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# A CLASS OF ENTIRE FUNCTIONS OF UNBOUNDED INDEX IN EACH DIRECTION

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We select a class of entire functions $f(z_1, z_2)$ with the property $\forall \mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ $\forall\ z_1^0,\ z_2^0 \in \mathbb{C}$ the function $f(z_1^0 + tb_1, z_2^0 + tb_2)$ is of bounded index as a function in variable $t \in \mathbb{C}$, but $f(z_1, z_2)$ is of unbounded index in every direction $\mathbf{b}$. Thus, it solves Problem 17 from the article A. I. Bandura, O. B. Skaskiv, *Open problems for entire functions of bounded index in direction,* Mat. Stud., **43** (2015), no.1, 103–109.

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Мы выделяем класс целых функций $f(z_1, z_2)$ со свойством $\forall \mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ $\forall\ z_1^0$, $z_2^0 \in \mathbb{C}$ функция $f(z_1^0 + tb_1, z_2^0 + tb_2)$ имеет ограниченный индекс, как функция переменной $t \in \mathbb{C}$, но $f(z_1, z_2)$ имеет неограниченный индекс по каждому направлению $\mathbf{b}$. Тем самым, нами решена Проблема 17 из статьи A. I. Bandura, O. B. Skaskiv, *Open problems for entire functions of bounded index in direction,* Mat. Stud., **43** (2015), no.1, 103–109.

**1. Introduction.** Let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a direction, $L\colon \mathbb{C}^n \to \mathbb{R}_+$ be a continuous function, $F\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function, $g_{z^0}(t) := F(z^0 + t\mathbf{b})$, $l_{z^0}(t) := L(z^0 + t\mathbf{b})$, $t \in \mathbb{C}$.

**Definition** ([1])**.** An entire function of $F(z)$, $z \in \mathbb{C}^n$, is called a *function of bounded $L$-index in the direction of* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ the next inequality is true

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \tag{1}$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\big), k \ge 2$.

The least such integer $m_0$ is called an $L$-index in the direction $\mathbf{b}$ of $F(z)$ and is denoted by $N_{\mathbf{b}}(F, L)$. If $m_0$ does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$ and $F$ is said to be of unbounded $L$-index in direction.

If $L(z) \equiv 1$ then $F$ is called a function of bounded index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F) \equiv N_{\mathbf{b}}(F, 1)$ is called the index in the direction $\mathbf{b}$. If $n = 1$, $\mathbf{b} = 1$ and $L(z) = l(z)$, $z \in \mathbb{C}$, inequality (1) defines a function of bounded $l$-index with the $l$-index $N(F, l) \equiv N_1(F, l)$

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([7]). And in the case $L(z) \equiv 1$ we get the definition of bounded index with the index $N(F) \equiv N_1(F, 1)$ ([8]).

Exploring properties of entire functions of bounded $L$-index in direction, we obtained the following assertion.

**Theorem 1** ([1])**.** *An entire function $F(z)$ is of bounded $L$-index in a direction $\mathbf{b}$ if and only if there exists a number $M > 0$ such that for all $z^0 \in \mathbb{C}^n$ the function $g_{z^0}(t)$ is of bounded $l_{z^0}$-index with $N(g_{z^0}, l_{z^0}) \le M < +\infty$ as a function of variable $t \in \mathbb{C}$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0})\colon z^0 \in \mathbb{C}^n\}$.*

In view of Theorem 1, a natural *question* ([2]): is there an entire function $F(z)$, $z \in \mathbb{C}^n$, with $N(g_{z^0}, l_{z^0}) < +\infty$ for every $z^0 \in \mathbb{C}^n$, but $N_{\mathbf{b}}(F, L) = +\infty$?

Below in this paper, we always assume that $L(z) \equiv 1$. We gave an affirmative answer ([2]) to the abovementioned question. There was proved that $F(z_1, z_2) = \cos\sqrt{z_1 z_2}$ is of unbounded index in direction $(1, 1)$. Recently, this result was generalized to the case of any each direction $\mathbf{b} \in \mathbb{C}^2 \setminus \{0\}$ ([3]).

