1. Introduction, definitions and results. In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard notation and fundamental results of the Nevanlinna Theory as described in [6, 13, 14]. For a nonconstant meromorphic function $f$, we denote by $T(r, f)$ the Nevanlinna characteristic of $f$ and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of a possible exceptional set $E$ of finite linear measure. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$, by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ ($r \to \infty$, $r \not\in E$). The meromorphic function $a$ is called a small function of $f$ if $T(r, a) = S(r, f)$.

Two nonconstant meromorphic functions $f$ and $g$ share a small function $a$ CM (counting multiplicities) provided that $f - a$ and $g - a$ have the same set of zeros with the same multiplicities; $f$ and $g$ share $a$ IM (ignoring multiplicities) if we do not consider the multiplicities. A finite value $z_0$ is called a fixed point of $f(z)$ if $f(z_0) = z_0$. We define

$$E_f = \{ z \in \mathbb{C} : f(z) = z, \text{counting multiplicities} \}.$$ 

Regarding a familiar question raised to W. K. Hayman ([5]), the following result was proved by M. L. Fang, X. H. Hua ([3]) in 1996.

**Theorem A.** Let $f$ and $g$ be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant $t$ such that $t^{n+1} = 1$. 

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In 2002 M. L. Fang ([2]) proved the following results extending Theorem A in which k-th derivative of $f^n$ and $g^n$ is taken into consideration.

**Theorem B.** Let $f$ and $g$ be two nonconstant entire functions, and let $n, k$ be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1, c_2$ and $c$ are three constants satisfying $(-1)^k(c_1 c_2)^n( nc)^{2k} = 1$ or $f = tg$ for a constant $t$ such that $t^n = 1$.

**Theorem C.** Let $f$ and $g$ be two nonconstant entire functions, and let $n, k$ be two positive integers with $n > 2k + 8$. If $[f^n(f - 1)]^{(k)}$ and $[g^n(g - 1)]^{(k)}$ share 1 CM, then $f = g$.

Natural question arises: What can be said if the value sharing in the above theorems is replaced by sharing a fixed point? Afterwards research works concerning the above question have been done by many mathematicians such as M. L. Fang, H. L. Qiu ([4]), W. C. Lin, H. X. Yi ([9]), X. G. Qi, L. Z. Yang ([10]), P. Sahoo ([11]), J. L. Zhang ([15]). In this direction, we recall the following results due to J. L. Zhang ([15]) proved in 2008.

**Theorem D.** Let $f$ and $g$ be two nonconstant entire functions, and $n, k$ be two positive integers with $n > 2k + 4$. If $E_{(f^n)^{(k)}} = E_{(g^n)^{(k)}}$, then either

(i) $k = 1$, $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1, c_2$ and $c$ are three constants satisfying $4(c_1 c_2)^n( nc)^{2} = -1$ or

(ii) $f = tg$ for a constant $t$ such that $t^n = 1$.

**Theorem E.** Let $f$ and $g$ be two nonconstant entire functions, and $n, k$ be two positive integers with $n > 2k + 6$. If $E_{(f^n(f - 1))^{(k)}} = E_{(g^n(g - 1))^{(k)}}$, then $f = g$.

In 2010 X. G. Qi and L. Z. Yang ([10]) and in 2011 J. Dou, X. G. Qi and L. Z. Yang ([11]) studied the uniqueness problem of entire functions concerning some general differential polynomials and proved the following results extending Theorems D and E, respectively.

