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O. Gutik, O. Ravsky

ON FEEBLY COMPACT INVERSE PRIMITIVE (SEMI)TOPOLOGICAL SEMIGROUPS

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We study the structure of inverse primitive feebly compact semitopological and topological semigroups. We find conditions under which the maximal subgroup of an inverse primitive feebly compact semitopological semigroup S is a closed subset of S and describe the topological structure of such semiregular semitopological semigroups. Later we describe the structure of feebly compact topological Brandt λ^0-extensions of topological semigroups and semiregular (quasi-regular) primitive inverse topological semigroups. In particular, we show that the inversion in a quasi-regular primitive inverse feebly compact topological semigroup is continuous. An analogue of the Comfort–Ross Theorem is proved for such semigroups: the Tychonoff product of an arbitrary family of primitive inverse semiregular feebly compact semitopological semigroups with closed maximal subgroups is feebly compact. We describe the structure of the Stone-Čech compactification of a Hausdorff primitive inverse countably compact semitopological semigroup S such that every maximal subgroup of S is a topological group.

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В работе изучается структура псевдокомпактных инверсных примитивных полутопологических и топологических полугрупп. Найдены условия, при которых максимальная подгруппа псевдокомпактной инверсной примитивной полутопологической полугруппы S является замкнутым подмножеством в S и описана топологическая структура таких полурегулярных полутопологических полугрупп. Далее мы описываем структуру псевдокомпактных λ^0-расширений Брандта топологических полугрупп и полурегулярных (квазирегулярных) примитивных инверсных топологических полугрупп. В частности мы показываем, что инверсия в квазирегулярной примитивной инверсной псевдокомпактной топологической полугруппе является непрерывным отображением. Мы также доказываем аналог теоремы Комфорта–Росса для таких полугрупп: тихоновское произведение произвольного семейства примитивных инверсных полурегулярных псевдокомпактных полутопологических полугрупп с замкнутыми максимальными подгруппами является псевдокомпактным просторанством. Описана структура стоун-чеховской компактификации хаусдорфовой примитивной инверсной счётно компактной полутопологической полугруппы S такой, что каждая максимальная подгруппа в S является топологической группой.

1. Introduction and preliminaries. Further we shall follow the terminology of [8, 9, 13, 25, 32]. By $\mathbb{N}$ we shall denote the set of all positive integers.

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A semigroup is a non-empty set with a binary associative operation. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element $y \in S$ is called *inverse* to x and is denoted by x^{-1}. The map assigning to each element x of an inverse semigroup S its inverse x^{-1} is called the *inversion*.

For a semigroup S by $E(S)$ we denote the subset of idempotents of S, and by S^1 (resp., S^0) we denote the semigroup S with the adjoined unit (resp., zero) (see [9, Section 1.1]). If a semigroup S has zero 0_S, then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$.

For a semilattice E the semilattice operation on E determines the partial order $\leqslant$ on E

$$e \leqslant f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. An element e of a partially ordered set X is called *minimal* if $f \leqslant e$ implies $f = e$ for $f \in X$. An idempotent e of a semigroup S without zero (with zero) is called *primitive* if e is a minimal element of $E(S)$ (of $(E(S))^*$).

Let S be a semigroup with zero and $\lambda \geqslant 1$ a cardinal. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \sqcup \{0\}$ we define a semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If S is a monoid, then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup* S ([15]). Obviously, $\mathcal{J} = \{0\} \cup \{(\alpha, \mathscr{O}, \beta) \colon \mathscr{O} \text{ is zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S) = B_\lambda(S)/\mathcal{J}$ and we shall call $B^0_\lambda(S)$ the *Brandt λ^0-extension of the semigroup S with zero* ([16]). Further, if $A \subseteq S$ then we shall denote $A_{\alpha,\beta} = \{(\alpha, s, \beta) \colon s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta) \colon s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in \lambda$. If $\mathcal{I}$ is a trivial semigroup (i.e., $\mathcal{I}$ contains only one element), then by $\mathcal{I}^0$ we denote the semigroup $\mathcal{I}$ with the adjoined zero. Obviously, for any $\lambda \geqslant 2$ the Brandt λ^0-extension of the semigroup $\mathcal{I}^0$ is isomorphic to the semigroup of $\lambda \times \lambda$-matrix units. Any Brandt λ^0-extension of a semigroup with zero contains the semigroup of $\lambda \times \lambda$-matrix units. Further by B_λ we shall denote the semigroup of $\lambda \times \lambda$-matrix units and by $B^0_\lambda(1)$ the subsemigroup of $\lambda \times \lambda$-matrix units of the Brandt λ^0-extension of a monoid S with zero.

A semigroup S with zero is called 0-*simple* if $\{0\}$ and S are its only ideals and $S \cdot S \neq \{0\}$, and *completely* 0-*simple* if it is 0-simple and has a primitive idempotent ([9]). A completely 0-simple inverse semigroup is called a *Brandt semigroup* ([25]). By Theorem II.3.5 ([25]), a semigroup S is a Brandt semigroup if and only if S is isomorphic to a Brandt λ-extension $B_\lambda(G)$ of a group G.

Let $\{S_\iota \colon \iota \in \mathscr{I}\}$ be a disjoint family of semigroups with zero such that 0_ι is zero in S_ι for any $\iota \in \mathscr{I}$. We put $S = \{0\} \cup \bigcup \{S^*_\iota \colon \iota \in \mathscr{I}\}$, where $0 \notin \bigcup \{S^*_\iota \colon \iota \in \mathscr{I}\}$, and define a semigroup operation " $\cdot$ " on S in the following way

$$s \cdot t = \begin{cases} st, & \text{if } st \in S^*_\iota \text{ for some } \iota \in \mathscr{I}; \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup S with the operation " $\cdot$ " is called an *orthogonal sum* of the semigroups $\{S_\iota \colon \iota \in \mathscr{I}\}$ and in this case we shall write $S = \sum_{\iota \in \mathscr{I}} S_\iota$.

A non-trivial inverse semigroup is called a *primitive inverse semigroup* if all its non-zero idempotents are primitive ([25]). A semigroup S is a primitive inverse semigroup if and only if S is an orthogonal sum of Brandt semigroups ([25, Theorem II.4.3]). We call a Brandt subsemigroup T of a primitive inverse semigroup S *maximal* if every Brandt subsemigroup of S which contains T, coincides with T.

In this paper all topological spaces are Hausdorff. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\operatorname{cl}_Y(A)$ and $\operatorname{int}_Y(A)$ we denote the topological closure and interior of A in Y, respectively.

A subset A of a topological space X is called *regular open* if $\operatorname{int}_X(\operatorname{cl}_X(A)) = A$.

We recall that a topological space X is:

- *semiregular* if X has a base consisting of regular open subsets;
- *quasiregular* if for any non-empty open set $U \subset X$ there exists a non-empty open set $V \subset U$ such that $\operatorname{cl}_X(V) \subseteq U$;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_i\}_{i\in\mathbb{N}}$ of X has a convergent subsequence in X;
- *countably compact* if each open countable cover of X has a finite subcover;
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;
- *feebly compact* if each locally finite open cover of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *k-space* if a subset $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K for every compact subspace $K \subseteq X$.

According to Theorem 3.10.22 of [13], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. A Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is feebly compact (see [2]).

We recall that the Stone-Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f\colon X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f}\colon \beta X \to Y$ ([13]).

A *(semi)topological semigroup* is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [12, Proposition II.1]). A Hausdorff topology τ on an (inverse) semigroup S is called an *(inverse) semigroup* if (S,τ) is a topological (inverse) semigroup. A *paratopological* (*semitopological*) *group* is a Hausdorff topological space with a jointly (separately) continuous group operation. A paratopological group with continuous inversion is a *topological group*.

Let $\mathfrak{STSG}_0$ be a class of semitopological semigroups.

Definition 1 ([15])**.** Let $\lambda \geqslant 1$ be a cardinal and $(S,\tau) \in \mathfrak{STSG}_0$ a semitopological monoid with zero. Let τ_B be a topology on $B_\lambda(S)$ such that:

a) $(B_\lambda(S), \tau_B) \in \mathfrak{STSG}_0$; and
b) for some $\alpha \in \lambda$ the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to (S,τ).

Then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ-extension of* (S,τ) *in* $\mathfrak{STSG}_0$.

Definition 2 ([16])**.** Let $\lambda \geqslant 1$ be a cardinal and $(S,\tau) \in \mathfrak{STSG}_0$. Let τ_B be a topology on $B^0_\lambda(S)$ such that

a) $(B^0_\lambda(S), \tau_B) \in \mathfrak{STSG}_0$;
b) the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to (S,τ) for some $\alpha \in \lambda$.

Then $(B^0_\lambda(S), \tau_B)$ is called a *topological Brandt λ^0-extension of* (S,τ) *in* $\mathfrak{STSG}_0$.

Later, if $\mathfrak{STSG}_0$ coincides with the class of all semitopological semigroups we shall say that $(B^0_\lambda(S), \tau_B)$ (resp., $(B_\lambda(S), \tau_B)$) is a *topological Brandt λ^0-extension* (resp., a *topological Brandt λ-extension*) of (S,τ).

Algebraic properties of Brandt λ^0-extensions of monoids with zero, non-trivial homomorphisms between them, and a category whose objects are ingredients of the construction of such extensions were described in [22]. In [19] and [22] a category whose objects are ingredients in the constructions of finite (resp., compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros were described.

O. Gutik and D. Repovš proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of a countably compact topological group H in the class of all topological inverse semigroups for some finite cardinal $\lambda \geqslant 1$ ([21]). Every 0-simple feebly compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of a feebly compact topological group H in the class of all topological inverse semigroups for some finite cardinal $\lambda \geqslant 1$ ([20]). Next O. Gutik and D. Repovš showed in [21] that the Stone-Čech compactification $\beta(T)$ of a 0-simple countably compact topological inverse semigroup T has a natural structure of a 0-simple compact topological inverse semigroup. It was proved in [20] that the same is true for 0-simple feebly compact topological inverse semigroups.

In [7] the structure of compact and countably compact primitive topological inverse semigroups was described and it was shown that any countably compact primitive topological inverse semigroup embeds to a compact primitive topological inverse semigroup.

W. W. Comfort and K. A. Ross in [10] proved that the Tychonoff product of an arbitrary family of pseudocompact topological groups is a pseudocompact topological group. They proved also that the Stone-Čech compactification of a pseudocompact topological group has a natural structure of a compact topological group. O. Ravsky in [29] generalized the Comfort–Ross Theorem and proved that the Tychonoff product of an arbitrary non-empty family of feebly compact paratopological groups is feebly compact.

In [17] the structure of feebly compact primitive topological inverse semigroups is described and it is shown that the Tychonoff product of an arbitrary non-empty family of feebly compact primitive topological inverse semigroups is feebly compact. It is proved also that the Stone-Čech compactification of a feebly compact primitive topological inverse semigroup has a natural structure of a compact primitive topological inverse semigroup.

In this paper we study the structure of inverse primitive feebly compact semitopological and topological semigroups. We find conditions under which a maximal subgroup of an inverse primitive feebly compact semitopological semigroup S is a closed subset of S and describe the topological structure of such a semiregular semigroup. Later we describe the structure of feebly compact topological Brandt λ^0-extensions of topological semigroups and semiregular (quasi-regular) primitive inverse topological semigroups. In particular we show that the inversion in a quasi-regular primitive inverse feebly compact topological semigroup is continuous. Moreover, an analogue of the Comfort–Ross Theorem is proved for such semigroups: the Tychonoff product of an arbitrary family of primitive inverse semiregular feebly compact semitopological semigroups with closed maximal subgroups is a feebly compact space. We describe the structure of the Stone-Čech compactification of a Tychonoff primitive inverse countably compact semitopological semigroup S such that every maximal subgroup of S is a topological group.