Traditionally a solution of a problem leads to new problems. In our case there is an interesting further question.

**Problem 1** ([4, Problem 17])**.** *What are conditions on zero sets and growth of entire functions providing boundedness of index $F(z_1^0 + b_1 t, z_2^0 + b_2 t)$ for every $(z_1^0, z_2^0) \in \mathbb{C}^2$ and unboundedness of index $F(z_1, z_2)$ in the direction $\mathbf{b} = (b_1, b_2)$?*

We note that definitions and results from our papers in Ukrainian ([1]–[2]) are included in a draft version of English monography ([5]).

**2. Auxiliary propositions.** To prove our main result, we need some theorems of other mathematicians. The following theorem was proved by M. M. Sheremeta ([9]) for entire functions of bounded $l$-index. It is a generalization and refinement of the similar result of W. K. Hayman ([6]). We recall the proposition of Sheremeta for the bounded index.

**Theorem 2** ([9])**.** *An entire function $f(t)$ is of bounded index if and only if there exist $p \in \mathbb{Z}_+$, $C > 0$ and $R > 0$ such that for all $t \in \mathbb{C}$, $|t| \ge R$, the inequality*

$$|f^{(p+1)}(t)| \le C \max\{|f^{(k)}(t)|\colon\ 0 \le k \le p\} \tag{2}$$

*holds.*

We need some notation. If for a given $z^0 \in \mathbb{C}^n$ one has $g_{z^0}(t) \ne 0$ for all $t \in \mathbb{C}$, then $G_r^{\mathbf{b}}(F, z^0) := \varnothing$; if for a given $z^0 \in \mathbb{C}^n$ we get $g_{z^0}(t) \equiv 0$, then $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$. And if for a given $z^0 \in \mathbb{C}^n$ we have $g_{z^0}(t) \not\equiv 0$ and $a_k^0$ are zeros of $g_{z^0}(t)$, then

$$G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \left\{z^0 + t\mathbf{b}\colon |t - a_k^0| \le r\right\}, \quad r > 0.$$

Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{C}^n} G_r^{\mathbf{b}}(F, z^0). \tag{3}$$

By $n\big(r, z^0, t_0, 1/F\big) = \sum_{|a_k^0 - t_0| \le r} 1$ we denote the counting function of the zero sequence $(a_k^0)$.

The following criterion is convenient for a proof boundedness of index in direction.

**Theorem 3** ([1]). *Let $F(z)$ be an entire function in $\mathbb{C}^n$. A function $F(z)$ is of bounded index in a direction* $\mathbf{b}$ *if and only if:*

*1) for every $r > 0$ there exists $P = P(r) > 0$ that for each $z \in \mathbb{C}^n \setminus G_r^{\mathbf{b}}(F)$*

$$\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}} \right| \le P; \tag{4}$$

*2) for every $r > 0$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ that for every $z^0 \in \mathbb{C}^n$, satisfying $F(z^0 + t\mathbf{b}) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$*

$$n\left(r, z^0, t_0, \frac{1}{F}\right) \le \widetilde{n}(r). \tag{5}$$

## 3. Main theorem.

**Theorem 4.** *Let $f(t)$, $t \in \mathbb{C}$, be an even entire transcendental function of bounded index. Then 1) for each direction $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ and for every fixed $z_1^0$, $z_2^0 \in \mathbb{C}$ the function $g(t) = f(\sqrt{(z_1^0 + b_1 t)(z_2^0 + b_2 t)})$ is an entire function of bounded index ($t \in \mathbb{C}$); 2) if $f(t)$ has no zeros or has a finite number of zeros, then $f(\sqrt{z_1 z_2})$ is of unbounded index in each direction $\mathbf{b}$; 3) if $\{c_k\}$ is an infinite sequence of zeros of $f(t)$, $|c_1| < |c_2| < \ldots < |c_k| < \ldots$ and for every $k \in \mathbb{N}$ $c_k^2 \in \mathbb{R}$ then the function $f(\sqrt{z_1 z_2})$ is of unbounded index in each direction $\mathbf{b}$.*

*Proof.* 1) Let $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$, $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ be given. Since $f(t)$ is an even entire function, the function $f(\sqrt{(z_1^0 + b_1 t)(z_2^0 + b_2 t)})$ is entire as well. We prove that $g(t)$ is of bounded index as a function of variable $t$.

