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ENDOMORPHISMS OF FREE ABELIAN MONOGENIC DIGROUPS

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We construct a free abelian monogenic digroup and describe its endomorphism semigroup.

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Определяется конструкция свободной абелевой моногенной дигруппы и описывается ее полугруппа эндоморфизмов.

1. Introduction. The notion of a digroup first appeared in the work of Jean-Louis Loday ([1]). An algebraic system $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called a *digroup* if for all $x, y, z \in D$ the following conditions hold:

(D_1) $(x \dashv y) \dashv z = x \dashv (y \vdash z)$,

(D_2) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

(D_3) $(x \dashv y) \vdash z = x \vdash (y \vdash z)$,

(D_4) there exists $e \in D$ such that for all $x \in D$, $e \vdash x = x = x \dashv e$,

(D_5) for all $x \in D$ there exists a unique element $x^{-1} \in D$ such that $x \vdash x^{-1} = e = x^{-1} \dashv x$.

An element e is called a *bar-unit* of $(D, \dashv, \vdash)$ and x^{-1} is said to be *inverse* to x with respect to e. It should be noted that this definition does not imply that e is the unique bar-unit of D. In general the digroup can have many bar-units. If operations of a digroup coincide, the digroup becomes a group. One of the first results about digroups is the proof of the fact that Cayley's theorem for groups has an analogue in the class of all digroups ([2]). M. K. Kinyon modified Loday's terminology to give a much cleaner definition of a digroup and then used semigroup theory to show that every digroup is the product of a group and a trivial digroup ([3]). An even simpler basis of independent axioms for the variety of digroups was obtained by J. D. Phillips ([4]). Some structural properties of digroups were studied in [5]. More information on digroups and their examples can be found, for example, in [6], [7].

It is well-known that the notion of a digroup is closely related with the notion of a dimonoid ([1]). Recall that a nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms (D_1)–(D_3) is called a *dimonoid*. Dimonoids have been studied

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by many authors (see, e.g., [8]–[10]). Dimonoids and in particular digroups play a prominent role in problems from the theory of Leibniz algebras. In this paper we study a free abelian monogenic digroup and its endomorphism semigroup.

The paper is organized as follows. In Section 2, we give necessary definitions and construct a free abelian monogenic digroup. In Section 3, we define the least congruence on a free dimonoid such that the corresponding quotient is isomorphic to the free abelian monogenic digroup. In Section 4, we describe all endomorphisms of the free abelian monogenic digroup and construct a semigroup which is isomorphic to the endomorphism monoid of the given free digroup.

2. The free abelian monogenic digroup. A digroup $(D, \dashv, \vdash)$ is called *abelian* if $x \dashv y = y \vdash x$ for all $x, y \in D$ [6]. A digroup generated by one element is called *monogenic*.

Let G be an arbitrary abelian additive group, $X_1, X_2, ..., X_{n-1}$ be non-empty subsets of G and $X_n = G$ $(n \geq 2)$. We denote by $\prod_{i=1}^n X_i$ the direct product $X_1 \times X_2 \times ... \times X_n$ and set $x^+ = x_1 + x_2 + ... + x_n$ for all $x = (x_1, x_2, ..., x_n) \in \prod_{i=1}^n X_i$.

We take arbitrary $x, y \in \prod_{i=1}^n X_i$ and define two binary operations $\dashv$ and $\vdash$ on $\prod_{i=1}^n X_i$ as follows: $x \dashv y = (x_1, x_2, ..., x_{n-1}, x_n + y^+)$, $\quad x \vdash y = (y_1, y_2, ..., y_{n-1}, y_n + x^+)$.

Proposition 1. *The algebraic system* $(\prod_{i=1}^n X_i, \dashv, \vdash)$ *is an abelian digroup.*

Proof. Let $x, y, z \in \prod_{i=1}^n X_i$. Then

$$
\begin{aligned}
(x \dashv y) \dashv z &= (x_1, ..., x_{n-1}, x_n + y^+) \dashv (z_1, z_2, ..., z_n) = (x_1, ..., x_{n-1}, x_n + y^+ + z^+) = \\
&= (x_1, x_2 ..., x_n) \dashv (y_1, ..., y_{n-1}, y_n + z^+) = x \dashv (y \dashv z), \\
(x \vdash y) \vdash z &= (y_1, ..., y_{n-1}, y_n + x^+) \vdash (z_1, z_2, ..., z_n) = (z_1, ..., z_{n-1}, z_n + y^+ + x^+) = \\
&= (x_1, x_2 ..., x_n) \vdash (z_1, ..., z_{n-1}, z_n + y^+) = x \vdash (y \vdash z).
\end{aligned}
$$

