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A NOTE-QUESTION ON PARTITIONS OF SEMIGROUPS

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Given a semigroup S and an n-partition $\mathcal{P}$ of S, $n \in \mathbb{N}$, do there exist $A \in \mathcal{P}$ and a subset F of S such that $S = F^{-1}\{x \in S \colon xA \cap A \neq \varnothing\}$ and $|F| \leq n$?

We give an affirmative answer provided that either S is finite or $n = 2$.

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Верно ли, что для любого n-разбиения полугруппы S найдутся подмножество A разбиения и подмножество F из S, такие, что $S = F^{-1}\{x \in S \colon xA \cap A \neq \varnothing\}$. Получен положительный ответ, если S конечна либо $n = 2$.

1. Introduction. In 1995, the first author asked the following question ([3, Problem 13.44]).

Given a group G and an n-partition $\mathcal{P}$, $n \in \mathbb{N}$ of G, do there exist $A \in \mathcal{P}$ and a subset F of G such that $G = FAA^{-1}$ and $|F| \leq n$?

For the current state of this open problem see the survey [1]. We mention only that an answer is positive if either G is amenable (in particular, finite), or $n \leq 3$, or $x^{-1}Ax = A$ for any $A \in \mathcal{P}$ and $x \in G$. If G is an arbitrary group and $\mathcal{P}$ is an n-partition of G then one can choose A, $B \in \mathcal{P}$ and subsets F, H of G such that $G = FAA^{-1}$, $|F| \leq n!$ and $G = HBB^{-1}B$ and $|H| \leq n$.

In this note, we formulate a semigroup version of above question and give positive answer provided that either S is finite or $n = 2$.

For systematic exposition of Ramsey theory of semigroups see [2].

For a semigroup S, $a \in S$, $A \subseteq S$ and $B \subseteq S$, we use the standard notations

$$a^{-1}B = \{x \in S \colon ax \in B\}, \quad A^{-1}B = \cup_{a \in A} a^{-1}B.$$

We set $\Delta(A) = \{x \in S \colon xA \cap A \neq \varnothing\}$ and observe that $\Delta(A) = \{x \in S \colon x^{-1}A \cap A \neq \varnothing\}$ and if S is a group then $\Delta(A) = AA^{-1}$.

We suppose that $S^{-1}A = S$ and define a *covering number*

$$\operatorname{cov} A = \min\{|X| \colon X \subseteq S, S = X^{-1}A\}.$$

If $S^{-1}A \neq S$ then $\operatorname{cov} A$ is not defined. Clearly, $\operatorname{cov} A$ is defined if and only if $Sx \cap A \neq \varnothing$ for every $x \in S$.

Now we are ready for promised question.

Given a semigroup S and an n-partition $\mathcal{P}$ of S, does there exist $A \in \mathcal{P}$ such that $\operatorname{cov} \Delta(A) \leq n$?

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2. Results.

Theorem 1. *For a semigroup S and an n-partition $\mathcal{P}$ of S, there exists $A \in \mathcal{P}$ such that* $\operatorname{cov} \Delta(A) \leq 2^{2^{n-1}-1}$.

If $n = 2$ then $\operatorname{cov} \Delta(A) \leq 2$. In a personal communication, G. Bergman answered the question positively for $n = 3$, and noticed that, we may suppose that a semigroup S is a monoid.

Theorem 2. *For a finite semigroup S and an n-partition $\mathcal{P}$ of S, there exists $A \in \mathcal{P}$ such that* $\operatorname{cov} \Delta(A) \leq n$.

Theorem 3. *If a subset A of a semigroup S contains either left or right zero then*

$$\operatorname{cov} \Delta(A) = 1.$$

3. Proofs. *Proof of Theorem 1.* We adopt arguments from [4, p. 120–121] proving this theorem for groups.

We define a function $f\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by

$$f(1, m) = m \quad \text{and} \quad f(n+1, m) = f(n, m + m^2).$$

By [4, Lemma 12.2], $f(n, m) \leq 2^{2^{n-1}-1} m^{2^{n-1}}$.

We use induction on n to prove the following auxiliary statement.