Let $g(t) = f(\sqrt{(z_1^0 + b_1 t)(z_2^0 + b_2 t)}) = f(\sqrt{at^2 + dt + c})$, where $a = b_1 b_2$, $d = z_1^0 b_2 + z_2^0 b_1$, $c = z_1^0 z_2^0$.

We calculate derivatives of this function

$$g'(t) = \frac{(2at + d) f'(\sqrt{at^2 + dt + c})}{2\sqrt{dt^2 + at + b}},$$

$$g''(t) = f''(\sqrt{at^2 + dt + c}) \left( \frac{2at + d}{2\sqrt{at^2 + dt + c}} \right)^2 + f'(\sqrt{at^2 + dt + c}) \frac{4ac - d^2}{4(at^2 + dt + c)^{3/2}}.$$

It is easy to check that for $s \in \mathbb{N} \setminus \{1\}$

$$g^{(s)}(t) = f^{(s)}(\sqrt{at^2 + dt + c}) \left( \frac{2at + d}{2\sqrt{at^2 + dt + c}} \right)^s + \sum_{k=1}^{s-1} f^{(k)}(\sqrt{at^2 + dt + c}) \frac{Q_{s,k}(t)}{(\sqrt{at^2 + dt + c})^{2s-k}}, \tag{6}$$

where $Q_{s,k}$ are polynomials with $deg\, Q_{s,k} \le s$ for all $k \in \{1, \ldots, s-1\}$.

By Hayman's Theorem (Theorem 2) applied to the function $f(\sqrt{at^2 + dt + c})$ there exist $p \in \mathbb{Z}_+$, $C > 0$ and $R > 0$ that for all $t \in \mathbb{C}$, $|\sqrt{at^2 + dt + c}| \ge R$, inequality (2) holds. A set $\{t \in \mathbb{C} \colon |\sqrt{at^2 + dt + c}| < R\}$ is bounded. Then there exists $R_1 > 0$ such that for all $t \in \mathbb{C}$, $|t| \ge R_1$, inequality (2) holds. But for these $t$ the expressions $\left| \frac{2at+d}{2\sqrt{at^2+dt+c}} \right|$ and $\left| \frac{Q_{p,k}(t)}{(\sqrt{at^2+dt+c})^{2p-k}} \right|$ ($k \in \{1, \ldots, p-1\}$) are bounded by same constant $M_1 > 0$, because their limits as $t \to \infty$ equal finite numbers. Thus and (6) imply that for $t \in \mathbb{C}$, $|t| \ge R_1$,

$$|g^{(p+1)}(t)| \le \left| f^{(p+1)}(\sqrt{at^2 + dt + c}) \right| \cdot \left| \frac{2at + d}{2\sqrt{at^2 + dt + c}} \right|^{p+1} + \sum_{k=1}^{p} |f^{(k)}(\sqrt{at^2 + dt + c})| \times$$

$$\times \frac{|Q_{p+1,k}(t)|}{|\sqrt{at^2+dt+c}|^{2p+2-k}} \leq (CM_1^{p+1}+pM_1)\max\{|f^{(k)}(\sqrt{at^2+dt+c})|\colon\ 0\leq k\leq p\}. \quad (7)$$

Using (6), we obtain the following equality for the $s$-th derivative of the function $f(\sqrt{at^2+dt+c})$

$$f^{(s)}(\sqrt{at^2+dt+c}) = g^{(s)}(t)\left(\frac{2\sqrt{at^2+dt+c}}{2at+d}\right)^s - \sum_{k=1}^{s-1} f^{(k)}(\sqrt{at^2+dt+c})\times$$

$$\times \frac{Q_{s,k}(t)}{(\sqrt{at^2+dt+c})^{2s-k}}\cdot\left(\frac{2\sqrt{at^2+dt+c}}{2at+d}\right)^s. \quad (8)$$

