h_1) + \eta(h_0),$$

and so η is a homomorphism.

Step 2. *η is surjective.* It suffices to construct $g \in \mathcal{K}$ with $\eta(g) = -1$. The choice of -1 simplify further exposition, however we could also construct g satisfying $\eta(g) = +1$.

Let J_ε be the same as in equation (18), and $\beta : J_\varepsilon \to [0, 1]$ a C^∞-function such that

$$\beta(z) = \begin{cases} -1, & |z| < \varepsilon/3, \\ 0, & |z| > 2\varepsilon/3. \end{cases}$$

Define another function $\sigma\colon T^2 \to \mathbb{R}$ by

$$\sigma(x, y) = \begin{cases} \beta(x), & x \in J_\varepsilon, \\ 0, & x \in S^1 \setminus J_\varepsilon, \end{cases}$$

and let $g = \mathbf{F}_\sigma$, so g is a map $T^2 \to T^2$ defined by $g(x, y) = \mathbf{F}(x, y, \sigma(x, y))$. Since $\sigma = 0$ outside $J_\varepsilon \times S^1$, it follows from equation (18) that $g = \mathbf{M}_\sigma$, i.e.

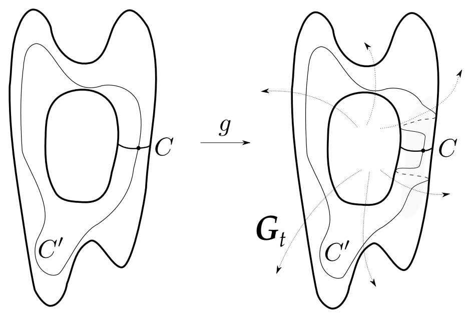
$$g(x, y) = \mathbf{M}(x, y, \sigma(x, y)). \tag{26}$$

Moreover, as periods of all points with respect to $\mathbf{M}$ are equal to 1, we can also write $g = \mathbf{M}_{\sigma+n}$ for any $n \in \mathbb{Z}$, i.e. $g(x, y) = \mathbf{M}(x, y, \sigma(x, y) + n)$. Thus $g = \mathbf{F}_\sigma = \mathbf{M}_\sigma = \mathbf{M}_{\sigma+n}$ for all $n \in \mathbb{Z}$. The following properties of g can easily be verified:

(i) g is fixed on $J_{\varepsilon/3} \times S^1$ and outsize $J_\varepsilon \times S^1$;

(ii) $g \in \mathcal{S}_{\mathrm{id}}(f)$ by Lemma 6, since $g = \mathbf{F}_\sigma$;

Fig. 3: Diffeomorphism g and the isotopy $\mathbf{G}_t$.



(iii) the following family of maps $\mathbf{G}_t = \mathbf{M}_{t(\sigma+1)}$, $t \in I$, is an isotopy between

$$\mathbf{G}_0 = \mathbf{M}_0 = \mathrm{id}_{T^2}, \quad \mathbf{G}_1 = \mathbf{M}_{\sigma+1} = g, \tag{27}$$

see Figure 3. This isotopy is fixed on C, as $t(\sigma+1) = 0$ on C. Hence $g \in \mathcal{D}^{\mathrm{id}}_C$ as well.

Thus by (ii) and (iii) $g \in \mathcal{S}(f) \cap \mathcal{D}^{\mathrm{id}}_C = \mathcal{K}$. It remains to note that $\eta(g) = \sigma|_C = -1$, and so η is surjective.

Step 3. $\ker \eta = \mathcal{S}^{\mathrm{id}}_C$. Suppose $h = \mathbf{F}_\delta \in \ker\eta$, so $\eta(h) = \delta|_C = 0$. Then by Lemma 6 an isotopy between h and id_{T^2} can be given by $g_t = \mathbf{F}_{t\delta}$, $t \in [0,1]$. Notice that $t\delta|_C = 0$ as well, whence $\{g_t\}$ is also fixed on C. Therefore $h \in \mathcal{S}^{\mathrm{id}}_C$.