2. An adjunction of zero to a compact like semitopological group. Given a topological space (X,τ) M. H. Stone ([33]) and M. Katětov ([23]) consider the topology τ_r on X generated by the base consisting of all regular open sets in the space (X,τ). This topology is called the *semiregularization* of the topology τ. If (X,τ) is a paratopological group then (X,τ_r) is a T_3 paratopological group ([26, Ex. 1.9], [27, p. 31], and [27, p. 28]).

Lemma 1 ([3, Theorem 1.7])**.** *Each paratopological group that is a dense G_δ-subset of a regular feebly compact space is a topological group.*

We recall that a group G endowed with a topology is *left* (resp. *right*) (ω-)*precompact*, if for each neighborhood U of unit of G there exists a (countable) finite subset F of G such that $FU=G$ (resp. $UF=G$). It is easy to check (see, for instance, [26, Proposition 3.1] or [26, Proposition 2.1]) that a paratopological group G is left precompact if and only if G is right precompact, so we shall call left precompact paratopological groups to be precompact. Moreover, it is well known ([1]) that a Hausdorff topological group G is precompact if and only if G is a subgroup of a compact topological group. Theorem 1 of [5] implies the following lemma.

Lemma 2. *A Hausdorff topological group G is precompact if and only if for any neighborhood W of unit of the group G there exists a finite set $F\subset G$ such that $G=FWF$.*

Lemma 3. *Let S be a Hausdorff left topological semigroup, 0 be a right zero of the semigroup S and $G=S\setminus\{0\}$ be a subgroup of the semigroup S. Then 0 is an isolated point of the semigroup S provided one of the following conditions holds:*

1. *the group G is left precompact;*
2. *the group G is a feebly compact paratopological group;*
3. *the group G is left ω-precompact and feebly compact;*
4. *S is a feebly compact topological semigroup;*
5. *S is a topological semigroup and for each neighborhood $U\subset G$ of unit of the group G there exists a finite subset F of the group G such that $G=FU^{-1}U$.*

Proof. Assume the contrary. Put $\mathcal{F}=\{U\cap G\colon U\subset S \text{ is a neighborhood of the point } 0\}$. Since 0 is a non-isolated point of the semigroup S, the family $\mathcal{F}$ is a filter. Let $x\in G$ be an arbitrary element and U an arbitrary member of the filter $\mathcal{F}$. Since $x0=0$ and left shifts on the semigroup S are continuous, there exists a member V of the filter $\mathcal{F}$ such that $xV \subset U$. Then $V \subset x^{-1}U$, so $x^{-1}U \in \mathcal{F}$. Since S is Hausdorff, there exists a neighborhood $W \subset G$ of unit such that $G \setminus W \in \mathcal{F}$.

Now we consider cases (1)–(5) separately.

(1) Since the group G is left precompact, there exists a finite subset F of the group G such that $FW = G$. But then

$$\varnothing = G \setminus \bigcup_{x \in F} xW = \bigcap_{x\in F} x(G \setminus W) \in \mathcal{F},$$

a contradiction.

(2) Since the semiregularization G_r of the group G is a feebly compact T_3 (and, hence, a regular) paratopological group, G_r is a topological group by Lemma 1. Therefore G_r is precompact. Thus there exists a finite subset F of the group G such that $F \cdot \mathrm{cl}_G(W) = G$. But then

$$\varnothing = G \setminus \bigcup_{x \in F} x \cdot \mathrm{cl}_G(W) = \bigcap_{x\in F} x(G \setminus \mathrm{cl}_G(W)) \in \mathcal{F},$$

a contradiction.

(3) Since the group G is left ω-precompact, there exists a countable subset $C = \{c_n \colon n \in \mathbb{N}\}$ of the group G such that $CW = G$. For each positive integer n put $C_n = \{c_i \colon 1 \leq i \leq n\}$ and $V_n = G \setminus C_nW$. Since the family $\mathcal{F}$ is a filter we have that $V_n \in \mathcal{F}$. Since 0 is a non-isolated point of the semigroup S, $\mathrm{int}_G(V_n)$ is a non-empty open subset of the space G. Since the space G is feebly compact, there exists a point $x \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_G\left(\mathrm{int}_G(V_n)\right)$. Since $G = CW$ we conclude that there exists a positive integer n such that $x \in c_nW$. But

$$c_nW \cap \mathrm{cl}_G\left(\mathrm{int}_G(V_n)\right) \subset c_nW \cap \mathrm{cl}_G(V_n) = c_nW \cap \mathrm{cl}_G\left(G \setminus C_nW\right) = c_nW \cap (G \setminus C_nW) = \varnothing,$$

a contradiction.

(4) First we suppose that the space of the semigroup S is regular. Lemma 1 implies that G is a topological group. If the group G is left precompact then 0 is an isolated point of the semigroup S by Case (1). So we assume that the group G is not left precompact. By Lemma 2 there exists a neighborhood $W_0 \subset G$ of unit such that $G \neq F_0W_0F_0$ for each finite subset F_0 of the group G. The multiplication on the semigroup S is continuous. Hence there exists a member V_1 of the filter $\mathcal{F}$ such that $V_1^2 \subset G \setminus W$. Moreover, there exist a symmetric open neighborhood W_1 of unit and a member V_2 of the filter $\mathcal{F}$ such that $W_1^5V_2 \subset V_1$ and $W_1^4 \subset W_0$. Let C be a maximal subset of the set $G \setminus V_2$ such that $W_1^2c \cap W_1^2c' = \varnothing$ for distinct elements c, c' of the set C. If z is an arbitrary element of the set $G \setminus V_2$ then $W_1^2c \cap W_1^2z \neq \varnothing$ for an element c of the set C. Hence $G \setminus V_2 \subset W_1^4C$. Put $F = \{c \in C \colon W_1c \cap V_2 = \varnothing\}$. Then we have $C \setminus F \subset W_1V_2$ and hence $G \setminus V_2 \subset W_1^4C \subset W_1^4F \cup W_1^5V_2$. Then we get that $G \setminus V_1 \subset G \setminus V_2 \subset W_1^4F \cup W_1^5V_2$ and hence $G \setminus V_1 \subset W_1^4F$, because $W_1^5V_2 \subset V_1$. Since $e \notin G \setminus W \supset V_1^2 \supset (G \setminus W_1^4F)^2$, we see that $x(G \setminus W_1^4F) \not\ni e$ for each element $x \in G \setminus W_1^4F$. Then we have $(G \setminus W_1^4F)^{-1} \subset W_1^4F$ and hence $G \subset W_1^4F \cup F^{-1}W_1^4$.

Since $W_1^4 \subset W_0$ we conclude that the set F is infinite. Let C' be an arbitrary countable infinite subset of the set F. Since the space S is feebly compact we have that there exists a point $x_0 \in S$ such that each neighborhood V' of the point x_0 intersects infinitely many members of the family $\{W_1c \colon c \in C'\}$ of the open subsets of the space S. Clearly, $x_0 \neq 0$. Then $x_0 \in G$. Put $V' = W_1x_0$. Then there exist distinct elements c and c' of the set C' such that $W_1c \cap W_1x_0 \neq \varnothing$ and $W_1c' \cap W_1x_0 \neq \varnothing$. This implies $x_0 \in W_1^2c \cap W_1^2c' \neq \varnothing$, a contradiction.

Now we consider the case where the space of the semigroup S is not necessarily regular. We claim that the semiregularization S_r of the semigroup S is a regular topological semigroup.

Indeed, let $U = \operatorname{int}_S(\operatorname{cl}_S(U))$ be an arbitrary regular open subset of the space S and $x \in U$ an arbitrary point. If $x \neq 0$ then there exists an open neighborhood $W \subset G$ of unit such that $0 \notin \operatorname{cl}_S(W)$ and $xW^2 \subset U$. Then $x \in xW^2 \subset xW \operatorname{cl}_S(W) \subset \operatorname{cl}_S(U)$. Since translations by elements of the group G are homeomorphisms of the space, the set $xW \operatorname{cl}_S(W)$ is open, and hence

$$x \in xW \subset \operatorname{cl}_S(xW) \subset xW \operatorname{cl}_S(W) \subset \operatorname{int}_S(\operatorname{cl}_S(U)).$$

If $x = 0$ then there exist an open neighbourhood $W \subset G$ of unit and an open neighbourhood $V \subset S$ of x such that $WV \subset U$. Then $x \in V \subset WV \subset W \operatorname{cl}_S(V) \subset \operatorname{cl}_S(U)$. We have that $x \in V \subset \operatorname{int}_S(\operatorname{cl}_S(U))$. Let $y \in \operatorname{cl}_S(V)$ be an arbitrary point distinct from 0. Then $Wy \subset \operatorname{cl}_S(U)$ is an open neighborhood of y. Hence $y \in Wy \subset \operatorname{int}_S(\operatorname{cl}_S(U))$. Therefore the space S_r is regular.

Now we show that multiplication on the semigroup S_r is continuous. Indeed, let $x, y \in S$ be arbitrary points and $O_{xy} = \operatorname{int}_S(\operatorname{cl}_S(O_{xy})) \ni xy$ be an arbitrary regular open subset of the space S. There exist open subsets $O_x \ni x$, $O_y \ni y$ of the semigroup S such that $O_xO_y \subset O_{xy}$. Since multiplication on the semigroup S is continuous, $\operatorname{cl}_S(O_x) \cdot \operatorname{cl}_S(O_y) \subset \operatorname{cl}_S(O_{xy})$. Let $x' \in \operatorname{cl}_S(O_x)$, $y' \in \operatorname{cl}_S(O_y)$ be arbitrary points. If $x' \neq 0$ then since left translations by elements of the group G are homeomorphisms of S onto itself, the set $x' \operatorname{int}_S(\operatorname{cl}_S(O_y))$ is open, so $x'y' \in \operatorname{int}_S(\operatorname{cl}_S(O_{xy}))$. Similarly, if $y' \neq 0$ then $x'y' \in \operatorname{int}_S(\operatorname{cl}_S(O_{xy}))$ too. If $x = y = 0$ does not hold then we can choose neighborhoods O_x and O_y so small that $\operatorname{cl}_S(O_x) \cap \operatorname{cl}_S(O_x) \not\ni 0$. Then necessarily $x' \neq 0$ or $y' \neq 0$. If $x = y = 0$ and $x' = y' = 0$ then $x'y' = xy \in \operatorname{int}_S(\operatorname{cl}_S(O_{xy}))$ by the choice of the neighborhood O_{xy}. Therefore, in all the cases we have $x'y' \in \operatorname{int}_S(\operatorname{cl}_S(O_{xy}))$. Thus $\operatorname{int}_S(\operatorname{cl}_S(O_x)) \cdot \operatorname{int}_S(\operatorname{cl}_S(O_y)) \subset \operatorname{int}_S(\operatorname{cl}_S(O_{xy}))$.

So, by the already proved case of the regular semigroup, 0 is an isolated point of the semigroup S_r. Since the topology of the semigroup S_r is weaker than the topology of the semigroup S, 0 is an isolated point of the semigroup S.