**Theorem F.** Let $f$ and $g$ be two transcendental entire functions, $n, m$ and $k$ be three positive integers with $n > 2k + m^* + 4$, $\lambda$ and $\mu$ be constants that satisfy $|\lambda| + |\mu| \neq 0$. If $[f^n(\lambda f^m + \mu)]^{(k)}$ and $[g^n(\lambda g^m + \mu)]^{(k)}$ share $z$ CM, then the following conclusions hold:

(i) if $\lambda \mu \neq 0$, then $f^{d}(z) = g^{d}(z)$, where $d = \gcd(n, m)$; especially when $d = 1$, $f = g$;

(ii) if $\lambda \mu = 0$, then either $f = tg$ for a constant $t$ that satisfies $t^{n+m^*} = 1$ or $k = 1$ and $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$ for three constants $c_1, c_2$ and $c$ that satisfy

$$4(\lambda + \mu)^2(c_1 c_2)^{n+m^*}((n + m^*)c)^{2} = -1,$$

where

$$m^* = \begin{cases} m & \text{if } \lambda \neq 0; \\
0 & \text{if } \lambda = 0. \end{cases}$$

**Theorem G.** Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$ or $P(z) = C$, where $a_0, a_1, \ldots, a_{m-1}, a_m(\neq 0), C(\neq 0)$ are complex constants. Suppose that $f$ and $g$ are two transcendental entire functions, and let $n, k$ and $m$ be three positive integers with $n > 2k + m^{**} + 4$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $z$ CM, then the following conclusions hold:
Theorem 1. Let $f$ and $g$ be two transcendental entire functions, $P_1(z)$ be a nonconstant polynomial of degree $p$, and let $n, k$ and $m$ be three positive integers with $n > 2k + 2p + m^* + 2$. Suppose further that $k > p$ when $p \geq 2$. If $[f^n(\lambda f^m + \mu)]^{(k)} - P_1$ and $[g^n(\lambda g^m + \mu)]^{(k)} - P_1$ share $(0, 2)$ where $\lambda, \mu$ are constants satisfying $|\lambda| + |\mu| \neq 0$, then the following conclusions hold:

(i) if $\lambda \mu \neq 0$, then $f^d(z) = g^d(z)$, where $d = \gcd(n, m)$; especially when $d = 1$, $f = g$;
(ii) if \( \lambda_\mu = 0 \), then either \( f = tg \) for a constant \( t \) that satisfies \( t^{n+m} = 1 \) or \( f(z) = b_1 e^{iQ(z)} \), \( g(z) = b_2 e^{-iQ(z)} \), where \( Q(z) \) is a polynomial without constant such that \( Q'(z) = P_1(z) \), \( b_1, b_2 \) and \( b \) are three constants satisfying \( \mu^2(nb)^2(b_1b_2)^n = -1 \) or \( \lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1 \).

**Theorem 2.** Let \( f \) and \( g \) be two transcendental entire functions, \( P_1(z) \) be a nonconstant polynomial of degree \( p \), and let \( n, k \) and \( m \) be three positive integers with \( n > 2k+2p+m^*+2 \). Let \( P(z) \) be defined as in Theorem G. If \( f^m P(f)^{(k)} - P_1 \) and \( g^n P(g)^{(k)} - P_1 \) share \((0, 2)\) then the following conclusions hold:

(i) if \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0 \) is not a monomial, then either \( f = tg \) for a constant \( t \) that satisfies \( t^d = 1 \), where \( d = (n+m, \ldots, n+m-i, \ldots, n) \), \( a_{m-i} \neq 0 \) for some \( i \in \{0, 1, 2, \ldots, m\} \); or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where

\[
R(w_1, w_2) = w_1^n (a_m w_1^m + \ldots + a_1 w_1 + a_0) - w_2^n (a_m w_2^m + \ldots + a_1 w_2 + a_0);
\]

(ii) when \( P(z) = C \) or \( P(z) = a_m z^m \), then either \( f = tg \) for a constant \( t \) that satisfies \( t^{n+m} = 1 \), or \( f(z) = b_1 e^{iQ(z)} \), \( g(z) = b_2 e^{-iQ(z)} \), where \( Q(z) \) is a polynomial without constant such that \( Q'(z) = P_1(z) \), \( b_1, b_2 \) and \( b \) are three constants satisfying \( C^2(nb)^2(b_1b_2)^n = -1 \) or \( a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1 \).

We now explain the following definitions and notations which are used in the paper.