Thus, operations $\dashv$ and $\vdash$ are associative. Show that axioms (D_1)–(D_3) hold:

$$
\begin{aligned}
(x \dashv y) \dashv z &= (x_1, ..., x_{n-1}, x_n + y^+ + z^+) = \\
&= (x_1, x_2 ..., x_n) \dashv (z_1, ..., z_{n-1}, z_n + y^+) = x \dashv (y \vdash z), \\
(x \vdash y) \dashv z &= (y_1, ..., y_{n-1}, y_n + x^+) \dashv (z_1, z_2, ..., z_n) = (y_1, ..., y_{n-1}, y_n + x^+ + z^+) = \\
&= (x_1, x_2 ..., x_n) \vdash (y_1, ..., y_{n-1}, y_n + z^+) = x \vdash (y \dashv z), \\
(x \dashv y) \vdash z &= (x_1, ..., x_{n-1}, x_n + y^+) \vdash (z_1, z_2, ..., z_n) = \\
&= (z_1, ..., z_{n-1}, z_n + y^+ + x^+) = x \vdash (y \vdash z).
\end{aligned}
$$

Therefore, $(\prod_{i=1}^n X_i, \dashv, \vdash)$ is a dimonoid.

Let e be an arbitrary bar-unit of $(\prod_{i=1}^n X_i, \dashv, \vdash)$. Then for all $x \in \prod_{i=1}^n X_i$ we obtain

$$e \vdash x = (x_1, ..., x_{n-1}, x_n + e^+) = (x_1, x_2, ..., x_n) = x \dashv e.$$

It follows that $e^+ = 0$. It is clear, if $e \in \prod_{i=1}^n X_i$ such that $e^+ = 0$, then e is a bar-unit of $(\prod_{i=1}^n X_i, \dashv, \vdash)$.

Fix a bar-unit e of $(\prod_{i=1}^n X_i, \dashv, \vdash)$ and assume that for some $x \in \prod_{i=1}^n X_i$ there exists $x^{-1} = (y_1, y_2, ..., y_n) \in \prod_{i=1}^n X_i$ such that $x \vdash x^{-1} = (y_1, ..., y_{n-1}, y_n + x^+) = (e_1, e_2, ..., e_n) = x^{-1} \dashv x$.

Hence $x^{-1} = (e_1, ..., e_{n-1}, e_n - x^+)$. Besides, x^{-1} is a unique inverse element to x with respect to e. So, $(\prod_{i=1}^n X_i, \dashv, \vdash)$ is a digroup.

Finally, we have $x \dashv y = (x_1, ..., x_{n-1}, x_n + y^+) = y \vdash x$ for all $x, y \in \prod_{i=1}^n X_i$. □

Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$ and $E = \{1, -1\}$. Denote by $(\mathbb{Z}, +)$ the additive group of all integer numbers.

By Proposition 1, the algebraic sysytem $(E \times \mathbb{Z}, \dashv, \vdash)$ is an abelian digroup. Bar-units of $(E \times \mathbb{Z}, \dashv, \vdash)$ are $(1, -1)$ and $(-1, 1)$.

For every element x of an arbitrary digroup $(D, \dashv, \vdash)$ we use denotations:

$$x^n_{\vdash} = \underbrace{x \vdash x \vdash \ldots \vdash x}_{n}, \quad x^n_{\dashv} = \underbrace{x \dashv x \dashv \ldots \dashv x}_{n} \ (n \in \mathbb{N}).$$

Lemma 1. *Each of sets $\{(1, 0)\}$, $\{(-1, 0)\}$ is generating for the digroup $(E \times \mathbb{Z}, \dashv, \vdash)$.*

Proof. Show that $\{(1, 0)\}$ is a generating set of $(E \times \mathbb{Z}, \dashv, \vdash)$. We take the bar-unit $(-1, 1)$ as the acting of a nullary operation on $(E \times \mathbb{Z}, \dashv, \vdash)$. Note that $(-1, 0)$ is inverse to $(1, 0)$ with respect to $(-1, 1)$. It is not hard to check by an induction that for all $n \in \mathbb{N}^0$,

$$(1, 0)^{n+1}_{\vdash} = (1, n) = (1, 0)^{n+1}_{\dashv}, \quad (-1, 0)^{n+1}_{\vdash} = (-1, -n) = (-1, 0)^{n+1}_{\dashv}.$$

Then for all $n \in \mathbb{N}^0$ we obtain $(1, 0) \dashv (-1, -n) = (1, -1 - n)$, $(-1, 0) \dashv (1, n) = (-1, 1 + n)$.

