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ON THE UNIVALENCE OF ENTIRE FUNCTIONS OF BOUNDED l-INDEX

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For an entire function f it is established a relation between the l-index boundedness of the derivative f' and the existence for each $z_0 \in \mathbb{C}$ of the derivative $f^{(k)}$ univalent in the disk $\{z : |z - z_0| < \delta/l(|z_0|)\}$.

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Для целой функции f установлена связь между ограниченностью l-индекса производной f' и существованием для каждого $z_0 \in \mathbb{C}$ однолистной в круге $\{z : |z - z_0| < \delta/l(|z_0|)\}$ производной $f^{(k)}$.

Let l be a positive continuous function on $[0, +\infty)$. An entire function f is said ([1; 2, p. 71]) to be of bounded l-index if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} :\ 0 \le k \le N\right\}. \tag{1}$$

The least such integer N is called the *l-index* and is denoted by $N(f,l)$. Denote

$$n\Big(r,\ z_0, \frac{1}{f}\Big) = \sum_{|a_k - z_0| \le r} 1,$$

where a_k are zeros of f. The function f is said [2, p. 49] to be of bounded value l-distribution if there exists $p \in \mathbb{N}$ such that for all $z_0 \in \mathbb{C}$ and $w \in \mathbb{C}$

$$n\Big(\frac{1}{l(|z_0|)},\ z_0, \frac{1}{f - w}\Big) \le p,$$

i. e. the equation $f(z) = w$ has in $\{z :\ |z - z_0| \le 1/l(|z_0|)\}$ at most p solution and, thus, f is p-valent in $\{z :\ |z - z_0| \le 1/l(|z_0|)\}$.

As in [1; 2, p. 71] by Q we denote the class of positive continuous functions l on $[0, +\infty)$ such that $l(x+O(1/l(x)) = O(l(x))$ as $x \to +\infty$. The following statement is true ([2, p. 49]): *an entire function f is of bounded value l-distribution with $l \in Q$ iff its derivative f' is of bounded l-index*. Here we investigate the univalence of entire functions of bounded l-index. Suppose that $l \in Q_1$ if $l \in Q$ and $l(x\gamma(x)) = O(l(x))$ as $x \to +\infty$ hold for an arbitrary continuous function γ satisfying the condition $0 < A \le \gamma(x) \le B < +\infty$ for all $x \ge 0$. We prove the following theorem.

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Theorem 1. *Let f be an entire function and $l \in Q_1$. Then f' is of bounded l-index if and only if there exist an integer $N > 0$ and a number $\delta > 0$ such that for every $z_0 \in \mathbb{C}$ there exists an integer k, $0 < k \le N$ such that the derivative $f^{(k)}$ is univalent in the disk $\{z: |z - z_0| < \delta/l(|z_0|)\}$.*

For the proof of this theorem we need the following three lemmas.

Lemma 1 ([1]). *If $l \in Q_1$ and an entire function f is of bounded l-index then for each $a \in \mathbb{C}$ and $b \in \mathbb{C}$ the function $f(az + b)$ is of bounded l-index.*

The condition $l \in Q_1$ can not be replaced by the condition $l \in Q$, because $N(f_0, l_0) = 0$ for $f_0(z) = \exp\{e^z\}$ and $l_0(x) = e^x$ but $f_0(az)$ is of unbounded l-index for each $a > 1$.

Lemma 2. *Let $F(z) = z + \sum_{j=2}^{\infty} b_j z^j$ be an analytic function in $\mathbb{D}_R = \{z: |z| < R\}$. If*

$$\sum_{j=2}^{\infty} j|b_j|R^{j-1} \le 1 \tag{2}$$

then F is univalent in $\mathbb{D}_R$.