Hence, there exist $R_2\geq R_1$ such that for $|t|\geq R_2$ the expressions $\left|\frac{2\sqrt{at^2+dt+c}}{2at+d}\right|$, $\frac{|Q_{s,k}(t)|}{|\sqrt{at^2+dt+c}|^{2s-k}}$ are bounded by same constant $M_2>0$. Applying (8), we estimate $|f^{(k)}(\sqrt{at^2+dt+c})|$ via $g^{(k)}(t)$ and $|f^{(j)}(\sqrt{at^2+dt+c})|$ $(j\in\{1,\ldots,k-1\})$ in (7)

$$|g^{(p+1)}(t)| \leq (CM_1^{p+1}+pM_1)\times \quad (9)$$

$$\times\max\left\{|g(t)|,\ |g'(t)|M_2,\ |g^{(k)}(t)|M_2^k+\sum_{j=1}^{k-1}|f^{(j)}(\sqrt{at^2+dt+c})|M_2^{k+1}\colon\ 2\leq k\leq p\right\}. \quad (10)$$

Instead of $|f^{(j)}(\sqrt{at^2+dt+c})|$ we again substitute (8) and estimate $|f^{(j)}(\sqrt{at^2+dt+c})|$ via $g^{(j)}(t)$ and derivatives $|f'(\sqrt{at^2+dt+c})|,\ \ldots,\ |f^{(j-1)}(\sqrt{at^2+dt+c})|$. Proceeding we reduce the order of the highest derivative of the function $|f(\sqrt{at^2+dt+c})|$ by one. Finally, there exists $M^*>0$ that for $|t|\geq R_2$

$$|g^{(p+1)}(t)| \leq M^*\max\left\{|g^{(k)}(t)|\colon\ 0\leq k\leq p\right\}. \quad (11)$$

Therefore, by Theorem 2, $g(t)$ is of bounded index.

2) Let $f(t)$ have no zeros and be of bounded index.

By Theorem 3, there exists $P>0$ such that for all $t\in\mathbb{C}$ $\left|\frac{f'(t)}{f(t)}\right|\leq P$. Put $P=\sup_{t\in\mathbb{C}}\left|\frac{f'(t)}{f(t)}\right|$. Then there exists a sequence $t_p\to\infty$ and $\left|\frac{f'(t_p)}{f(t_p)}\right|\to P$ as $p\to\infty$.

Calculate the modulus of the directional logarithmic derivative of $f(\sqrt{z_1z_2})$ :

$$\frac{1}{|f(\sqrt{z_1z_2})|}\cdot\left|\frac{\partial f(\sqrt{z_1z_2})}{\partial\mathbf{b}}\right| = \left|\frac{f'(\sqrt{z_1z_2})}{f(\sqrt{z_1z_2})}\right|\cdot\frac{1}{2}\left|b_1\sqrt{\frac{z_2}{z_1}}+b_2\sqrt{\frac{z_1}{z_2}}\right|. \quad (12)$$

Let $b_1\neq 0$. We take $z_1=1$, $z_2=t_p^2$ and calculate the limit in (12):

$$\lim_{p\to\infty}\frac{1}{|f(\sqrt{t_p^2\cdot 1})|}\cdot\left|\frac{\partial f(\sqrt{t_p^2\cdot 1})}{\partial\mathbf{b}}\right| = \frac{P}{2}\lim_{n\to\infty}\left(b_1t_p+\frac{b_2}{t_p}\right)=\infty.$$

If $b_1=0$ then $b_2\neq 0$. We take $z_2=1$, $z_1=t_p^2$ and again prove that the directional logarithmic derivative is unbounded. By Theorem 3 the function $f(\sqrt{z_1z_2})$ is of unbounded index in the direction $\mathbf{b}$.

If $f(t)$ has a finite number of zeros then there exists $R>0$ such that $\forall t\in\mathbb{C}$, $|t|\geq R$, we have $f(t)\neq 0$. Therefore we replace $t\in\mathbb{C}$ with $|t|\geq R$ and repeat other considerations of item 2) without changes.