**Definition 2** ([6]). Let \( a \in \mathbb{C} \cup \{\infty\} \). We denote by \( N(r, a; f) \) the counting function of simple \( a \) points of \( f \). For a positive integer \( p \) we denote by \( N(r, a; f \leq p) \) the counting function of those \( a \)-points of \( f \) (counted with proper multiplicities) whose multiplicities are not greater than \( p \). By \( \overline{N}(r, a; f \leq p) \) we denote the corresponding reduced counting function.

Analogously we can define \( N(r, a; f \geq p) \) and \( \overline{N}(r, a; f \geq p) \).

**Definition 3.** Let \( a \) be any value in the extended complex plane, and let \( k \) be an arbitrary nonnegative integer. We denote by \( N_k(r, a; f) \) the counting function of \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k \) times if \( m > k \). Then

\[
N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \geq 2) + \ldots + \overline{N}(r, a; f \geq k).
\]

Clearly \( N_1(r, a; f) = \overline{N}(r, a; f) \).

2. **Lemmas.**

**Lemma 1** ([12]). Let \( f \) be a nonconstant meromorphic function and let \( a_n(z)(\neq 0), a_{n-1}(z), \ldots, a_0(z) \) be small functions of \( f \). Then

\[
T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).
\]

**Lemma 2** ([16]). Let \( f \) be a nonconstant meromorphic function, and \( p, k \) be positive integers. Then

\[
N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),
\]

\[
N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).
\]
Lemma 3 ([8]). Let $f$ and $g$ be two nonconstant meromorphic functions sharing $(1, 2)$. Then one of the following cases hold:

(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,

(ii) $f = g$,

(iii) $fg = 1$.

Lemma 4 ([6]). Let $f$ be a transcendental meromorphic function, and let $a_1(z), a_2(z)$ be two distinct meromorphic functions such that $T(r, a_1(z)) = S(r, f), i \in \{1, 2\}$. Then

$$T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 5 ([6]). Suppose that $f$ is a nonconstant meromorphic function, $k \geq 2$ is an integer. If $N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \frac{L}{T})$, then $f = e^{az+b}$, where $a(\neq 0), b$ are constants.

Lemma 6 ([11]). Let $f$ and $g$ be two nonconstant meromorphic functions and let $n, k$ be two positive integers. Suppose that $F_1 = (f^n P(f))^{(k)}$ and $G_1 = (g^n P(g))^{(k)}$ where $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0, a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0)$ are complex constants. If there exist two nonzero constants $c_1$ and $c_2$ such that $N(r, c_1; F_1) = N(r, 0; G_1)$ and $\overline{N}(r, c_2; G_1) = \overline{N}(r, 0; F_1)$, then $n \leq 2k + m + 2$.

Lemma 7. Let $f$ and $g$ be two nonconstant entire functions, $n, m$ and $k$ be three positive integers. Suppose that $F_1 G_1 = P_1^2$, where $F_1, G_1$ are defined as in Lemma 6 and $P_1(z)$ is defined as in Theorem 1. Then $n \leq k + 2p$.

Proof. If possible, we assume that $n > k + 2p$. From $F_1 G_1 = P_1^2$, we have

$$(f^n P(f))^{(k)}(g^n P(g))^{(k)} = P_1^2.$$

Let $z_0$ be a zero of $f$ with multiplicity $l$. Then $z_0$ is a zero of $(f^n P(f))^{(k)}$ with multiplicity $nl - k$. Since $g$ is an entire function and $n > k + 2p$, $z_0$ is a zero of $P_1^2$ with multiplicity at least $2p + 1$, which is absurd. Thus $f$ has no zeros. We put $f = e^\alpha$, where $\alpha$ is a nonconstant entire function. Now

$$(a_m f^{n+m})^{(k)} = t_m(\alpha', \alpha'', \ldots, \alpha^{(k)}) e^{(n+m)\alpha}, \quad (3)$$

$$\ldots$$

$$(a_0 f^n)^{(k)} = t_0(\alpha', \alpha'', \ldots, \alpha^{(k)}) e^{n\alpha}, \quad (4)$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}) (i \in \{0, 1, \ldots, m\})$ are differential polynomials in $\alpha', \alpha'', \ldots, \alpha^{(k)}$. Obviously $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}) \neq 0$ for $i \in \{0, 1, 2, \ldots, m\}$, and $(f^n P(f))^{(k)} \neq 0$. Therefore from (3) and (4) we obtain

$$t_m(\alpha', \alpha'', \ldots, \alpha^{(k)}) e^{n\alpha} + \ldots + t_0(\alpha', \alpha'', \ldots, \alpha^{(k)}) \neq 0. \quad (5)$$