$(*)$ *Let $F, A_1, A_2, \ldots, A_n$ be subsets of a semigroup S such that $S = F^{-1}(A_1 \cup A_2 \cup \ldots \cup A_n)$ and $|F| \leq m$. Then there exist $i \in \{1, 2, \ldots, n\}$ and a subset K of S such that $S = K^{-1}\Delta(A_i)$ and $|K| \leq f(n, m)$.*

For $n = 1$, we have $S = F^{-1}A_1$. We take an arbitrary $x \in S$ and choose $g \in F$ such that $xA_1 \cap g^{-1}A_1 \neq \varnothing$. Then $A_1 \cap x^{-1}g^{-1}A_1 \neq \varnothing$, $A_1 \cap (gx)^{-1}A_1 \neq \varnothing$ so $gx \in \Delta(A)$, $x \in g^{-1}\Delta(A)$, $x \in F^{-1}\Delta(A)$ and $S = F^{-1}\Delta(A)$.

Let $S = F^{-1}(A_1 \cup A_2 \cup \ldots \cup A_{n+1})$. We consider two cases.

Case 1. $gA_1 \subseteq F^{-1}(A_2 \cup \ldots \cup A_{n+1})$ for some $g \in S$. Then $A_1 \subseteq g^{-1}F^{-1}(A_2 \cup \ldots \cup A_{n+1})$ and $S = (F^{-1} \cup F^{-1}g^{-1}F^{-1})(A_2 \cup \ldots \cup A_{n+1})$. Since $| F \cup FgF | \leq m + m^2$, by the inductive hypothesis, there exist $i \in \{2, 3, \ldots, n+1\}$ and a subset K of S such that

$$S = K^{-1}\Delta(A_i), \quad |K| \leq f(n, m + m^2) = f(n+1, m).$$

Case 2. $xA_1 \cap F^{-1}A_1 \neq \varnothing$ for every $x \in S$. Then $A_1 \cap x^{-1}F^{-1}A_1 \neq \varnothing$, $x^{-1}g^{-1}A_1 \cap A_1 \neq \varnothing$ for some $g \in F$, $gx \in \Delta(A_1)$ and $x \in g^{-1}\Delta(A)$, $S = F^{-1}\Delta(A)$. We set $K = F$ and note that $|K| \leq m \leq f(n+1, m)$.

To conclude the proof, we assume that $S = A_1 \cup \ldots \cup A_n$, take an arbitrary $g \in S$, put $F = \{g\}$, note that $S = F^{-1}(A_1 \cup \ldots \cup A_n)$ and apply $(*)$.

Proof of Theorem 2. Let S be a finite semigroup and $S = A_1 \cup \ldots \cup A_n$. We take a minimal right ideal R of S, choose $r \in R$ and note that $rS \subseteq R$, $S \subseteq r^{-1}R$, so we may suppose that $S = R$. By [2, Theorem 1.63(g)], S is a direct product of a group G and a right zero semigroup I. We take $a \in I$ and put $H = G \times \{a\}$. For each $i \in \{1, \ldots, n\}$, we denote $B_i = A_i \cap H$. Since H is a finite group, there are $j \in \{1, \ldots, n\}$ and $K \in H$ such that $|K| \leq n$ and $H = K^{-1}\Delta_H(B_j)$, where $\Delta_H(B_j) = \{x \in H\colon xB_j \cap B_j \neq \varnothing\}$. We take an arbitrary $(g, b) \in G \times I$ and choose $z \in K$ such that $z(g, a) \in \{x \in H\colon xB_j \cap B_j \neq \varnothing\}$. Since I is a right zero semigroup, we have $z(g, b)B_j \cap B_j \neq \varnothing$. Hence $(g, b) \in z^{-1}\{x \in S\colon xA_j \cap A_j \neq \varnothing\}$ and $S = K^{-1}\Delta(A_j)$. $\square$

Proof of Theorem 3. If $a \in A$ is left zero then, for every $x \in S$, we have $S = a^{-1}a = a^{-1}\{x \in A\colon xA \cap A \neq \varnothing\}$ and $S = a^{-1}\Delta(A)$.

If $a \in A$ is right zero then, for every $x \in S$, $a \in xA \cap A$ so $xA \cap A \neq \varnothing$ and $S = \Delta(A)$ and $S = g^{-1}\Delta(A)$ for each $g \in S$. $\square$

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