Indeed, let $|z_1| < R$, $|z_2| < R$ and $z_1 \ne z_2$. Then

$$|F(z_2) - F(z_2)| = \Big|z_2 - z_1 + \sum_{j=2}^{\infty} b_j(z_2^j - z_2^j)\Big| \ge |z_2 - z_1|\Big(1 - \sum_{j=2}^{\infty} |b_j|\Big|\frac{z_2^j - z_2^j}{z_2 - z_1}\Big|\Big) \ge$$

$$\ge |z_2 - z_1|\Big(1 - \sum_{j=2}^{\infty} |b_j||z_2^{j-1} + z_2^{j-2}z_1 + \cdots + z_1^{j-1}|\Big) > |z_2 - z_1|\Big(1 - \sum_{j=2}^{\infty} |b_j|jR^{j-1}\Big) \ge 0,$$

that is $F(z_2) \ne F(z_2)$.

Lemma 3. *If a function $\Phi(z) = z - z_0 + \sum_{j=2}^{\infty} b_j(z - z_0)^j$ is analytic and univalent in $\{z: |z - z_0| < R\}$ then $|b_j| \le j/R^{j-1}$ for all $j \ge 1$.*

Indeed, the function $F(z) = z + \sum_{j=2}^{\infty} b_j z^j$ is analytic and univalent in $\{z: |z| < R\}$ and, thus, the function

$$F(Rz) = R\Big(z + \sum_{j=2}^{\infty} b_j R^{j-1} z^j\Big)$$

is analytic and univalent in $\{z: |z| < 1\}$. Therefore, by Bieberbach's conjecture (proved in [3]) $|b_j|R^{j-1} \le j$ for all $j \ge 1$.

Now we prove Theorem. Suppose that f' is of bounded l-index and $l \in Q_1$. By Lemma 1 the function $\phi(z) = f'(2z)$ is of bounded l-index $N = N(\phi, l)$ and for each z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Then

$$\frac{2^n|f^{(n+1)}(2z)|}{n!l^n(|z|)} \le \max\left\{\frac{2^k|f^{(k+1)}(2z)|}{k!l^k(|z|)} : 0 \le k \le N\right\}$$

for all $z \in \mathbb{C}$ and, thus,

$$\frac{2^n|f^{(n+1)}(z_0)|}{n!l^n(|z_0/2|)} \le \max\left\{\frac{2^k|f^{(k+1)}(z_0)|}{k!l^k(|z_0/2|)} : 0 \le k \le N\right\},$$

whence it follows that there exists $0 \le k \le N$ such that for $j \ge 1$

$$\frac{|f^{(k+1)}(z_0)|}{k!} \ge \frac{2^j|f^{(k+j+1)}(z_0)|}{(k+j)!l^j(|z_0/2|)}.$$

That is

$$\frac{|f^{(k+1)}(z_0)|}{(k+1)!} \ge \frac{2^j(k+j+1)}{k+1}\frac{|f^{(k+j+1)}(z_0)|}{(k+j+1)!l^j(|z_0/2|)} \ge 2^j\frac{|f^{(k+j+1)}(z_0)|}{(k+j+1)!l^j(|z_0/2|)}. \tag{3}$$

Clearly,

$$f^{(k)}(z) = f^{(k)}(z_0) + f^{(1+k)}(z_0)\Big(z - z_0 + \sum_{j=2}^{\infty}\frac{f^{(k+j)}(z_0)}{j!f^{(k+1)}(z_0)}(z-z_0)^j\Big). \tag{4}$$

Consider the function

$$F(z) = z + \sum_{j=2}^{\infty}\frac{f^{(k+j)}(z_0)}{j!f^{(k+1)}(z_0)}z^j = z + \sum_{j=2}^{\infty}b_jz^j. \tag{5}$$