(5) Since the multiplication on the semigroup S is continuous, there exist a neighborhood $W_1 \subset W$ of unit and a member V of the filter $\mathcal{F}$ such that $W_1V \subset G \setminus W$. Then $W_1V \cap W = \varnothing$, so $V \cap W_1^{-1}W_1 \subset V \cap W_1^{-1}W = \varnothing$. Hence $G \setminus W_1^{-1}W_1 \in \mathcal{F}$. By the assumption, there exists a finite subset F of the group G such that $G = FW^{-1}W$. Then

$$\mathcal{F} \ni \bigcap_{x \in F} x(G \setminus W_1^{-1}W_1) = G \setminus \bigcup_{x \in F} xW_1^{-1}W \neq \varnothing,$$

a contradiction. □

Remark 1. Authors do not know, if a counterpart of Lemma 3 holds if the group G is a countably compact semitopological group.

3. Feebly compact topological Brandt λ^0-extensions of topological semigroups and primitive inverse semitopological semigroups.

Proposition 1. *Let S be a Hausdorff semitopological semigroup such that S is an orthogonal sum of the family $\{B^0_{\lambda_i}(S_i) : i \in \mathscr{I}\}$ of topological Brandt λ_i^0-extensions of semitopological monoids with zeros. Then for every non-zero element $(\alpha_i, g_i, \beta_i) \in (S_i)_{\alpha_i,\beta_i} \subseteq B^0_{\lambda_i}(S_i) \subseteq S$ there exists an open neighborhood $U_{(\alpha_i,g_i,\beta_i)}$ of (α_i, g_i, β_i) in S such that $U_{(\alpha_i,g_i,\beta_i)} \subseteq (S_i)^*_{\alpha_i,\beta_i}$ and hence every set $(S_i)^*_{\alpha_i,\beta_i}$ is an open subset of S.*

Proof. Suppose the contrary that $U_{(\alpha_i,g_i,\beta_i)} \nsubseteq (S_i)^0_{\alpha_i,\beta_i}$ for every open neighborhood $U_{(\alpha_i,g_i,\beta_i)}$ of (α_i, g_i, β_i) in S. Hausdorffness of S implies that there exists an open neighborhood $V_{(\alpha_i,g_i,\beta_i)}$ of (α_i, g_i, β_i) in S such that $0 \notin V_{(\alpha_i,g_i,\beta_i)}$. By the separate continuity of the multiplication in S there exists an open neighborhood $W_{(\alpha_i,g_i,\beta_i)}$ of (α_i, g_i, β_i) in S such that

$$W_{(\alpha_i,g_i,\beta_i)} \cdot (\beta_i, e_i, \beta_i) \subseteq V_{(\alpha_i,g_i,\beta_i)} \text{ and } (\alpha_i, e_i, \alpha_i) \cdot W_{(\alpha_i,g_i,\beta_i)} \subseteq V_{(\alpha_i,g_i,\beta_i)}.$$

Then condition $W_{(\alpha_i,g_i,\beta_i)} \nsubseteq (S_i)^*_{\alpha_i,\beta_i}$ implies that either $W_{(\alpha_i,g_i,\beta_i)}\cdot(\beta_i, e_i, \beta_i) \ni 0$ or $(\alpha_i, e_i, \alpha_i)\cdot W_{(\alpha_i,g_i,\beta_i)} \ni 0$, a contradiction. The obtained contradiction implies the statement of the proposition. □

Corollary 1. *Let S be a Hausdorff primitive inverse semitopological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Then the following statements hold:*

- (i) *for every non-zero element $(\alpha_i, g_i, \beta_i) \in (G_i)_{\alpha_i,\beta_i} \subseteq B_{\lambda_i}(G_i) \subseteq S$ there exists an open neighborhood $U_{(\alpha_i,g_i,\beta_i)}$ of (α_i, g_i, β_i) in S such that $U_{(\alpha_i,g_i,\beta_i)} \subseteq (G_i)_{\alpha_i,\beta_i}$ and hence every subset $(G_i)_{\alpha_i,\beta_i}$ is an open subset of S;*
- (ii) *every non-zero idempotent of S is an isolated point of $E(S)$.*

Proof. Assertion (i) follows from Proposition 1 and (ii) follows from (i). □

Proposition 2. *Let S be a Hausdorff countably compact semitopological semigroup such that S is an orthogonal sum of the family $\{B^0_{\lambda_i}(S_i) : i \in \mathscr{I}\}$ of topological Brandt λ^0_i-extensions of semitopological monoids with zeros. Then for every open neighborhood $U(0)$ of zero 0 in S the set of pairs of indices (α_i, β_i) such that $(S_i)_{\alpha_i,\beta_i} \nsubseteq U(0)$ is finite. Moreover, every maximal topological Brandt λ^0_i-extension $B^0_{\lambda_i}(S_i)$, $i \in \mathscr{I}$, is countably compact.*

Proof. Suppose to the contrary that there exists an open neighborhood $U(0)$ of zero 0 in S such that $(S_i)_{\alpha_i,\beta_i} \nsubseteq U(0)$ for infinitely many pairs of indices (α_i, β_i). Then for every such $(S_i)_{\alpha_i,\beta_i}$ we choose a point $x_{\alpha_i,\beta_i} \in (S_i)_{\alpha_i,\beta_i} \setminus U(0)$ and put $A = \bigcup\{x_{\alpha_i,\beta_i}\}$. Then A is infinite and Proposition 1 implies that the set A has no accumulation point of S. This contradicts Theorem 3.10.3 of [13]. The obtained contradiction implies the first statement of the proposition.

The second statement follows from Proposition 1, because by Theorem 3.10.4 of [13] every closed subspace of a countably compact space is countably compact. □

Proposition 2 implies the following corollary.

Corollary 2. *Let S be a Hausdorff primitive inverse countably compact semitopological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathscr{I}\}$ of semitopological Brandt's semigroups with zeros. Then for every open neighborhood $U(0)$ of zero 0 in S the set of pairs of indices (α_i, β_i) such that $(S_i)_{\alpha_i,\beta_i} \nsubseteq U(0)$ is finite. Moreover, every maximal Brandt subsemigroup $B_{\lambda_i}(G_i)$, $i \in \mathscr{I}$, is countably compact.*

Proposition 3. *Let S be a Hausdorff feebly compact semitopological semigroup such that S is an orthogonal sum of the family $\{B^0_{\lambda_i}(S_i) : i \in \mathscr{I}\}$ of topological Brandt λ^0_i-extensions of semitopological monoids with zeros. Then*

- (i) *every maximal topological Brandt λ^0_i-extension $B^0_{\lambda_i}(S_i)$, $i \in \mathscr{I}$, is feebly compact;*
- (ii) *the subspace $(S_i)_{\alpha_i,\beta_i}$ is feebly compact for all $\alpha_i, \beta_i \in \lambda_i$.*

Proof. (i) Let $\mathscr{F}=\{U_\alpha\colon \alpha\in\mathscr{J}\}$ be a infinite family of open non-empty subsets of $B^0_{\lambda_i}(S_i)$. If 0 is contained in infinitely many members of the family $\mathscr{F}$ then it is not locally finite. In the opposite case the family $\mathscr{F}$ contains an infinite subfamily $\mathscr{F}'$ no member of which contains 0. Since the space S is feebly compact, there exists a point $x\in S$ such that each neighborhood of x intersects infinitely many members the family $\mathscr{F}'$. Suppose that $x\in U=B^0_{\lambda_j}(S_j)\setminus\{0\}$ for some index $j\neq i$. By Proposition 1, U is an open subset of S. But $U\cap U_\alpha=\varnothing$ for each member U_α of the family $\mathscr{F}'$. Hence $x\in B^0_{\lambda_i}(S_i)$, a contradiction. Thus the family $\mathscr{F}'$ is not locally finite in $B^0_{\lambda_i}(S_i)$.

(ii) Since the semigroup operation in S is separately continuous the map $f_{\alpha_i,\beta_i}\colon S\to S\colon x\mapsto(\alpha_i,1_{S_i},\alpha_i)\cdot x\cdot(\beta_i,1_{S_i},\beta_i)$ is continuous too, and hence $(S_i)_{\alpha_i,\beta_i}$ is a feebly compact subspace of S as a continuous image of a feebly compact space. □

Proposition 3 implies the following corollary.

Corollary 3. *Let S be a Hausdorff primitive inverse feebly compact semitopological semigroup and S an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i\in\mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Then*

- (i) *every maximal Brandt semigroup $B_{\lambda_i}(G_i)$, $i\in\mathscr{I}$, is feebly compact;*
- (ii) *$(G_i)^0_{\alpha_i,\beta_i}$ is feebly compact for all $\alpha_i,\beta_i\in\lambda_i$.*

Proposition 4. *Let S be a semiregular feebly compact semitopological semigroup such that S is an orthogonal sum of the family $\{B^0_{\lambda_i}(S_i)\colon i\in\mathscr{I}\}$ of topological Brandt λ^0_i-extensions of semitopological monoids with zeros. Then for every open neighborhood $U(0)$ of zero 0 in S the set of pairs of indices (α_i,β_i) such that $(S_i)_{\alpha_i,\beta_i}\nsubseteq U(0)$ is finite.*

Proof. Since the semigroup S is semiregular, there exists a regular open neighborhood $V(0)$ of zero 0 in S such that $V(0)\subset U(0)$. Let $\mathscr{A}=\{(\alpha_i,\beta_i)\colon(S_i)_{\alpha_i,\beta_i}\nsubseteq V(0)\}$. Let $(\alpha_i,\beta_i)\in\mathscr{A}$ be an arbitrary pair. The set $(S_i)'_{\alpha_i,\beta_i}=(S_i)^*_{\alpha_i,\beta_i}\setminus\operatorname{cl}_S V(0)$ is a non-empty open subset of the topological space S. Indeed, in the opposite case $(S_i)_{\alpha_i,\beta_i}\subseteq\operatorname{cl}_S V(0)$ and since by Proposition 1 the set $(S_i)^*_{\alpha_i,\beta_i}$ is open and the set $V(0)$ is regular open, we have $(S_i)_{\alpha_i,\beta_i}\subseteq\operatorname{int}_S(\operatorname{cl}_S(V(0)))=V(0)$, a contradiction. One can easily check that the family $\mathscr{P}=\{(S_i)'_{\alpha_i,\beta_i}\colon(\alpha_i,\beta_i)\in\mathscr{A}\}$ is a locally finite family of open subsets of the topological space S. Since S is feebly compact, the family $\mathscr{P}$ is finite, so the family $\mathscr{A}$ is finite too. □

Proposition 4 implies the following corollary.

Corollary 4. *Let S be a semiregular primitive inverse feebly compact semitopological semigroup and S an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i\in\mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Then for every open neighborhood $U(0)$ of zero 0 in S the set of pairs of indices (α_i,β_i) such that $(G_i)_{\alpha_i,\beta_i}\nsubseteq U(0)$ is finite.*

The structure of primitive Hausdorff feebly compact topological inverse semigroup is described in [17]. It is proved that every primitive Hausdorff feebly compact topological inverse semigroup S is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}}B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of pseudocompact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i\geqslant 1$. Moreover, [17] contains a description of a base of the topology of a primitive Hausdorff feebly compact topological inverse semigroup. Similar results for the primitive Hausdorff countably compact topological inverse semigroups and Hausdorff compact topological inverse semigroups were obtained in [7].

The following example shows that counterparts of these results do not hold for primitive Hausdorff compact (and hence countably compact and feebly compact) semitopological inverse semigroups with continuous inversion.

Example 1. Let $\mathbb{Z}(+)$ be the discrete additive group of integers and $\mathcal{O} \notin \mathbb{Z}(+)$. We put Z^0 to be $\mathbb{Z}(+)$ with adjoined zero $\mathcal{O}$ and consider the topology of the one-point Alexandroff compactification on Z^0 with the remainder $\{\mathcal{O}\}$. Simple verifications show that Z^0 is a Hausdorff compact semitopological inverse semigroup with continuous inversion.