Conversely, let $h \in \mathcal{S}^{\mathrm{id}}_C$, so h is isotopic to id_{T^2} in $\mathcal{S}^{\mathrm{id}}$ via an isotopy $\{h_t\}_{t\in[0,1]}$ fixed on C and such $h_0 = \mathrm{id}_{T^2}$ and $h_1 = h$. Then $h_t = \mathbf{F}_{\delta_t}$, $t \in [0,1]$, for some smooth function $\delta_t \colon T^2 \to \mathbb{R}$. Since C is a non-fixed trajectory of $\mathbf{F}$, it follows from [20, Theorem 25], that the values of δ_t on C continuously depend on t. But each δ_t takes a constant integer value on C, and $\mathrm{id}_{T^2} = \mathbf{F}_0$, whence $\eta(h) = \delta_1|_C = \delta_t|_C = \delta_0|_C = 0$, that is $h \in \ker\eta$. □

6.3. Inverse of boundary isomorphism $\partial_C^{-1} \colon \pi_0\mathcal{S}_C \longrightarrow \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C)$. Thus we have that both $\ker i_0$ and $\mathcal{M}$ are isomorphic to $\mathbb{Z}$. By Lemma 9 $\ker i_0$ is generated by the homotopy class of the diffeomorphism $g = \mathbf{F}_\sigma = \mathbf{M}_\sigma \in \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, C) \subset \mathcal{S}_C$ defined by equation (26) and satisfying $\eta(g) = -1$.

On the other hand, $q(\mathcal{M})$ is generated by the homotopy class of the following map

$$q(\mathbf{M}) \colon (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2}), \quad q(\mathbf{M})(t) = \mathbf{M}_t. \tag{28}$$

Therefore in order to complete the proof of Proposition 2 it suffices to prove the following lemma.

Lemma 10. $i_1 \circ \partial_C^{-1}[g] = [q(\mathbf{M})]$. *Hence* $i_1 \circ \partial_C$ *isomorphically maps* $\ker i_0$ *onto* $q(\mathcal{M})$.

Proof. Recall that the boundary homomorphism $\partial_C \colon \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C) \longrightarrow \pi_0\mathcal{S}_C$ is defined as follows: if $\alpha \in \pi_1(\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C)$ and $\omega \colon (I, \partial I, 0) \to (\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C, \mathrm{id}_{T^2})$ is a representative of α, then $\partial_C(\alpha) = [\omega(1)] \in \pi_0\mathcal{S}_C$. Now let $\mathbf{G}_t = \mathbf{M}_{t(\sigma+1)}$ be an isotopy between $\mathbf{G}(0) = \mathrm{id}_{T^2}$ and $\mathbf{G}(1) = g$ fixed on C, see equation (27). Regard it as a map of triples $\mathbf{G} \colon (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}_C, \mathcal{S}_C, \mathrm{id}_{T^2})$.

Then $\partial([\mathbf{G}]) = [\mathbf{G}(1)] = [g]$, and so $\partial_C^{-1}[g] = [\mathbf{G}]$. As ∂_C is an *isomorphism*, $\partial_C^{-1}[g]$ *does not depend* on a particular choice of such an isotopy $\mathbf{G}$. Furthermore, $i_1 \circ \partial_C^{-1}[g]$ is a homotopy class of $\mathbf{G}$ regarded as a map

$$\mathbf{G} \colon (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2}), \quad \mathbf{G}(t) = \mathbf{M}_{t(\sigma+1)}. \tag{29}$$

Therefore it remains to show that $[\mathbf{G}] = [q(\mathbf{M})]$, that is maps (28) and (29) are homotopic as maps of triples.

In fact the homotopy between them can be defined as follows

$$\mathbf{H}\colon (I, \partial I, 0) \times I \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2}), \quad \mathbf{H}(t,s) = \mathbf{M}_{t(s\sigma+1)}.$$

1) First we verify that $\mathbf{H}(t,s)$ *is a diffeomorphism for all* $t, s \in I$. As $g = \mathbf{M}_\sigma$ is a diffeomorhism, it follows from Lemma 6 that $\mathbf{M}_{s\sigma} = \mathbf{M}_{s\sigma+1}$ is also a diffeomorphism for all $s \in I$. But then by the same lemma $\mathbf{H}(t,s) = \mathbf{M}_{t(s\sigma+1)}$ is a diffeomorphism for all $t, s \in I$ as well.

2) Now let us show that $\mathbf{H}$ is a homotopy of maps of triples. Indeed, for each $s \in I$ we have that $\mathbf{H}_{0,s} = \mathbf{M}_0 = \mathrm{id}_{T^2}$, and $\mathbf{H}_{1,s} = \mathbf{M}_{s\sigma+1} = \mathbf{M}_{s\sigma} \in \mathcal{S}$.

3) Finally, $\mathbf{H}_{t,0} = \mathbf{M}_t = q(\mathbf{M})(t)$, and $\mathbf{H}_{t,1} = \mathbf{M}_{t(\sigma+1)} = \mathbf{G}(t)$ for all $t \in I$. Thus $\mathbf{H}$ is a homotopy between $q(\mathbf{M})$ and $\mathbf{G}$. Lemma 10 and Proposition 2 are proved. □

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