We choose δ_* such that

$$\sum_{j=1}^{\infty}\frac{(j+1)^N\delta_*^j}{2^j} \le 1.$$

Since $\frac{(k+j+1)!}{j!(k+1)!} \le (j+1)^k \le (j+1)^N$ from (3) we obtain

$$\sum_{j=1}^{\infty}(j+1)|b_{j+1}|\Big(\frac{\delta_*}{l(|z_0/2|)}\Big)^j = \sum_{j=1}^{\infty}(j+1)\frac{|f^{(k+j+1)|}(z_0)}{(j+1)!|f^{(k+1)|}(z_0)|}\Big(\frac{\delta_*}{l(|z_0/2|)}\Big)^j \le$$
$$\le \sum_{j=1}^{\infty}\frac{(k+j+1)!}{j!(k+1)!}\frac{\delta_*^j}{2^j} \le \sum_{j=1}^{\infty}\frac{(j+1)^N\delta_*^j}{2^j} \le 1.$$

Therefore, by Lemma 2 the function F is univalent in $\{z : |z| < \delta_*/l(|z_0/2|)\}$ and, thus, $f^{(k)}(z)$ is univalent in $\{z : |z-z_0| < \delta_*/l(|z_0/2|)\}$. If $l \in Q_1$ then $l(|z_0/2|) \asymp l(|z_0|)$ and in view of Lemma 1 the necessity is proved.

Conversely, if there exist an integer $N > 0$ and a number $\delta > 0$ such that for every $z_0 \in \mathbb{C}$ there exists an integer $0 < k \le N$ such that the derivative (4) is univalent in the disk $\{z : |z-z_0| < \delta/l(|z_0|)\}$. Then $f^{(1+k)}(z_0) \ne 0$ and the function (5) is univalent in the disk $\{z : |z| < \delta/l(|z_0|)\}$, that is by Lemma 3 $|b_j| \le j(l(|z_0|)/\delta)^{j-1}$ and, thus,

$$\frac{|f^{(k+j)}(z_0)|}{j!|f^{(k+1)}(z_0)|} \le j\Big(\frac{l(|z_0|)}{\delta}\Big)^{j-1}$$

for all $j \ge 1$. Hence

$$\frac{|f^{(k+j)}(z_0)|}{(k+j)!l^{k+j}(|z_0|)} \le \Big(\frac{1}{\delta}\Big)^{j-1}\frac{j!(k+1)!}{(k+j)!}\frac{j|f^{(k+1)}(z_0)|}{(k+1)!l^{k+1}(|z_0|)} \le \frac{j}{\delta^{j-1}}\frac{|f^{(k+1)}(z_0)|}{(k+1)!l^{k+1}(|z_0|)},$$

that is

$$\frac{\delta^{j-1}}{j}\frac{|f^{(k+j)}(z_0)|}{(k+j)!l^{k+j}(|z_0|)} \le \max\left\{\frac{|f^{(k+1)}(z_0)|}{(k+1)!l^{k+1}(|z_0|)} : 0 \le k \le N\right\}$$

for all $j \ge 1$ and all $z_0 \in \mathbb{C}$. We choose $\Delta \in (0, +\infty)$ such that $(\Delta\delta)^{j-1} \ge j$ for all $j \ge 1$. Then for $z_0 \in \mathbb{C}$

$$\frac{|f^{(k+j)}(\Delta z_0)|}{(k+j)!l^{k+j}(|z_0|)} \le \max\left\{\frac{|f^{(k+1)}(\Delta z_0)|}{(k+1)!l^{k+1}(|z_0|)} : 0 \le k \le N\right\}$$

and, thus,

$$\frac{|f^{(n)}(\Delta z_0)|}{n!l^{n}(|z_0|)} \le \max\left\{\frac{|f^{(k+1)}(\Delta z_0)|}{(k+1)!l^{k+1}(|z_0|)} : 0 \le k \le N\right\}$$

for all $n \ge 1$ and all $z_0 \in \mathbb{C}$. Hence it follows that the function $f'(\Delta z)$ is a function of bounded l-index and by Lemma 1 $f'(z)$ is of bounded l-index. The proof of Theorem 1 is complete.

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