We fix an arbitrary cardinal $\lambda \geqslant 1$. Define a topology τ_B on $B^0_\lambda(Z^0)$ as follows:

(i) all non-zero elements of $B^0_\lambda(Z^0)$ are isolated points;

(ii) the family $\mathscr{P}(0) = \{U(\alpha,\beta,n)\colon \alpha,\beta\in\lambda,\ n\in\mathbb{N}\}$, where

$$U(\alpha,\beta,n) = B^0_\lambda(Z^0)\setminus(\{-n,-n+1,\ldots,n-1,n\})_{\alpha,\beta},$$

forms a pseudobase of the topology τ_B at zero.

Simple verifications show that $(B^0_\lambda(Z^0),\tau_B)$ is a Hausdorff compact semitopological inverse semigroup with continuous inversion, and moreover the space $(B^0_\lambda(Z^0),\tau_B)$ is homeomorphic to the one-point Alexandroff compactification of the discrete space of cardinality $\max\{\lambda,\omega\}$ with the remainder zero of the semigroup $B^0_\lambda(Z^0)$.

Theorem 1. *Let S be a Hausdorff primitive inverse countably compact semitopological semigroup and S an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i\in\mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i\in\mathscr{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i\in\lambda_i$, such that at least the one of the following conditions holds:*

(1) the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;

(2) $(G_i)_{\alpha_i,\alpha_i}$ is a feebly compact paratopological group;

(3) the group $(G_i)_{\alpha_i,\alpha_i}$ is left ω-precompact feebly compact;

(4) the semigroup $S_{\alpha_i,\alpha_i} = (G_i)_{\alpha_i,\alpha_i}\cup\{0\}$ is a topological semigroup.

Then the following assertions hold:

(i) every maximal subgroup of S is a closed subset of S and hence is countably compact;

(ii) for every $i\in\mathscr{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a countably compact topological Brandt λ-extension of a countably compact semitopological group G_i;

(iii) if $\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$ is a base of the topology at unit (α_i,e_i,α_i) of a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$ of S, $i\in\mathscr{I}$, such that $U\subseteq(G_i)_{\alpha_i,\alpha_i}$ for any $U\in\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$, then the family

$$\mathscr{B}_{(\beta_i,x,\gamma_i)} = \left\{(\beta_i,x,\alpha_i)\cdot U\cdot(\alpha_i,e_i,\gamma_i)\colon U\in\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}\right\}$$

is a base of the topology of S at the point $(\beta_i,x,\gamma_i)\in(G_i)_{\beta_i,\gamma_i}\subseteq B_{\lambda_i}(G_i)$, for all $\beta_i,\gamma_i\in\lambda_i$;

(iv) the family

$$\mathscr{B}_0 = \big\{S\setminus\big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}}\cup\cdots\cup(G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big)\colon i_1,\ldots,i_k\in\mathscr{I},\alpha_{i_k},\beta_{i_k}\in\lambda_{i_k},\ k\in\mathbb{N},\{(\alpha_{i_1},\beta_{i_1}),\ldots,(\alpha_{i_k},\beta_{i_k})\}\text{ is finite}\big\}$$

is a base of the topology at zero of S.

Proof. (i) Fix an arbitrary maximal subgroup G of S. Without loss of generality we can assume that G is a non-zero subgroup of S. Then there exists a maximal Brand subsemigroup $B_{\lambda_i}(G_i)$, $i \in \mathscr{I}$, which contains G. The separate continuity of the multiplication in S implies that for all $\alpha_i, \beta_i, \gamma_i, \delta_i \in \lambda_i$ the map $\psi^{\alpha_i,\beta_i}_{\gamma_i,\delta_i}\colon S \to S$ defined by the formula $\psi^{\alpha_i,\beta_i}_{\gamma_i,\delta_i}(x) = (\gamma_i, e_i, \alpha_i) \cdot x \cdot (\beta_i, e_i, \delta_i)$, where e_i is unit of the group G_i, is continuous. Since for all $\alpha_i, \beta_i, \gamma_i, \delta_i \in \lambda_i$ the restrictions $\psi^{\alpha_i,\beta_i}_{\gamma_i,\delta_i}|_{(G_i)_{\alpha_i,\beta_i}}\colon (G_i)_{\alpha_i,\beta_i} \to (G_i)_{\gamma_i,\delta_i}$ and $\psi^{\gamma_i,\delta_i}_{\alpha_i,\beta_i}|_{(G_i)_{\gamma_i,\delta_i}}\colon (G_i)_{\gamma_i,\delta_i} \to (G_i)_{\alpha_i,\beta_i}$ are bijective continuous maps we conclude that $(G_i)_{\alpha_i,\beta_i}$ and $(G_i)_{\gamma_i,\delta_i}$ are homeomorphic subspaces of S, and moreover the semitopological subgroups $(G_i)_{\alpha_i,\alpha_i}$ and $(G_i)_{\gamma_i,\gamma_i}$ are topologically isomorphic for all indices $\alpha_i, \gamma_i \in \lambda_i$. Therefore G is topologically isomorphic to the semitopological subgroup $(G_i)_{\alpha_i,\alpha_i}$ for any $\alpha_i \in \lambda_i$. For any $\alpha_i, \beta_i \in \lambda_i$ we put $S_{\alpha_i,\beta_i} = (G_i)_{\alpha_i,\beta_i} \cup \{0\}$. Then for all $\alpha_i, \beta_i, \gamma_i, \delta_i \in \lambda_i$ the restrictions $\psi^{\alpha_i,\beta_i}_{\gamma_i,\delta_i}|_{S_{\alpha_i,\beta_i}}\colon S_{\alpha_i,\beta_i} \to S_{\gamma_i,\delta_i}$ and $\psi^{\gamma_i,\delta_i}_{\alpha_i,\beta_i}|_{S_{\gamma_i,\delta_i}}\colon S_{\gamma_i,\delta_i} \to S_{\alpha_i,\beta_i}$ are bijective continuous maps. Hence S_{α_i,β_i} and S_{γ_i,δ_i} are homeomorphic subspaces of S, and moreover the semitopological subsemigroups S_{α_i,α_i} and S_{γ_i,γ_i} are topologically isomorphic for all indices $\alpha_i, \gamma_i, \in \lambda_i$. Now Lemma 3 implies that 0 is an isolated point of S_{α_i,α_i}. Indeed, if one of conditions (1)–(3) of the theorem is satisfied then we can directly apply Lemma 3 and if condition (4) of the theorem is satisfied then we observe that for each λ_i and $\alpha_i \in \lambda_i$ the subsemigroup S_{α_i,α_i} of S is countably compact as a retract of S. Hence S_{α_i,α_i} is feebly compact and then again Lemma 3 applies. By Corollary 1, $(G_i)_{\alpha_i,\alpha_i}$ is a closed subspace of S and by Theorem 3.10.4 of [13] $(G_i)_{\alpha_i,\alpha_i}$ is countably compact, and hence so is G, too.

(ii) The arguments presented in the proof of assertion (i) imply that for every $i \in \mathscr{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a topological Brandt λ-extension of a countably compact semitopological group G_i. By Corollary 1 we have that for every $i \in \mathscr{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a closed subset of S. By Theorem 3.10.4 of [13] $B_{\lambda_i}(G_i)$ is countably compact.

Assertion (iii) follows from (ii).

(iv) follows from Corollary 2 and assertions (i) and (ii). □

The proof of the following theorem is similar to the proof of Theorem 1 and makes use of Corollary 3 and Proposition 4.

Theorem 2. *Let S be a Hausdorff primitive inverse feebly compact semitopological semigroup and S be the orthogonal sum of a family $\{B_{\lambda_i}(G_i)\colon i \in \mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i \in \mathscr{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least one of the following conditions holds:*

(1) the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;

(2) $(G_i)_{\alpha_i,\alpha_i}$ is a feebly compact paratopological group;

(3) the group $(G_i)_{\alpha_i,\alpha_i}$ is left ω-precompact feebly compact;

(4) the semigroup $S_{\alpha_i,\alpha_i} = (G_i)_{\alpha_i,\alpha_i} \cup \{0\}$ is a topological semigroup.

Then the following assertions hold:

(i) every maximal subgroup of S is an open-and-closed subset of S and hence is pseudocompact;

(ii) for every $i \in \mathscr{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a feebly compact topological Brandt λ-extension of a feebly compact semitopological group G_i;

(*iii*) *if* $\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$ *is a base of the topology at unit* $(\alpha_i, e_i, \alpha_i)$ *of a maximal non-zero subgroup* $(G_i)_{\alpha_i,\alpha_i}$ *of* S, $i \in \mathscr{I}$, *such that* $U \subseteq (G_i)_{\alpha_i,\alpha_i}$ *for any* $U \in \mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$, *then the family*

$$\mathscr{B}_{(\beta_i,x,\gamma_i)} = \left\{(\beta_i, x, \alpha_i) \cdot U \cdot (\alpha_i, e_i, \gamma_i) \colon U \in \mathscr{B}_{(\alpha_i,e_i,\alpha_i)}\right\}$$

is a base of the topology at the point $(\beta_i, x, \gamma_i) \in (G_i)_{\beta_i,\gamma_i} \subseteq B_{\lambda_i}(G_i)$, *for all* $\beta_i, \gamma_i \in \lambda_i$;

if, in addition, the topological space S *is semiregular then*

(*iv*) *the family*

$$\begin{aligned}\mathscr{B}_0 = \big\{S\backslash\big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big) \colon i_1, \ldots, i_k \in \mathscr{I}, \alpha_{i_k}, \beta_{i_k} \in \lambda_{i_k}, k \in \mathbb{N};\\ \text{if } \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\} \text{ is finite}\big\}\end{aligned} \qquad (1)$$

is a base of the topology at zero of S.

The following example shows that in the case of the primitive Hausdorff feebly compact semitopological inverse semigroups with compact maximal subgroups and continuous inversion statement (*iii*) of Theorem 2 does not hold.

Example 2. Let λ be an infinite cardinal and $\mathbb{T}$ unit circle with the usual multiplication of complex numbers and the usual topology $\tau_{\mathbb{T}}$. It is obvious that $(\mathbb{T}, \tau_{\mathbb{T}})$ is a topological group. The base of the topology τ_B on the Brandt semigroup $B_\lambda(\mathbb{T})$ we define as follows:

1) for every non-zero element (α, x, β) of the semigroup $B_\lambda(\mathbb{T})$ the family

$$\mathscr{B}_{(\alpha,x,\beta)} = \{(\alpha, U(x), \beta) \colon U(x) \in \mathscr{B}_{\mathbb{T}}(x)\},$$

where $\mathscr{B}_{\mathbb{T}}(x)$ is a base of the topology $\tau_{\mathbb{T}}$ at the point $x \in \mathbb{T}$, is the base of the topology τ_B at $(\alpha, x, \beta) \in B_\lambda(\mathbb{T})$;

2) the family

$$\mathscr{B}_0 = \{U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) \colon \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \lambda, x_1, \ldots, x_k \in \mathbb{T}, n, k \in \mathbb{N}\},$$

where

$$\begin{aligned}U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) =\\ = B_\lambda(\mathbb{T}) \setminus (\mathbb{T}_{\alpha_1,\beta_1} \cup \cdots \cup \mathbb{T}_{\alpha_n,\beta_n} \cup \{(\alpha, x_i, \beta) \colon \alpha, \beta \in \lambda, i \in \{1, \ldots, k\}\}),\end{aligned}$$

is the base of the topology τ_B at zero $0 \in B_\lambda(\mathbb{T})$.