Therefore, $\langle (1, 0) \rangle = E \times \mathbb{Z}$. Analogously we can prove that $\{(-1, 0)\}$ is a generating set of the digroup $(E \times \mathbb{Z}, \dashv, \vdash)$. □

From this lemma immediately follows

Corollary 1. *Let $(i, 0)^0_{\dashv}$ be the fixed bar-unit of $(E \times \mathbb{Z}, \dashv, \vdash)$ for all $i \in E$. Each element (a, m) of $(E \times \mathbb{Z}, \dashv, \vdash)$ can be uniquely represented as $(a, m) = (a, 0) \dashv (i, 0)^m_{\dashv}$ for suitable $i \in E$.*

It is not hard to check that for every element x of an abelian digroup $(D, \dashv, \vdash)$ we have $x^n_{\vdash} = x^n_{\dashv}$ for all $n \in \mathbb{N}$. Therefore, for abelian digroups we write x^n instead of $x^n_{\vdash}$.

Remark 1. For abelian digroups the identity $x^m \dashv x^n = x^{m+n}$ is not true for integers m, n. In order to satisfy this identity, it is enough that both its sides would have one common multiplier on the left (on the right) with respect to an operation $\dashv$ (respect. $\vdash$).

For example, take the digroup $(E \times \mathbb{Z}, \dashv, \vdash)$, $x = (1, 0) \in E \times \mathbb{Z}$ and $m = 2, n = -4$. Then with respect to the bar-unit $(-1, 1)$ we have $x^{m+n} = (1, 0)^{-2} = (-1, -1) \neq (1, -3) = (1, 0)^2 \dashv (1, 0)^{-4} = x^m \dashv x^n$, however for all $(a, b) \in E \times \mathbb{Z}$, $(a, b) \dashv x^{m+n} = (a, b - 2) = (a, b) \dashv x^m \dashv x^n$.

For arbitrary digroups $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$, a mapping $\varphi : D_1 \to D_2$ is called *a homomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$ if for all $x, y \in D_1$ we have $(x \dashv_1 y)\varphi = x\varphi \dashv_2 y\varphi$, $(x \vdash_1 y)\varphi = x\varphi \vdash_2 y\varphi$.

A bijective homomorphism $\varphi \colon D_1 \to D_2$ is called *an isomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$. In this case digroups $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are called *isomorphic.*

Theorem 1. *The digroup $(E \times \mathbb{Z}, \dashv, \vdash)$ is a free abelian monogenic digroup.*

Proof. Let $(D', \dashv', \vdash')$ be an arbitrary abelian digroup, $(1, 0)\xi = t \in D'$, and $(-1, 0)\xi = t^{-1}$, where t^{-1} is inverse to t with respect to the fixed bar-unit $e' \in D'$. Further, we naturally extend ξ to a mapping Ξ of $E \times \mathbb{Z}$ into D' using the fact that $\{(1, 0)\}$ is the generating set of

$(E\times\mathbb{Z},\dashv,\vdash)$ and $(1,0)^{-1}=(-1,0)$ (see Lemma 1), that is, $(a,m)\Xi=((a,0)\dashv(i,0)^m)\Xi=t^a\dashv' t^{i|m|}=t^a\dashv' t^m$ for all $(a,m)\in E\times\mathbb{Z}$.