Since $\alpha$ is an entire function, we have $T(r, \alpha^{(j)}) = S(r, f)$ for $j \in \{1, 2, \ldots, k\}$, and hence $T(r, t_i) = S(r, f)$ for $i \in \{0, 1, 2, \ldots, m\}$. Therefore using (5), Lemmas 1 and 4 we deduce that

$$mT(r, f) = T(r, t_m e^{n\alpha} + \ldots + t_1 e^{\alpha}) + S(r, f) \leq$$

$$\leq \overline{N}(r, 0; t_m e^{n\alpha} + \ldots + t_1 e^{\alpha}) + \overline{N}(r, 0; t_m e^{n\alpha} + \ldots + t_1 e^{\alpha} + t_0) + S(r, f) \leq$$

$$\leq \overline{N}(r, 0; t_m e^{(m-1)\alpha} + \ldots + t_1) + S(r, f) \leq (m - 1)T(r, f) + S(r, f),$$

a contradiction. Hence $n \leq k + 2p$ and the lemma follows. \qed
Lemma 8 ([10]). Let \( f \) and \( g \) be two nonconstant entire functions, \( n \), \( m \) and \( k \) be three positive integers, and let \( F_2 = [f^n(\lambda f^m + \mu)]^{(k)} \) and \( G_2 = [g^n(\lambda g^m + \mu)]^{(k)} \) where \( |\lambda| + |\mu| \neq 0 \), and \( \lambda \mu = 0 \). If there exist two nonzero constants \( c_1 \) and \( c_2 \) such that \( \overline{N}(r, c_1; F_2) = \overline{N}(r, 0; G_2) \) and \( \overline{N}(r, c_2; G_2) = \overline{N}(r, 0; F_2) \), then \( n \leq 2k + m^* + 2 \).

Lemma 9. Let \( f \) and \( g \) be two nonconstant entire functions, \( n \), \( m \) and \( k \) be three positive integers with \( n > 2k + 2p + m^* + 2 \). Further assume that \( k > p \) when \( p \geq 2 \). Suppose that \( F_2G_2 = P^2_1 \), where \( F_2, G_2 \) are defined as in Lemma 8, \( |\lambda| + |\mu| \neq 0 \) and \( P_1(z) \) is defined as in Theorem 1. Then \( f(z) = b_1 e^{bQ(z)}, g(z) = b_2 e^{-bQ(z)} \), where \( b_1, b_2 \) and \( b \) are three constants satisfying \( \lambda^2((n + m)b)^2(b_1b_2)^{n+m} = -1 \) or \( \mu^2(nb)^2(b_1b_2)^n = -1 \) and \( Q(z) \) is same as in Theorem 1.

Proof. We discuss the following two cases separately.

Case I. Let \( \lambda \mu = 0 \). Since \( |\lambda| + |\mu| \neq 0 \), we may take \( \mu = 0, \lambda \neq 0 \) and therefore \( m^* = m \). The case \( \mu \neq 0, \lambda = 0 \) can be proved similarly. First we assume that \( k = 1 \). Then \( F_2G_2 = P^2_1 \) gives

\[
(\lambda f^{n+m})'(\lambda g^{n+m})' = P^2_1. \tag{6}
\]

Since \( f \) and \( g \) are entire functions and \( n > 2k + 2p + m + 2 \), we deduce from (6) that \( f \) and \( g \) have no zeros. We put

\[
f = e^\alpha, \quad g = e^\beta, \tag{7}
\]

where \( \alpha \) and \( \beta \) are two nonconstant entire functions. Therefore

\[
\lambda^2(n + m)^2 \alpha' \beta' e^{(n+m)(\alpha + \beta)} = P^2_1. \tag{8}
\]