Simple verifications show that $(B_\lambda(\mathbb{T}), \tau_B)$ is a non-semiregular Hausdorff feebly compact topological space for every infinite cardinal λ. We show that multiplication on $(B_\lambda(\mathbb{T}), \tau_B)$ is separately continuous. The proof of the separate continuity of multiplication in the cases $0 \cdot 0$ and $(\alpha, x, \beta) \cdot (\gamma, y, \delta)$, where $\alpha, \beta, \gamma, \delta \in \lambda$ and $x, y \in \mathbb{T}$, is trivial. Hence we only consider the cases $(\alpha, x, \beta) \cdot 0$ and $0 \cdot (\alpha, x, \beta)$.

Then we have

$$\begin{aligned}(\alpha, x, \beta) \cdot U(\beta, \beta_1; \ldots; \beta, \beta_n; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) \subseteq\\ \subseteq \{0\} \cup \bigcup \{\mathbb{T}_{\alpha,\gamma} \setminus \{(\alpha, xx_1, \gamma), \ldots, (\alpha, xx_k, \gamma)\} : \gamma \in \lambda \setminus \{\beta_1, \ldots, \beta_n\}\} \subseteq\\ \subseteq U(\alpha, \beta_1; \ldots; \alpha, \beta_n; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; xx_1, \ldots, xx_k) \subseteq U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; xx_1, \ldots, xx_k)\end{aligned}$$

and similarly

$$U(\alpha_1,\alpha;\ldots;\alpha_n,\alpha;\alpha_1,\beta_1;\ldots;\alpha_n,\beta_n;x_1,\ldots,x_k)\cdot(\alpha,x,\beta)\subseteq$$
$$\subseteq\{0\}\cup\bigcup\{\mathbb{T}_{\gamma,\beta}\setminus\{(\gamma,x_1x,\beta),\ldots,(\gamma,x_kx,\beta)\}:\gamma\in\lambda\setminus\{\alpha_1,\ldots,\alpha_n\}\}\subseteq$$
$$\subseteq U(\alpha_1,\beta;\ldots;\alpha_n,\beta;\alpha_1,\beta_1;\ldots;\alpha_n,\beta_n;x_1x,\ldots,x_kx)\subseteq U(\alpha_1,\beta_1;\ldots;\alpha_n,\beta_n;x_1x,\ldots,x_kx),$$

for all $U(\alpha_1,\beta_1;\ldots;\alpha_n,\beta_n;xx_1,\ldots,xx_k)$, $U(\alpha_1,\beta_1;\ldots;\alpha_n,\beta_n;x_1x,\ldots,x_kx)\in\mathscr{B}_0$. This completes the proof of the separate continuity of multiplication in $(B_\lambda(\mathbb{T}),\tau_B)$.

Proposition 5. *The space $(B_\lambda(\mathbb{T}),\tau_B)$ is countably pracompact if and only if $\lambda\leqslant\mathfrak{c}$.*

Proof. ($\Leftarrow$) Suppose that $\lambda\leqslant\mathfrak{c}$. Then there exists a countable dense subgroup H of $\mathbb{T}$. Let $\mathfrak{H}_H$ be the family of all distinct conjugate classes of subgroup H in $\mathbb{T}$. Since the subgroup H is countable we conclude that the cardinality of $\mathfrak{H}_H$ is $\mathfrak{c}$. This implies that there exists a one-to-one (not necessary bijective) map $f\colon\lambda\times\lambda\to\mathfrak{H}_H\colon(\alpha,\beta)\mapsto g_{\alpha,\beta}H$. Then by the definition of the topology τ_B we have that $A=\bigcup_{\alpha,\beta\in\lambda}(g_{\alpha,\beta}H)_{\alpha,\beta}$ is a dense subset of the topological space $(B_\lambda(\mathbb{T}),\tau_B)$. Fix an arbitrary infinite countable subset Q of A. If the set $Q\cap\mathbb{T}_{\alpha,\beta}$ is infinite for some $\alpha,\beta\in\lambda$ then the compactness of $\mathbb{T}$ implies that Q has an accumulation point in $\mathbb{T}_{\alpha,\beta}$, and hence in $(B_\lambda(\mathbb{T}),\tau_B)$. In the other case, by the definition of the topology τ_B we have that zero 0 is an accumulation point of Q. Therefore the space $(B_\lambda(\mathbb{T}),\tau_B)$ is countably compact at A, and hence it is countably pracompact.

($\Rightarrow$) Suppose that there exists a cardinal $\lambda>\mathfrak{c}$ such that the space $(B_\lambda(\mathbb{T}),\tau_B)$ is countably pracompact. Then there exists a dense subset A of $(B_\lambda(\mathbb{T}),\tau_B)$ such that the space $(B_\lambda(\mathbb{T}),\tau_B)$ is countably compact at A. By the definition of the topology τ_B we have that $A\cap\mathbb{T}_{\alpha,\beta}$ is a dense subset in $\mathbb{T}_{\alpha,\beta}$ for all $\alpha,\beta\in\lambda$. Since $\lambda>\mathfrak{c}$ and $|\mathbb{T}|=\mathfrak{c}$ we conclude that there exists a point $x\in\mathbb{T}$ such that $(\alpha,x,\beta)\in A$ for infinitely many distinct pairs (α,β) of indices in λ. Put $\mathcal{K}=\{(\alpha,\beta)\in\lambda\times\lambda\colon(\alpha,x,\beta)\in A\}$. By the definition of the topology τ_B one has that for every infinite countable subset $\mathcal{K}_0\subseteq\mathcal{K}$ the set $\{(\alpha,x,\beta)\colon(\alpha,\beta)\in\mathcal{K}_0\}$ has no accumulation point in $(B_\lambda(\mathbb{T}),\tau_B)$, a contradiction. □

The proof of the following proposition is similar to the proof of Proposition 22 of [18].

Proposition 6. *Let S be a semiregular feebly compact (Hausdorff countably compact) semitopological semigroup such that S is an orthogonal sum of the family $\{B^0_{\lambda_i}(S_i)\colon i\in\mathscr{I}\}$ of topological Brandt λ_i^0-extensions of semitopological monoids with zeros, i.e. $S=\sum_{i\in\mathscr{I}}B^0_{\lambda_i}(S_i)$. Then the following assertions hold:*

- (*i*) *the topological space S is regular if and only if the space S_i is regular for each $i\in\mathscr{I}$;*
- (*ii*) *the topological space S is Tychonoff if and only if the space S_i is Tychonoff for each $i\in\mathscr{I}$;*
- (*iii*) *the topological space S is normal if and only if the space S_i is normal for each $i\in\mathscr{I}$.*

The following theorem characterizes feebly compact topological Brandt λ^0-extensions of topological monoids with zero in the class of Hausdorff topological semigroups.

Theorem 3. *A topological Brandt λ^0-extension $(B^0_\lambda(S),\tau_B)$ of a topological monoid (S,τ_S) with zero in the class of Hausdorff topological semigroups is feebly compact if and only if the cardinal λ is finite and the space (S,τ_S) is feebly compact.*

Proof. ($\Leftarrow$) The continuity of multiplication in $(B^0_\lambda(S),\tau_B)$ implies that for all $\alpha,\beta,\gamma,\delta\in\lambda$ the map $\psi^{\alpha,\beta}_{\gamma,\delta}\colon B^0_\lambda(S)\to B^0_\lambda(S)$ defined by the formula $\psi^{\alpha,\beta}_{\gamma,\delta}(x)=(\gamma,1_S,\alpha)\cdot x\cdot(\beta,1_S,\delta)$, where 1_S is unit of the semigroup S, is continuous. Since for all $\alpha,\beta,\gamma,\delta\in\lambda$ the restrictions $\psi^{\alpha,\beta}_{\gamma,\delta}|_{S_{\alpha,\beta}}\colon S_{\alpha,\beta}\to S_{\gamma,\delta}$ and $\psi^{\gamma,\delta}_{\alpha,\beta}|_{S_{\gamma,\delta}}\colon S_{\gamma,\delta}\to S_{\alpha,\beta}$ are bijective continuous maps we conclude that $S_{\alpha,\beta}$ and $S_{\gamma,\delta}$ are homeomorphic subspaces of $(B^0_\lambda(S),\tau_B)$. Therefore the space $(B^0_\lambda(S),\tau_B)$ is the union of finitely many copies of the feebly compact topological space (S,τ_S), and hence it is feebly compact.

($\Rightarrow$) Suppose that a topological Brandt λ^0-extension $(B^0_\lambda(S),\tau_B)$ of a topological monoid (S,τ_S) with zero in the class of topological semigroups is feebly compact. Then by Proposition 3(ii) the space (S,τ_S) is feebly compact.

Suppose on the contrary that there exists a feebly compact topological Brandt λ^0-extension $(B^0_\lambda(S),\tau_B)$ of a topological monoid (S,τ_S) with zero in the class of Hausdorff topological semigroups such that the cardinal λ is infinite. Then the Hausdorffness of $(B^0_\lambda(S),\tau_B)$ implies that for every $\alpha\in\lambda$ there exist open disjoint neighborhoods U_0 and $U_{(\alpha,1_S,\alpha)}$ of zero and $(\alpha,1_S,\alpha)$ in $(B^0_\lambda(S),\tau_B)$, respectively. Without loss of generality we may assume that $U_{(\alpha,1_S,\alpha)}=(U(1_S))_{\alpha,\alpha}$ for some open neighborhood $U(1_S)$ of unit 1_S in (S,τ_S) (see Proposition 1). By the continuity of multiplication in $(B^0_\lambda(S),\tau_B)$ there exists an open neighborhood V_0 of zero in $(B^0_\lambda(S),\tau_B)$ such that $V_0\cdot V_0\subseteq U_0$. Furthermore the continuity of multiplication in (S,τ_S) implies that there exists an open neighborhood $V(1_S)$ of unit 1_S in (S,τ_S) such that $V(1_S)\cdot V(1_S)\subseteq U(1_S)$ in S.

Then the feeble compactness of $(B^0_\lambda(S),\tau_B)$ implies that zero 0 is an accumulation point of each infinite subfamily of $\{(V(1_S))_{\alpha,\beta}\colon \alpha,\beta\in\lambda\}$. Hence $V_0\cap(V(1_S))_{\alpha,\beta}=\varnothing$ only for finitely many pairs of indices (α,β). So, by the definition of multiplication on $B^0_\lambda(S)$ we have $(V_0\cdot V_0)\cap U_{(\alpha,1_S,\alpha)}\neq\varnothing$. This contradicts the assumption $U_0\cap U_{(\alpha,1_S,\alpha)}=\varnothing$. The obtained contradiction implies that cardinal λ is finite. $\square$

Theorem 3 implies the following corollary.

Corollary 5. *A feebly compact topological Brandt λ^0-extension of a topological inverse monoid with zero in the class of Hausdorff topological semigroups is a topological inverse semigroup.*

The following example shows that there exists a compact topological semigroup with a non-pseudocompact topological Brandt 2^0-extension in the class of topological semigroups and hence the counterpart of Theorem 3 does not necessarily hold for semigroups without a non-zero idempotent.