Assume that $(a,m),(b,n)\in E\times\mathbb{Z}$. Taking into account Remark 1,

$$((a,m)\dashv(b,n))\Xi=(a,m+b+n)\Xi=t^a\dashv' t^{m+b+n}=t^a\dashv'(t^m\dashv' t^b\dashv' t^n)=$$
$$=(t^a\dashv' t^m)\dashv'(t^b\dashv' t^n)=(a,m)\Xi\dashv'(b,n)\Xi.$$

By Proposition 1, the digroup $(E\times\mathbb{Z},\dashv,\vdash)$ is abelian. Since $(D',\dashv',\vdash')$ is an abelian digroup also, we obtain

$$((a,m)\vdash(b,n))\Xi=((b,n)\dashv(a,m))\Xi=(b,n)\Xi\dashv'(a,m)\Xi=(a,m)\Xi\vdash'(b,n)\Xi.$$

Thus, Ξ is a homomorphism of $(E\times\mathbb{Z},\dashv,\vdash)$ into $(D',\dashv',\vdash')$. In addition, $(E\times\mathbb{Z})\Xi$ is generated by one element t. □

3. The least abelian digroup congruence. Let $(D,\dashv,\vdash)$ be an arbitrary dimonoid, ρ be an equivalence relation on D which is stable on the left and on the right with respect to each of operations $\dashv,\vdash$. In this case ρ is called a *congruence* on $(D,\dashv,\vdash)$.

For a congruence ρ on a dimonoid $(D,\dashv,\vdash)$ the corresponding quotient-dimonoid is denoted by $(D,\dashv,\vdash)/\rho$. A congruence ρ on a dimonoid $(D,\dashv,\vdash)$ is called *abelian digroup* if the quotient-dimonoid $(D,\dashv,\vdash)/\rho$ is an abelian digroup.

Now we define a free dimonoid on an arbitrary set Y. Put $\widetilde{Y}=\{\widetilde{y}|y\in Y\}$. Two binary operations $\dashv$ and $\vdash$ are defined on the set

$$\mathrm{Fd}(Y)=\widetilde{Y}\cup(\widetilde{Y}\times Y)\cup(Y\times\widetilde{Y})\cup(\widetilde{Y}\times Y\times Y)\cup(Y\times\widetilde{Y}\times Y)\cup(Y\times Y\times\widetilde{Y})\cup...$$

as follows:

$$(y_1,...,\widetilde{y_i},...,y_k)\prec(y_{k+1},...,\widetilde{y_j},...,y_l)=(y_1,...,\widetilde{y_i},...,y_l),$$
$$(y_1,...,\widetilde{y_i},...,y_k)\succ(y_{k+1},...,\widetilde{y_j},...,y_l)=(y_1,...,\widetilde{y_j},...,y_l).$$

The algebra $(\mathrm{Fd}(Y),\prec,\succ)$ is the *free dimonoid* (see [1]). Elements of $\mathrm{Fd}(Y)$ are called *words* and $\widetilde{Y}$ is the *generating set* of $(\mathrm{Fd}(Y),\prec,\succ)$.

Let $X=\{x,x^{-1}\}$ and $(\mathrm{Fd}(X),\prec,\succ)$ be the free dimonoid on X. By $q_{\widetilde{t}}(w), t\in X$, we denote the quantity of all letters $\widetilde{t}$ that are included in the canonical form of $w=(w_1,...,\widetilde{w_k},...,w_l)$: $w=\widetilde{w_1}\succ...\succ\widetilde{w_k}\prec...\prec\widetilde{w_l}$.

For every $w\in\mathrm{Fd}(X)$ we put $q(w)=q_{\widetilde{x}}(w)-q_{\widetilde{x^{-1}}}(w)$. Define a binary relation σ on $\mathrm{Fd}(X)$ as follows: words $u=(u_1,...,\widetilde{u_i},...,u_n)$ and $v=(v_1,...,\widetilde{v_j},...,v_m)$ of $\mathrm{Fd}(X)$ are *σ-equivalent* if $u_i=v_j$ and $q(u)=q(v)$.

A word $u=(\widetilde{u_1},...,u_i,...,u_n)\in\mathrm{Fd}(X)$ we call *irreducible* if it do not contain any subword of the form $(x,x^{-1}),(x^{-1},x)$. For example, irreducible words of $\mathrm{Fd}(X)$ are $\widetilde{x}$, $(\widetilde{x},x,x),(\widetilde{x},x^{-1},x^{-1},x^{-1})$ and $\widetilde{x^{-1}}$, $(\widetilde{x^{-1}},x^{-1})$, $(\widetilde{x^{-1}},x,x,x,x)$.