3) Let $(c_l)_{l=1}^{\infty}$ be an infinite sequence of zeros of $f(t)$, $|c_1| < |c_2| < \ldots < |c_k| < \ldots$ and for every $k \in \mathbb{N}$ $c_k^2 \in \mathbb{R}$. It remains to prove that the function $f(\sqrt{z_1 z_2})$ is of unbounded index in direction $\mathbf{b}$. We show that condition (5) of Theorem 3 does not hold.

**Case 1.** Let $b_1 \neq 0$, $b_2 \neq 0$ and $a_k \in \mathbb{R}$, $a_k \to \infty$. Later we impose more conditions on the sequence $(a_k)_{k=1}^{\infty}$. Put $\varphi = \arg(b_1 b_2)$, $z^0 = (z_1^0, z_2^0)$, where $z_1^0$ is an arbitrary complex number

$$z_2^0 = \frac{b_2 z_1^0 + (1 - a_k^2) e^{i\varphi/2}}{b_1}, \ t_0 = \frac{a_k^2 e^{i\varphi/2} - b_2 z_1^0}{b_1 b_2}. \tag{13}$$

The zeros of the function $F(z^0 + t\mathbf{b})$ can be found from the equation

$$(z_1^0 + b_1 t)(z_2^0 + b_2 t) = b_1 b_2 t^2 + (z_1^0 b_2 + z_2^0 b_1) t + z_1^0 z_2^0 = c_l^2, l \in \mathbb{N}.$$

Consider its roots

$$t_l = \frac{-(b_2 z_1^0 + b_1 z_2^0) + \sqrt{(b_2 z_1^0 - b_1 z_2^0)^2 + 4 c_l^2 b_1 b_2}}{2 b_1 b_2}.$$

A condition equivalent to $|t_l - t_0| < r$ is

$$r|b_1 b_2| > \left| a_k^2 e^{i\varphi/2} - b_2 z_1^0 - \frac{-(2 b_2 z_1^0 + (1 - a_k^2) e^{i\varphi/2}) + \sqrt{(a_k^2 - 1)^2 e^{i\varphi} + 4|b_1 b_2| e^{i\varphi} c_l^2}}{2} \right| \iff$$

$$2r|b_1| \cdot |b_2| > \left| a_k^2 + 1 \pm \sqrt{(a_k^2 - 1)^2 + 4 c_l^2 |b_1 b_2|} \right| \Longrightarrow$$

$$a_k^2 + 1 - 2r|b_1| \cdot |b_2| < \sqrt{(a_k^2 - 1)^2 + 4 c_l^2 |b_1 b_2|} < a_k^2 + 1 + 2r|b_1| \cdot |b_2|.$$

For positivity of the left hand side we assume $r < \frac{1}{2|b_1 b_2|}$. Hence we have

$$a_k^4 + 1 + 4r^2 |b_1 b_2|^2 + 2a_k^2 - 4r|b_1 b_2| - 4r|b_1 b_2| a_k^2 < a_k^4 - 2a_k^2 + 1 + 4|b_1 b_2| c_l^2 <$$
$$< a_k^4 + 1 + 4r^2 |b_1 b_2|^2 + 2a_k^2 + 4r|b_1 b_2| + 4r|b_1 b_2| a_k^2 \iff$$
$$r^2 |b_1 b_2|^2 + a_k^2 - r|b_1 b_2| - r a_k^2 |b_1 b_2| < |b_1 b_2| c_l^2 < r^2 |b_1 b_2|^2 + a_k^2 + r|b_1 b_2| + r a_k^2 |b_1 b_2|.$$