Example 3. Let X be any infinite Hausdorff compact topological space. Fix an arbitrary $z\in X$ and define multiplication on X in the following way: $x\cdot y=z$ for all $x,y\in X$. It is obvious that this operation is continuous on X and z is zero of X. The set X endowed with such an operation is called a *semigroup with zero-multiplication*. We define the topology τ_B on the Brandt 2^0-extension $B^0_2(X)$ of the semigroup X as follows:

(i) the family $\mathscr{B}(0)=\{U_{1,1}\cup U_{2,2}\colon U\in\mathscr{B}(z)\}$, where $\mathscr{B}(z)$ is a base of the topology of X at z, is the base of topology τ_B at zero of $B^0_2(X)$;

(ii) for $i\in\{1,2\}$ and any $x\in X\setminus\{z\}$ the family $\mathscr{B}_{(i,x,i)}=\{U_{i,i}\colon U\in\mathscr{B}(x)\}$, where $\mathscr{B}(x)$ is a base of the topology of X at the point x and is the base of topology τ_B at the point $(i,x,i)\in B^0_2(X)$;

(iii) all points of the subsets $X^*_{1,2}$ and $X^*_{2,1}$ are isolated points in $(B^0_2(X),\tau_B)$.

It is obvious that $B_2^0(X)$ is a semigroup with zero-multiplication. Simple verifications show that τ_B is a Hausdorff topology on $B_2^0(X)$. Hence $(B_2^0(X),\tau_B)$ is a topological semigroup and $(B_2^0(X),\tau_B)$ is a topological Brandt 2^0-extension of X in the class of topological semigroups. Since $X_{1,2}^*$ and $X_{2,1}^*$ are discrete open-and-closed subspaces of $(B_2^0(X),\tau_B)$ we have that the topological space $(B_2^0(X),\tau_B)$ is not feebly compact.

Also, the following example shows that there exists a compact topological semigroup S such that for every infinite cardinal λ there exists a compact (and hence feebly compact) topological Brandt λ^0-extension $B_\lambda^0(S)$ of the semigroup S in the class of topological semigroups.

Example 4. Let X be a compact topological semigroup defined in Example 3 and λ an arbitrary infinite cardinal. We define the topology τ_B on the Brandt λ^0-extension $B_\lambda^0(X)$ of the semigroup X as follows:

(i) the family $\mathscr{B}_B(0)=\big\{U_A(0)=\bigcup_{(\alpha,\beta)\in(\lambda\times\lambda)\setminus A}X_{\alpha,\beta}\cup\bigcup_{(\gamma,\delta)\in A}(U(z))_{\gamma,\delta}\colon A$ is a finite subset of $\lambda\times\lambda$ and $U(z)\in\mathscr{B}_X(z)\big\}$, where $\mathscr{B}_X(z)$ is a base of the topology $x\in X$, is a base of topology τ_B at zero of $B_\lambda^0(X)$;

(ii) for all $\alpha,\beta\in\lambda$ and any $x\in X\setminus\{z\}$ the family $\mathscr{B}_{(\alpha,x,\beta)}=\{U_{\alpha,\beta}\colon U\in\mathscr{B}(x)\}$, where $\mathscr{B}_X(x)$ is a base of the topology of X at the point x, is the base of topology τ_B at the point $(\alpha,x,\beta)\in B_\lambda^0(X)$.

It is obvious that $B_\lambda^0(X)$ is a semigroup with zero-multiplication. Simple verifications show that τ_B is a Hausdorff compact topology on $B_\lambda^0(X)$. Hence $(B_\lambda^0(X),\tau_B)$ is a topological semigroup and $(B_\lambda^0(X),\tau_B)$ is a compact topological Brandt λ^0-extension of X in the class of topological semigroups.

The following proposition extends Theorem 3.

Proposition 7. *Let S be a Hausdorff feebly compact topological semigroup such that S is an orthogonal sum of the family $\{B_{\lambda_i}^0(S_i)\colon i\in\mathscr{I}\}$ of topological Brandt λ_i^0-extensions of topological semigroups with zeros, i.e. $S=\sum_{i\in\mathscr{I}}B_{\lambda_i}^0(S_i)$. If for some $i\in\mathscr{I}$ the semigroup S_i has a non-zero idempotent then the cardinal λ_i is finite.*

Proof. Assume on the contrary that there exists $i\in\mathscr{I}$ such that the cardinal λ_i is infinite. Let e be a non-zero idempotent of S_i. Then the Hausdorffness of S implies that for every $\alpha_i\in\lambda_i$ there exist open disjoint neighborhoods U_0 and $U_{(\alpha_i,e,\alpha_i)}$ of zero and (α_i,e,α_i) in S, respectively. By the continuity of multiplication in S there exists an open neighborhood $V_{(\alpha_i,e,\alpha_i)}$ of (α_i,e,α_i) in S such that $(\alpha_i,e,\alpha_i)\cdot V_{(\alpha_i,e,\alpha_i)}\cdot(\alpha_i,e,\alpha_i)\subseteq U_{(\alpha_i,e,\alpha_i)}$. This implies that $V_{(\alpha_i,e,\alpha_i)}\subseteq(S_i^*)_{\alpha_i,\alpha_i}$. Therefore without loss of generality we may assume that $U_{(\alpha_i,e,\alpha_i)}=(U(e))_{\alpha_i,\beta_i}$ for some open neighborhood $U(e)$ of the idempotent e in S_i. By the continuity of multiplication in S there exists an open neighborhood V_0 of zero in S such that $V_0\cdot V_0\subseteq U_0$. Also the continuity of the semigroup operation in S_i implies that there exists an open neighborhood $V(e)$ of the idempotent e in S_i such that $V(e)\cdot V(e)\subseteq U(e)$ in S_i.

Then the feeble compactness of S implies that zero 0 is an accumulation point of each infinite subfamily of $\{(V(1_S))_{\alpha_i,\beta_i}\colon\alpha_i,\beta_i\in\lambda_i,i\in\mathscr{I}\}$. Hence $V_0\cap(V(1_S))_{\alpha_i,\beta_i}=\varnothing$ only for finitely many pairs if indices (α_i,β_i) from λ_i, $i\in\mathscr{I}$. Therefore, by the definition of multiplication on S we have that $(V_0\cdot V_0)\cap U_{(\alpha_i,e,\alpha_i)}\neq\varnothing$. This contradicts the assumption $U_0\cap U_{(\alpha_i,e,\alpha_i)}=\varnothing$. The obtained contradiction implies that cardinal λ_i is finite. □

Theorem 2 and Proposition 7 imply the following statement.

Theorem 4. *Let S be a Hausdorff primitive inverse feebly compact topological semigroup and S an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i\in\mathscr{I}\}$ of topological Brandt semigroups with zeros. Then the following assertions hold:*

(i) every cardinal λ_i is finite;

(ii) every maximal subgroup of S is open-and-closed subset of S and hence is feebly compact;

(iii) for every $i\in\mathscr{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a feebly compact topological Brandt λ-extension of the feebly compact paratopological group G_i;

(iv) if $\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$ is a base of the topology at unity (α_i,e_i,α_i) of a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$ of S, $i\in\mathscr{I}$, such that $U\subseteq(G_i)_{\alpha_i,\alpha_i}$ for any $U\in\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$, then the family

$$\mathscr{B}_{(\beta_i,x,\gamma_i)}=\big\{(\beta_i,x,\alpha_i)\cdot U\cdot(\alpha_i,e_i,\gamma_i)\colon U\in\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}\big\}$$

is a base of the topology at the point $(\beta_i,x,\gamma_i)\in(G_i)_{\beta_i,\gamma_i}\subseteq B_{\lambda_i}(G_i)$, for all $\beta_i,\gamma_i\in\lambda_i$.

If, in addition, the topological space S is semiregular then

(v) the family

$$\mathscr{B}_0=\big\{S\backslash\big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}}\cup\cdots\cup(G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big)\colon i_1,\ldots,i_k\in\mathscr{I},\alpha_{i_k},\beta_{i_k}\in\lambda_{i_k},\\ k\in\mathbb{N},\{(\alpha_{i_1},\beta_{i_1}),\ldots,(\alpha_{i_k},\beta_{i_k})\}\text{ is finite}\big\}$$

is a base of the topology at zero of S.

The following example shows that statement (v) of Theorem 4 does not necessarily hold when the semigroup S is functionally Hausdorff and countably pracompact but it is not semiregular.

Example 5. In [29, Example 3] a functionally Hausdorff ω-precompact first countable paratopological group (G,τ_R) is constructed such that each power of (G,τ_R) is countably pracompact but (G,τ_R) is not a topological group. Moreover, the group (G,τ_R) contains an open dense subsemigroup S. Let $\mathscr{I}$ be an infinite set of indices. For any $i\in\mathscr{I}$ let λ_i be any finite cardinal $\geqslant 1$. Let $B_{\lambda_i}(G)$ be the algebraic Brandt λ_i-extension of the algebraic group G for each $i\in\mathscr{I}$. Put $R(G,\{\lambda_i\}_{ß\in\mathscr{I}})=\sum_{i\in\mathscr{I}}B_{\lambda_i}(G)$. For any subset C of the group G and all $i,i_1,\ldots,i_k\in\mathscr{I}$, $k\in\mathbb{N}$, put

$$B_{\lambda_i}(C)=\{0\}\cup\{(\alpha_i,x,\beta_i)\in B_{\lambda_i}(G)\colon x\in C,\alpha_i,\beta_i\in\lambda_i\},\ R(C,\{\lambda_i\}_{ß\in\mathscr{I}})=\sum_{i\in\mathscr{I}}B_{\lambda_i}(C),$$
$$U(i_1,\ldots,i_k)=R(S,\{\lambda_i\}_{ß\in\mathscr{I}})\setminus\big((B_{\lambda_1}(S))^*\cup\cdots\cup(B_{\lambda_k}(S))^*\big).$$

We define the topology τ_{RB} on $R(G,\{\lambda_i\}_{ß\in\mathscr{I}})$ in the following way:

(i) if $\mathscr{B}_e$ is a base of the topology τ_R at unit e of the group G then the family

$$\mathscr{B}_{(\beta_i,x,\gamma_i)}=\{(\beta_i,xU,\gamma_i)\colon U\in\mathscr{B}_e\}$$

is a base of the topology τ_{RB} at the point $(\beta_i,x,\gamma_i)\in G_{\beta_i,\gamma_i}\subseteq B_{\lambda_i}(G_i)$, for all $\beta_i,\gamma_i\in\lambda_i$;

(ii) the family $\mathscr{B}_0=\{U(i_1,\ldots,i_k)\colon i_1,\ldots,i_k\in\mathscr{I}\}$ is a base of the topology at zero of $R(G,\lambda_i,\mathscr{I})$.

It is obvious that $(R(G,\{\lambda_i\}_{ß\in\mathscr{I}}),\tau_{RB})$ is a Hausdorff topological space. Since S is a dense open subsemigroup of (G,τ_R) we conclude that $(R(G,\{\lambda_i\}_{ß\in\mathscr{I}}),\tau_{RB})$ is not semiregular. Since the space (G,τ_R) is functionally Hausdorff and G_{β_i,γ_i} is an open-and-closed subspace of $(R(G,\{\lambda_i\}_{ß\in\mathscr{I}}),\tau_{RB})$, for all $\beta_i,\gamma_i\in\lambda_i$, the space $(R(G,\{\lambda_i\}_{ß\in\mathscr{I}}),\tau_{RB})$ is functionally Hausdorff too.

Now, the definition of the semigroup $R(G,\{\lambda_i\}_{ß\in\mathscr{I}})$ implies that

$$U(i_1,\ldots,i_k)\cdot B_{\lambda_m}(G)=B_{\lambda_m}(G)\cdot U(i_1,\ldots,i_k)=\{0\},$$

for each $i_m\in\{i_1,\ldots,i_k\}$ and $U(i_1,\ldots,i_k)\cdot U(i_1,\ldots,i_k)\subseteq U(i_1,\ldots,i_k)$ for all $i_1,\ldots,i_k\in\mathscr{I}$, $k\in\mathbb{N}$, because S is a subsemigroup of the group G. This and the continuity of multiplication in (G,τ_R) imply that multiplication in $(R(G,\{\lambda_i\}_{ß\in\mathscr{I}})$ is continuous.