From (8) it follows that \( \alpha, \beta \) must be polynomials and \( \alpha + \beta \equiv C \), where \( C \) is a constant. Thus

\[
\deg(\alpha) = \deg(\beta).
\]

Therefore \( \alpha' + \beta' \equiv 0 \) and \( \lambda^2(n + m)^2 \alpha' \beta' e^{(n+m)C} = P^2_1 \). Simplifying we obtain \( \alpha' = bP_1(z) \) and \( \beta' = -bP_1(z) \), where \( b(\neq 0) \) is a constant. This gives \( \alpha = bQ(z) + d_1 \) and \( \beta = -bQ(z) + d_2 \), where \( Q(z) \) is a polynomial without constant such that \( Q'(z) = P_1(z) \) and \( d_1, d_2 \) are constants. Therefore \( f = b_1 e^{bQ(z)}, g = b_2 e^{-bQ(z)} \), where \( b_1, b_2 \) and \( b \) are three constants satisfying \( \lambda^2((n + m)b)^2(b_1b_2)^{n+m} = -1 \). Next we assume that \( k \geq 2 \). Then \( F_2G_2 = P^2_1 \) gives

\[
(\lambda f^{n+m})^{(k)}(\lambda g^{n+m})^{(k)} = P^2_1. \tag{9}
\]

Since \( f \) and \( g \) are transcendental entire function, from (9) we obtain \( N(r, 0; (\lambda f^{n+m})^{(k)}) = O\{\log r\} \). From this and (7) we get

\[
N(r, \infty; \lambda f^{n+m}) + N(r, 0; \lambda f^{n+m}) + N(r, 0; (\lambda f^{n+m})^{(k)}) = O\{\log r\}.
\]

Suppose that \( \alpha \) is a transcendental entire function. Then by Lemma 5 we deduce that \( \alpha \) is a polynomial, a contradiction. Next we assume that \( \alpha, \beta \) are polynomials of degree \( p_1 \) and \( p_2 \) respectively. If \( p_1 = p_2 = 1 \), then \( f = e^{A z + B}, g = e^{C z + D}, \) where \( A(\neq 0), B, C(\neq 0) \) and \( D \) are constants. So from (9) we obtain

\[
\lambda^2(A^2 C^k (n + m)^2 e^{(n+m)((A+C)z+(B+D))}) = P^2_1.
\]
which is not possible. Hence \( \max \{ p_1, p_2 \} > 1 \). We assume that \( p_1 > 1 \). Then \( (\lambda f^{n+m})^{(k)} = Q_1 e^{(n+m)\alpha} \) and \( (\lambda g^{n+m})^{(k)} = Q_2 e^{(n+m)\beta} \), where \( Q_1, Q_2 \) are polynomials of degree \( k(p_1 - 1) \) and \( k(p_2 - 1) \), respectively. So from (9) we obtain \( \alpha + \beta \equiv k_1 \), a constant, and hence \( p_1 = p_2 \) and \( k(p_1 - 1) = p \). This shows that \( p \geq k \geq 2 \), contradicting with the assumption that \( k > p \) when \( p \geq 2 \).

Case II. Let \( \lambda \mu \neq 0 \). Since \( n > 2k + 2p + m + 2 > k + 2p \), using the argument similar as in Lemma 7 we obtain a contradiction. \( \square \)

**Lemma 10** ([10]). Suppose that \( F_2 \) and \( G_2 \) are given as in Lemma 8 where \( \lambda \mu \neq 0 \). If \( n > 2k + m \) and \( F_2 = G_2 \), then \( f^d(z) = g^d(z) \) where \( d = \gcd(n, m) \).

**Lemma 11** ([10]). Suppose that \( F_2 \) and \( G_2 \) are given as in Lemma 8 where \( \lambda \mu = 0 \). If \( n > 2k + m^* \) and \( F_2 = G_2 \), then \( f = tg \) for a constant \( t \) satisfying \( r^{n + m^*} = 1 \).