Then

$$c_l^2 \in \left( r^2 |b_1 b_2| + \frac{a_k^2}{|b_1 b_2|} - r - r a_k^2 ; r^2 |b_1 b_2| + \frac{a_k^2}{|b_1 b_2|} + r + r a_k^2 \right) \equiv (A_k; B_k).$$

for $r \in (0; \frac{1}{2|b_1 b_2|})$. But $B_k - A_k = 2r + 2r a_k^2$. Thus, for a given $n^* \in \mathbb{N}$ we require $c_{l+n^*+1}^2 - c_l^2 < 2r + 2r a_k^2$. Hence, we pick $a_k^2 > \frac{c_{l+n^*+1}^2 - c_l^2 - 2r}{2r}$. It means $\exists r > 0$ (we have $r \in (0; \frac{1}{2|b_1 b_2|})$) $\forall n^* \in \mathbb{N}$ $\exists z^0 \in \mathbb{C}^n$ $\exists t^0 \in \mathbb{C}$ (see (13)) that $n(r, z^0, t^0, \frac{1}{f}) \geq n^* + 1 > n^*$.

Therefore, $f(\sqrt{z_1 z_2})$ is of unbounded index in the direction $\mathbf{b}$.

**Case 2.** Let $b_1 \neq 0$, $b_2 = 0$ and $a_k \in \mathbb{R}$, $a_k \to \infty$. Later we impose more conditions on the sequence $(a_k)_{k=1}^{\infty} (k \to \infty)$. We put $\varphi = \arg(b_1)$, $z^0 = (z_1^0, z_2^0)$, where

$$z_1^0 = e^{i\varphi}, \ z_2^0 = a_k^2 \cdot e^{-i\varphi}, \ t_0 = 1. \tag{14}$$

The zeros of the function $F(z^0 + t\mathbf{b})$ are found from the equation

$$(z_1^0 + b_1 t) z_2^0 = z_2^0 b_1 t + z_1^0 z_2^0 = c_l^2, l \in \mathbb{Z}.$$

Consider its root $t_l = \frac{c_l^2 - z_1^0 z_2^0}{b_1 z_2^0}$. The condition that a zero $t_l$ belongs to $r$–neighborhood of the point $t_0$ has the form

$$|t_l - t_0| < r \Longleftrightarrow t_0 - r < \frac{c_l^2 - z_1^0 z_2^0}{b_1 z_2^0} < t_0 + r \text{ for } r \in (0, t_0).$$

We remark that $t_0$, $r$, $b_1 z_2^0$, $z_1^0 z_2^0 \in \mathbb{R}$. Hence, for $r \in (0,1)$ the following inequalities hold

$$(1-r)|b_1|a_k^2 + a_k^2 < c_l^2 < (1+r)|b_1|a_k^2 + a_k^2$$

Then $c_l^2 \in ((1-r)|b_1|a_k^2 + a_k^2; (1+r)|b_1|a_k^2 + a_k^2) \equiv (A_k; B_k)$. But $B_k - A_k = 2r|b_1|a_k^2$. Thus, for given $n^* \in \mathbb{N}$ we require $c_{l+n^*+1}^2 - c_l^2 < 2r|b_1|a_k^2$. Hence, we choose $a_k^2 > \frac{c_{l+n^*+1}^2 - c_l^2}{2r|b_1|}$. It means $\exists r > 0$ (we have $r \in (0;1)$) $\forall n^* \in \mathbb{N}$ $\exists z^0 \in \mathbb{C}^n$ $\exists t^0 \in \mathbb{C}$ (see (14)) that $n(r, z^0, t^0, \frac{1}{f}) \geq n^* + 1 > n^*$.

Thus, the function $f(\sqrt{z_1 z_2})$ is of unbounded index in the direction $\mathbf{b}$. □

**Remark 1.** The condition $c_k^2 \in \mathbb{R}$ can be replaced by the existence of an infinite subsequence of zeros of the form $c_k' = |c_k'| \cdot e^{i\theta}$, i.e. all $c_k'$ lay on some ray. Then in the case $b_1 \neq 0$, $b_2 \neq 0$ we choose $\varphi = 2\theta + \arg(b_1 b_2)$ and in the case $b_1 \neq 0$, $b_2 = 0$ we choose $z_2^0 = a_k^2 \cdot e^{-i\varphi + 2i\theta}$. Other considerations remained without changes in the proof.

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