We claim that the topological space $(R(G,\{\lambda_i\}_{ß\in\mathscr{I}}),\tau_{RB})$ is countably pracompact. Indeed, there exists a set $A\subset S\subset G$ such that A is dense in the space (G,τ_R) and this space is countably compact at A ([29, Example 3]). Then the set $R(A,\{\lambda_i\}_{ß\in\mathscr{I}})$ is dense in $R(G,\{\lambda_i\}_{ß\in\mathscr{I}})$. We claim that the space $R(G,\{\lambda_i\}_{ß\in\mathscr{I}})$ is countably compact at $R(A,\lambda_i,\mathscr{I})$. Indeed, let A' be an arbitrary countable infinite subset of the set $R(A,\{\lambda_i\}_{ß\in\mathscr{I}})$. If 0 is not an accumulation point of the set A' then there exist indices $i_1,\ldots,i_k\in\mathscr{I}$ such that the set $U(i_1,\ldots,i_k)\cap A'$ is finite. Since $A\subset S$ then $A'\subset R(A,\{\lambda_i\}_{ß\in\mathscr{I}})\subset R(S,\{\lambda_i\}_{ß\in\mathscr{I}})$ and the set $A'\cap\big((B_{\lambda_1}(S))^*\cup\cdots\cup(B_{\lambda_k}(S))^*\big)\subset A'\setminus U(i_1,\ldots,i_k)$ is infinite. Since for each $1\leq j\leq k$ the cardinal λ_{i_j} is finite, there exists an index $1\leq j\leq k$ and elements $\alpha,\beta\in\lambda_{i_j}$ such that the intersection $A'\cap S_{\alpha,\beta}\subset B_{\lambda_{i_j}}(A)\subset B_{\lambda_{i_j}}(G)$ is infinite. Since the space G is countably compact at A, $B_{\lambda_{i_j}}(G)$ is countable compact at $B_{\lambda_{i_j}}(A)$. Therefore the set $A'\cap S_{\alpha,\beta}$ has an accumulation point in $B_{\lambda_{i_j}}(G)$.

Unlike functional Hausdorffness, the quasiregularity guaranties stronger properties of primitive inverse feebly compact topological semigroups and this follows from the next two propositions.

Theorem 5. *Let S be a quasiregular primitive inverse feebly compact topological semigroup and S be the orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i\in\mathscr{I}\}$ of topological Brandt semigroups with zeros. Then the family*

$$\begin{aligned}\mathscr{B}_0=\big\{S\setminus\big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}}\cup\cdots\cup(G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big)\colon i_1,\ldots,i_k\in\mathscr{I},\alpha_{i_k},\beta_{i_k}\in\lambda_{i_k},\\ k\in\mathbb{N},\{(\alpha_{i_1},\beta_{i_1}),\ldots,(\alpha_{i_k},\beta_{i_k})\}\text{ is finite}\big\}\end{aligned}$$

is a base of the topology at zero of S.

Proof. Assume on the way of contradiction that there exists an open subset $W\ni 0$ of S such that $U\nsubseteq W$ for any $U\in\mathscr{B}_0$. There exists an open neighborhood $V\subseteq W$ of zero in S such that $V\cdot V\cdot V\subseteq W$. Since every non-zero maximal subgroup of S is an open-and-closed subset of S and the space S is feebly compact, there exist finitely many indices $i_1,\ldots,i_k\in\mathscr{I}$ such that $V\cap\big(S\setminus\big((B_{\lambda_1}(S))^*\cup\cdots\cup(B_{\lambda_k}(S))^*\big)\big)$ is a dense open subset of the space $S\setminus\big((B_{\lambda_1}(S))^*\cup\cdots\cup(B_{\lambda_k}(S))^*\big)$. Then every non-zero maximal subgroup of S is a quasi-regular space and hence by Proposition 3 of [30] (see also [31]) every maximal subgroup of S is a topological group. Now, Proposition 2.5 of [17] implies that

$$V\cdot V\cdot V\supseteq S\setminus((B_{\lambda_1}(S))^*\cup\cdots\cup(B_{\lambda_k}(S))^*)\nsubseteq W.$$

The obtained contradiction implies the required conclusion. □

Since by Proposition 3 of [30] inversion on a quasiregular feebly compact paratopological group is continuous, Proposition 6, Theorems 4 and 5 imply the following corollary.

Corollary 6. *Inversion on a quasi-regular primitive inverse feebly compact topological semigroup S is continuous and hence S is Tychonoff.*

Remark 2. Example 1 of [6] shows that inversion on a quasi-regular inverse countably compact topological semigroup in which maximal subgroups are topological groups is not continuous. Corollary 6 and Proposition 2.8 of [17] imply that a quasi-regular primitive inverse feebly compact topological semigroup is Tychonoff.

Corollary 6 implies such statement.

Corollary 7. *Every quasi-regular feebly compact Brandt topological semigroup is a Tychonoff topological inverse semigroup.*

Theorem 1 implies the following theorem.

Theorem 6. *Let S be a Hausdorff primitive inverse countably compact topological semigroup and S the orthogonal sum of a family $\{B_{\lambda_i}(G_i)\colon i \in \mathscr{I}\}$ of topological Brandt semigroups with zero. Then the family*

$$\mathscr{B}_0 = \big\{S\backslash\big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big)\colon i_1, \ldots, i_k \in \mathscr{I}, \alpha_{i_k}, \beta_{i_k} \in \lambda_{i_k}, \\ k \in \mathbb{N}, \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\} \textit{ is finite}\big\}$$

is a base of the topology at zero of S.

By Definition 1, Theorem 6 and arguments presented in the proof of Theorem 1 imply the following corollary.

Corollary 8. *Inversion on a Hausdorff primitive inverse countably compact topological semigroup S is continuous if and only if every maximal subgroup of S is a topological group.*

Remark 3. The second named author, using a result of P. Koszmider, A. Tomita and S. Watson ([24]), constructed under MA an example of a Hausdorff countably compact paratopological group failing to be a topological group ([28, 29]).

4. Products of feebly compact inverse primitive semitopological semigroups and their Stone-Čech compactification. The counterparts of the following four statements for the Tychonoff spaces are proved in [13, Section 3.10]. But since the proofs which are based on the function theory are not applicable for our case, we present straightforward proofs here.

Proposition 8. *Let X be a feebly compact topological space and Y be a sequentially compact topological space. Then $X \times Y$ is feebly compact.*

Proof. We have to prove that any infinite family $\{U_n\colon n \in \mathbb{N}\}$ of non-empty open subsets of the space $X \times Y$ is not locally finite. For this purpose we find a point $(x, y) \in X \times Y$ such that every open neighborhood of (x, y) intersects infinitely many elements of the family $\{U_n\colon n \in \mathbb{N}\}$. Let n be a positive integer. There exist non-empty open subsets $V_n \subset X$ and $W_n \subset Y$ such that $V_n \times W_n \subset U_n$. Choose a point $y_n \in W_n$. Since the space Y is sequentially compact, the sequence $\{y_n\colon n \in \mathbb{N}\}$ has a subsequence $\{y_{n_k}\colon k \in \mathbb{N}\}$ converging to a point

$y \in Y$. Since the space X is feebly compact, there exists a point $x \in X$ such that every open neighborhood of the point x in X intersects V_{n_k} for infinitely many numbers k. Then each open neighborhood of the point $(x, y) \in X \times Y$ intersects U_n for infinitely many numbers n. Hence (x, y) is the required point. □

Proposition 9. *Let X be a Hausdorff feebly compact topological space. Then $X \times Y$ is feebly compact for any feebly compact k-space Y.*

Proof. It suffices to observe that every non-empty open subset of the Cartesian product $X \times Y$ contains an open subset $U \times V$, where U and V are non-empty open subset of X and Y, respectively, and then Lemma 3.10.12 of [13] implies the statement of the proposition. □

Proposition 9 implies the following two corollaries.

Corollary 9. *The Cartesian product $X \times Y$ of a feebly compact space X and a compactum Y is feebly compact.*

Corollary 10. *The Cartesian product $X \times Y$ of a feebly compact space X and a feebly compact sequential space Y is feebly compact.*

Proposition 10. *Let S be a primitive semitopological inverse semigroup such that every maximal subgroup of S is a feebly compact paratopological (topological) group. Then S is a continuous[1] image of the product $\widetilde{E}_S \times G_S$, where $\widetilde{E}_S$ is a compact semilattice and G_S is a feebly compact paratopological (topological) group provided one of the following conditions holds:*

(1) S is semiregular and feebly compact;

(2) S is Hausdorff and countably compact.

Proof. We only consider the case where S is a semiregular feebly compact space and every maximal subgroup of S is a paratopological group because in case (2) the proof is similar.

By Theorem 2 the topological semigroup S is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}} B_{\lambda_i}(G_i)$ of the topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of feebly compact paratopological groups G_i in the class of Hausdorff semitopological semigroups for some cardinals $\lambda_i \geqslant 1$. The family defined by formula (1) in Theorem 2(iv) determines the base of a topology at zero of S.

Fix an arbitrary $i \in \mathscr{I}$. Then by Corollary 4 the space $E(B_{\lambda_i}(G_i))$ is compact. First we consider the case where the cardinal λ_i is finite. Suppose that $|E(B_{\lambda_i}(G_i))| = n_i + 1$ for some integer n_i. Then $\lambda_i = n_i \geqslant 1$. On the set $E_i = (\lambda_i \times \lambda_i) \cup \{0\}$, where $0 \notin \lambda_i \times \lambda_i$ we define multiplication in the following way

$$(\alpha, \beta) \cdot (\gamma, \delta) = \begin{cases} (\alpha, \beta), & \text{if } (\alpha, \beta) = (\gamma, \delta); \\ 0, & \text{otherwise}, \end{cases}$$

and $0 \cdot (\alpha, \beta) = (\alpha, \beta) \cdot 0 = 0 \cdot 0 = 0$ for all $\alpha, \beta, \gamma, \delta \in \lambda_i$. Simple verifications show that E_i with this multiplication is a semilattice and every non-zero idempotent of E_i is primitive. If the cardinal λ_i is infinite then on the set $E_i = (\lambda_i \times \lambda_i) \cup \{0\}$ we define the semilattice operation in a similar way.

[1] not necessarily a homomorphic image

We denote by $\widetilde{E}_S$ the orthogonal sum $\sum_{i\in\mathscr{I}} E_i$. It is obvious that $\widetilde{E}_S$ is a semilattice and every non-zero idempotent of $\widetilde{E}_S$ is primitive. We determine on $\widetilde{E}_S$ the topology of the Alexandroff one-point compactification τ_A: all non-zero idempotents of $\widetilde{E}_S$ are isolated points in $\widetilde{E}_S$ and the family $\mathscr{B}(0)=\{U\colon U\ni 0 \text{ and } \widetilde{E}_S\setminus U \text{ is finite}\}$ is the base of the topology τ_A at zero $0\in\widetilde{E}_S$. Simple verifications show that $\widetilde{E}_S$ with the topology τ_A is a Hausdorff compact topological semilattice. Later we denote $(\widetilde{E}_S,\tau_A)$ by $\widetilde{E}_S$.