3. Proof of the Theorems 1 and 2.

**Proof of Theorem 2.** We discuss the following three cases separately.

Case (i) Let \( P(z) = a_m z^n + a_{m-1} z^{m-1} + \ldots + a_2 z^2 + a_1 z + a_0 \), where \( a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0) \) are complex constants. We consider \( F = \frac{(n P(f))^{(k)}}{P_1(z)} \) and \( G = \frac{(g^n P(g))^{(k)}}{P_1(z)} \). Then \( F \) and \( G \) are transcendental meromorphic functions that share \( (1, 2) \). Now from Lemma 1 and (1) we obtain

\[
N_2(r, 0; F) \leq N_2(r, 0; (f^n P(f))^{(k)}) + S(r, f) \leq T(r, (f^n P(f))^{(k)}) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) + S(r, f) \leq T(r, F) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) + S(r, f). \tag{10}
\]

Similarly

\[
N_2(r, 0; G) \leq T(r, G) - (n + m)T(r, g) + N_{k+2}(r, 0; g^n P(g)) + S(r, g). \tag{11}
\]

Again by (2) we have

\[
N_2(r, 0; F) \leq N_{k+2}(r, 0; f^n P(f)) + S(r, f), \tag{12}
\]

\[
N_2(r, 0; G) \leq N_{k+2}(r, 0; g^n P(g)) + S(r, g). \tag{13}
\]

From (10) and (11) we get

\[
(n + m)\{T(r, f) + T(r, g)\} \leq T(r, F) + T(r, G) + N_{k+2}(r, 0; f^n P(f)) + N_{k+2}(r, 0; g^n P(g)) - N_2(r, 0; F) - N_2(r, 0; G) + S(r, f) + S(r, g). \tag{14}
\]

We assume that the conclusion (i) of Lemma 3 holds. Then using Lemma 1, (12) and (13) we obtain from (14)

\[
(n + m)\{T(r, f) + T(r, g)\} \leq N_2(r, 0; F) + N_2(r, 0; G) + 2N_2(r, \infty; F) + 2N_2(r, \infty; G) + N_{k+2}(r, 0; f^n P(f)) + N_{k+2}(r, 0; g^n P(g)) + S(r, f) + S(r, g) \leq 2N_{k+2}(r, 0; f^n P(f)) + 2N_{k+2}(r, 0; g^n P(g)) + S(r, f) + S(r, g) \leq 2(k + m + 2)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).
\]
From this we get \((n - m - 2k - 4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)\), which leads to a contradiction as \(n > 2k + 2p + m + 2\). Hence by Lemma 3 we have either \(FG = 1\) or \(F = G\). If \(FG = 1\), then \((f^n P(f))^{(k)}(g^n P(g))^{(k)} = P_1^k\), a contradiction by Lemma 7. Hence \(F = G\). That is \([f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}\). Integrating we get \([f^n P(f)]^{(k-1)} = [g^n P(g)]^{(k-1)} + c_{k-1}\), where \(c_{k-1}\) is a constant. If \(c_{k-1} \neq 0\), from Lemma 6 we obtain \(n \leq 2k + m\), a contradiction. Hence \(c_{k-1} = 0\). Repeating \(k\)-times, we obtain \(f^n P(f) = g^n P(g)\). Then

\[
 f^n (a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0) = g^n (a_m g^m + a_{m-1} g^{m-1} + \ldots + a_1 g + a_0). \tag{15}
\]

Let \(h = \frac{f}{g}\). If \(h\) is a constant, by putting \(f = gh\) in (15) we get

\[
a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \ldots + a_0 g^n(h^n - 1) = 0,
\]

which implies \(h^d = 1\), where \(d = (n + m, \ldots, n + m - i, \ldots, n + 1, n)\), \(a_{m-i} \neq 0\) for some \(i \in \{0, 1, \ldots, m\}\). Thus \(f = tg\) for a constant \(t\) such that \(t^d = 1\), \(d = (n + m, \ldots, n + m - i, \ldots, n + 1, n)\), \(a_{m-i} \neq 0\) for some \(i \in \{0, 1, \ldots, m\}\).

If \(h\) is not a constant, then from (15) we can say that \(f\) and \(g\) satisfy the algebraic equation \(R(f, g) = 0\), where \(R(w_1, w_2) = w_1^n (a_m w_1^m + \ldots + a_1 w_1 + a_0) - w_2^n (a_m w_2^m + \ldots + a_1 w_2 + a_0)\).

**Case (i)** Now we assume that \(P(z) = a_m z^m\), where \(a_m \neq 0\) is a complex constant. Let \(F = \frac{(a_m f^{n+m})^{(k)}}{P_1(z)}\) and \(G = \frac{(a_m g^{n+m})^{(k)}}{P_1(z)}\). Then \(F\) and \(G\) are transcendental meromorphic functions that share the value 1 with weight two. Proceeding in the similar manner as in Case (i) above we obtain either \(FG = 1\) or \(F = G\).

If \(FG = 1\), then \((a_m f^{n+m})^{(k)}(a_m g^{n+m})^{(k)} = P_1^k\). So by Lemma 9 we obtain \(f(z) = b_1 e^{bQ(z)}\), \(g(z) = b_2 e^{-bQ(z)}\), where \(b_1, b_2\) and \(b\) are three constants satisfying \(a_m^2 ((n + m)b)^2 (b_1 b_2)^{n+m} = -1\) and \(Q(z)\) is same as in Theorem 1. If \(F = G\), then using Lemmas 8 and 11 we obtain \(f = tg\) for a constant \(t\) such that \(t^{n+m} = 1\).

**Case (ii)** Let \(P(z) = C\). Taking \(F = \frac{(Cf^n)^{(k)}}{P_1(z)}\), \(G = \frac{(Cg^n)^{(k)}}{P_1(z)}\) and arguing similarly as in Case (ii) we obtain either \(f(z) = b_1 e^{bQ(z)}\), \(g(z) = b_2 e^{-bQ(z)}\), where \(b_1, b_2\) and \(b\) are three constants satisfying \(C^2 (nb)^2 (b_1 b_2)^n = -1\), \(Q(z)\) is same as in Theorem 1 or \(f = tg\) for a constant \(t\) satisfying \(t^n = 1\). 

**Proof of Theorem 1.** Let \(F = \frac{[f^n (\lambda^m + \mu)]^{(k)}}{P_1(z)}\) and \(G = \frac{[g^n (\lambda^m + \mu)]^{(k)}}{P_1(z)}\). Then \(F\) and \(G\) are transcendental meromorphic functions that share the value 1 with weight two. Proceeding similarly as in Theorem 2 we obtain either \(FG = 1\) or \(F = G\). First we assume that \(\lambda \mu \neq 0\). Then \(FG \neq 1\), by Lemma 7. Hence \(F = G\) and so by Lemmas 6 and 10 we obtain \(f^d(z) = g^d(z)\) where \(d = \text{gcd}(n, m)\). Next we assume that \(\lambda \mu = 0\). Let \(\lambda \neq 0\) and \(\mu = 0\). Then if \(FG = 1\), by Lemma 9 we have \(f(z) = b_1 e^{bQ(z)}\), \(g(z) = b_2 e^{-bQ(z)}\), where \(b_1, b_2\) and \(b\) are three constants satisfying \(\lambda^2 ((n + m)b)^2 (b_1 b_2)^{n+m} = -1\) and \(Q(z)\) is defined as in Theorem 1. Similar result holds when \(\mu \neq 0\) and \(\lambda = 0\). If \(F = G\), by Lemmas 8 and 11 we conclude that \(f = tg\) for a constant \(t\) that satisfies \(t^{n+m} = 1\).

**REFERENCES**


Department of Mathematics, University of Kalyani
Department of Mathematics, Ghurni High School(H.S), Krishnagar
West Bengal, India
sahoolak@yahoo.com, sahoopulak1@gmail.com
sahaanjanii@gmail.com

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