Let $G_S=\prod_{i\in\mathscr{I}} G_i$ be the direct product of feebly compact paratopological groups G_i, $i\in\mathscr{I}$, with the Tychonoff topology. Then Proposition 24 of [29] implies that G_S is a feebly compact paratopological group. By Corollary 9 we have that the product $\widetilde{E}_S\times G_S$ is a feebly compact space.

For every $i\in\mathscr{I}$ we denote by $\pi_i\colon G_S=\prod_{i\in\mathscr{I}} G_i\to G_i$ the projection on the i-th factor.

Now, for every $i\in\mathscr{I}$ we define a map $f_i\colon E_i\times G_S\to B_{\lambda_i}(G_i)$ by the formulae $f_i((\alpha,\beta),g)=(\alpha,\pi_i(g),\beta)$ and $f_i(0,g)=0_i$ is zero of the semigroup $B_{\lambda_i}(G_i)$, and put $f=\bigcup_{i\in\mathscr{I}} f_i$. It is obvious that the map $f\colon \widetilde{E}_S\times G_S\to S$ is well defined. The definition of the topology τ_A on $\widetilde{E}_S$ implies that for every $((\alpha,\beta),g)\in E_i\times G_i\subseteq \widetilde{E}_S\times G_i$ the set $\{(\alpha,\beta)\}\times G_i$ is open in $\widetilde{E}_S\times G_S$ and hence the map f is continuous at the point $((\alpha,\beta),g)$. For every $U(0)=S\setminus\big(B_{\lambda_{i_1}}(G_{i_1})\cup B_{\lambda_{i_2}}(G_{i_2})\cup\cdots\cup B_{\lambda_{i_n}}(G_{i_n})\big)^*$ the set $f^{-1}(U(0))=\big(\widetilde{E}_S\setminus((\lambda_{i_1}\times\lambda_{i_1})\cup\ldots\cup(\lambda_{i_n}\times\lambda_{i_n}))\big)\times G_S$ is open in $\widetilde{E}_S\times G_S$ and hence the map f is continuous.

We observe that in the case where all maximal subgroups of S are topological groups, $G_S=\prod_{i\in\mathscr{I}} G_i$ is a pseudocompact topological group by Comfort–Ross theorem (see Theorem 1.4 in [10]).

In the case of a Hausdorff semitopological semigroup S the proof is similar. □

The following result is an extension of the Comfort–Ross Theorem for primitive feebly compact semitopological inverse semigroups.

Theorem 7. *Let $\{S_j\colon j\in\mathscr{J}\}$ be a family of primitive semitopological inverse semigroups such that for each $j\in\mathscr{J}$ the semigroup S_j is either semiregular feebly compact or Hausdorff countably compact, and suppose that each maximal subgroup of S_j is a feebly compact paratopological group. Then the direct product $\prod_{j\in\mathscr{J}} S_j$ with the Tychonoff topology is a feebly compact semitopological inverse semigroup.*

Proof. Since the direct product of a family of semitopological inverse semigroups is a semitopological inverse semigroup, it is sufficient to show that the space $\prod_{j\in\mathscr{J}} S_j$ is feebly compact. For each $j\in\mathscr{J}$ let $\widetilde{E}_{S_j}$, G_{S_j}, and $f_j\colon \widetilde{E}_{S_j}\times G_{S_j}\to S_j$ be the semilattice, the group and the map, respectively, defined in the proof of Proposition 10. Since the space $\prod_{j\in\mathscr{J}}(\widetilde{E}_{S_j}\times G_{S_j})$ is homeomorphic to the product $\prod_{j\in\mathscr{J}}\widetilde{E}_{S_j}\times\prod_{j\in\mathscr{J}} G_{S_j}$ we conclude that by Theorem 3.2.4 of [13], Corollary 9 and Proposition 24 of [29] the space $\prod_{j\in\mathscr{J}}(\widetilde{E}_{S_j}\times G_{S_j})$ is feebly compact. Now, since the map $\prod_{j\in\mathscr{J}} f_j\colon \prod_{j\in\mathscr{J}}(\widetilde{E}_{S_j}\times G_{S_j})\to\prod_{j\in\mathscr{J}} S_j$ is continuous $\prod_{j\in\mathscr{J}} S_j$ is a feebly compact topological space. □

The proofs of the following two propositions are similar to Proposition 10 and Theorem 7; they generalize Proposition 2.11 and Theorem 2.12 of [17].

Proposition 11. *Let S be a primitive inverse topological semigroup. Then S is a continuous (not necessarily homomorphic) image of the product $\widetilde{E}_S\times G_S$, where $\widetilde{E}_S$ is a compact semilattice and G_S is a feebly compact paratopological group provided one of the following conditions holds:*

(1) S is semiregular feebly compact;

(2) S is Hausdorff countably compact.

Theorem 8. *Let $\{S_i \colon i \in \mathscr{J}\}$ be a family of primitive inverse semiregular feebly compact (Hausdorff countably compact) topological semigroups. Then the direct product $\prod_{j\in\mathscr{J}} S_j$ with the Tychonoff topology is a feebly compact inverse topological semigroup.*

Let a Tychonoff topological space X be the topological sum of subspaces A and B, i.e., $X = A\bigoplus B$. It is obvious that every continuous map $f\colon A \to K$ from A to a compact space K (resp., $f\colon B \to K$ from B to a compact space K) extends to a continuous map $\widehat{f}\colon X \to K$. This implies the following proposition.

Proposition 12. *If a Tychonoff topological space X is the topological sum of subspaces A and B, then βX is equivalent to the topological sum $\beta A\bigoplus \beta B$.*

The following theorem follows from Corollary 6 and Theorem 3.2 of [17], and describes the structure of the Stone-Čech compactification of a primitive inverse feebly compact quasi-regular topological semigroup.

Theorem 9. *Let S be a primitive inverse feebly compact quasi-regular topological semigroup. Then the Stone-Čech compactification of S admits the structure of a primitive topological inverse semigroup with respect to which the inclusion mapping of S to βS is a topological isomorphism. Moreover, βS is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$ of the topological Brandt λ_i-extensions $B_{\lambda_i}(\beta G_i)$ of compact topological groups βG_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geqslant 1$.*

Theorem 10. *Let S be a regular primitive inverse countably compact semitopological semigroup and S be the orthogonal sum of a family $\{B_{\lambda_i}(G_i)\colon i \in \mathscr{I}\}$ of the semitopological Brandt semigroups with zeros. Suppose that for every $i \in \mathscr{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least the one of the following conditions holds:*

(1) the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;

(2) the group $(G_i)_{\alpha_i,\alpha_i}$ is left ω-precompact feebly compact;

(3) the semigroup $S_{\alpha_i,\alpha_i} = (G_i)_{\alpha_i,\alpha_i} \cup \{0\}$ is a topological semigroup.

Then the Stone-Čech compactification of S admits the structure of a primitive inverse semitopological semigroup with continuous inversion with respect to which the inclusion mapping of S to βS is a topological isomorphism. Moreover, βS is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$ of compact topological Brandt λ_i-extensions $B_{\lambda_i}(\beta G_i)$ of compact topological groups βG_i in the class of semitopological semigroups for some cardinals $\lambda_i \geqslant 1$.

Proof. By Theorem 1, the semigroup S is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}} B_{\lambda_i}(G_i)$ of the topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact paratopological groups G_i in the class of semitopological semigroups for some cardinals $\lambda_i \geqslant 1$, such that any non-zero $\mathscr{H}$-class of S is an open-and-closed subset of S. The family $\mathscr{B}(0)$ defined by formula (1) in Theorem 2(iv) determines a base of the topology at zero 0 of S. Since the space S is regular and any non-zero $\mathscr{H}$-class of S is an open-and-closed subset of S, every maximal subgroup of S is a topological group ([29, Proposition 3]). Hence S is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}} B_{\lambda_i}(G_i)$ of topological the Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of semitopological semigroups for some cardinals $\lambda_i \geqslant 1$. Then by Proposition 6 the semigroup S is Tychonoff, and hence the Stone-Čech compactification of S exists.

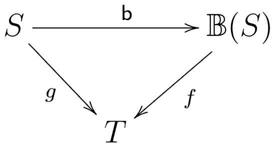
By Theorem 7, $S \times S$ is a pseudocompact topological space. Now by Theorem 1 of [14], we have that $\beta(S \times S)$ is equivalent to $\beta S \times \beta S$, and hence by Theorem 1.1 of [4], S is a subsemigroup of the compact semitopological semigroup βS.

By Proposition 12 for every non-zero $\mathscr{H}$-class $(G_i)_{k,l}$, $k, l \in \lambda_i$, we have that $\operatorname{cl}_{\beta S}((G_i)_{k,l})$ is equivalent to $\beta(G_i)_{k,l}$, and hence is equivalent to βG_i. Therefore we may naturally consider the space $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$ as a subspace of the space βS. Suppose that $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i) \neq \beta S$. We fix an arbitrary $x \in \beta S \setminus \sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$. Then the Hausdorffness of βS implies that there exist open neighborhoods $V(x)$ and $V(0)$ of the points x and *the zero* zero 0 in βS, respectively, and there exist finitely many indices $i_1, \ldots, i_k \in \mathscr{I}$ and finitely many pairs of indices $(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})$ such that $V(0) \cap \beta S \supseteq S \setminus \big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big)$. Then we have

$$V(x) \cap S \subseteq \big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big) \subseteq \big((\beta G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (\beta G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big).$$

But this contradicts that x is an accumulation point of $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$ in βS, which does not belong to $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$, because $(\beta G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (\beta G_{i_k})_{\alpha_{i_k},\beta_{i_k}}$ is a compact subset of βS. □

Recall [11] that the *Bohr compactification* of a semitopological semigroup S is the pair $(\mathsf{b}, \mathbb{B}(S))$ such that $\mathbb{B}(S)$ is a compact semitopological semigroup, $\mathsf{b}\colon S \to \mathbb{B}(S)$ is a continuous homomorphism, and if $g\colon S \to T$ is a continuous homomorphism of S to a compact semitopological semigroup T, then there exists a unique continuous homomorphism $f\colon \mathbb{B}(S) \to T$ such that the diagram



commutes. In the sequel, similar to that in General Topology, by the Bohr compactification of a semitopological semigroup S we mean not only pair $(\mathsf{b}, \mathbb{B}(S))$ but also the compact semitopological semigroup $\mathbb{B}(S)$.

By the definitions of the Stone-Čech compactification and the Bohr compactification, Theorem 10 imply the following corollary.

Corollary 11. *Let S be a Hausdorff primitive inverse countably compact semitopological semigroup such that every maximal subgroup of S is a pseudocompact topological group and S be the orthogonal sum of a family $\{B_{\lambda_i}(G_i)\colon i \in \mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Then the Bohr compactification of S admits the structure of a primitive inverse semitopological semigroup with continuous inversion with respect to which the inclusion mapping of S to $(\mathsf{b}, \mathbb{B}(S))$ is a topological isomorphism. Moreover, $(\mathsf{b}, \mathbb{B}(S))$ is topologically isomorphic to the orthogonal sum $\sum_{i\in\mathscr{I}} B_{\lambda_i}(\beta G_i)$ of the topological Brandt λ_i-extensions $B_{\lambda_i}(\beta G_i)$ of compact topological groups βG_i in the class of semitopological semigroups for some cardinals $\lambda_i \geqslant 1$.*

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Faculty of Mathematics, Ivan Franko National University of Lviv
o_gutik@franko.lviv.ua, ovgutik@yahoo.com
Pidstrygach Institute for Applied Problems of Mechanics and Mathematics of NASU
oravsky@mail.ru

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