Lemma 2. *The relation σ is a congruence on the free dimonoid $(\mathrm{Fd}(X),\prec,\succ)$ such that for any class $[w]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$ there exists a unique irreducible word $w'\in[w]$ of the form $w'=\widetilde{y}v,\ y\in X, v\in X^*\cup(X^{-1})^*$, where X^* and $(X^{-1})^*$ are free monoids on X and X^{-1}, respectively.*

Proof. It easy to see that σ is an equivalence relation. Assume that $u=(u_1,...,\widetilde{u_i},...,u_n), v=(v_1,...,\widetilde{v_j},...,v_m)\in \mathrm{Fd}(X)$ such that $u\sigma v$ and $w=(w_1,...,\widetilde{w_k},...,w_l)\in \mathrm{Fd}(X)$. Then

$$u\prec w=(u_1,...,\widetilde{u_i},...,u_n,w_1,...,w_l),\quad v\prec w=(v_1,...,\widetilde{v_j},...,v_m,w_1,...,w_l),$$
$$u\succ w=(u_1,...,u_n,w_1,...,\widetilde{w_k},...,w_l),\quad v\succ w=(v_1,...,v_m,w_1,...,\widetilde{w_k},...,w_l).$$

Since $u_i=v_j$ and $q(u\prec w)=q(v\prec w)$, $q(u\succ w)=q(v\succ w)$, we have $(u\prec w)\sigma(v\prec w)$ and $(u\succ w)\sigma(v\succ w)$. Analogously we can show that $(w\prec u)\sigma(w\prec v)$ and $(w\succ u)\sigma$ $(w\succ v)$. Thus, σ is a congruence.

Let $[w]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$ be an arbitrary congruence class, $w=(w_1,...,\widetilde{w_k},...,w_l)$. By the definition of σ, such words as $w''=(\widetilde{w_k},w_1,...,w_{k-1},w_{k+1},...,w_l)$ and

$$(\widetilde{w_k},x,x^{-1},w_1,...,w_{k-1},w_{k+1},...,w_l),...,(\widetilde{w_k},w_1,...,w_{k-1},w_{k+1},...,w_l,x,x^{-1}),$$
$$(\widetilde{w_k},x^{-1},x,w_1,...,w_{k-1},w_{k+1},...,w_l),...,(\widetilde{w_k},w_1,...,w_{k-1},w_{k+1},...,w_l,x^{-1},x)$$

are σ-equivalent to w. Thus, deleting from w'' all subwords of the form $(x,x^{-1}),(x^{-1},x)$ (in the case if such subwords there exist), we obtain the irreducible word w' which is σ-equivalent to w. Uniqueness of w' is obvious, besides w' can be represented as $(\widetilde{x^m},\underbrace{x^n,x^n,...,x^n}_{s})$, where $m,n=\pm1, s\geq 0$. $\square$

For convenient irreducible words of $\mathrm{Fd}(X)$ we will write as $\widetilde{x^\alpha}x^\beta$, where $\alpha\in E,\beta\in\mathbb{Z}$, in particular $\widetilde{x^\alpha}x^0=\widetilde{x^\alpha}$.

Lemma 3. *The quotient-dimonoid* $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ *is an abelian digroup isomorphic to the free digroup* $(E\times\mathbb{Z},\dashv,\vdash)$.

Proof. Lemma 2 implies $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ is a dimonoid. Since $(u\dashv v)\sigma(v\vdash u)$ for all $u,v\in\mathrm{Fd}(X)$, we have $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ is abelian. Let $[e]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$ such that $q_{\widetilde{x}}(e)=q_{\widetilde{x^{-1}}}(e)$. Then for all $[w]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$, $[e]\succ[w]=[e\succ w]=[w]=[w]\prec[e]$.

Thus, $[e]$ is a bar-unit of $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ for all $e\in\mathrm{Fd}(X)$ with $q(e)=0$. Moreover, for the fixed bar-unit $[e]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma, e=\widetilde{x^{e_1}}x^{e_2}$, and $[w],[u]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$, where $w=\widetilde{x^{w_1}}x^{w_2}$, $u=\widetilde{x^{u_1}}x^{u_2}$, the equalities $[w]\succ[u]=[\widetilde{x^{w_1}}x^{w_2}\succ\widetilde{x^{u_1}}x^{u_2}]=[\widetilde{x^{u_1}}x^{w_1+w_2+u_2}]=[\widetilde{x^{e_1}}x^{e_2}]=[u]\prec[w]$ imply $[u]=[\widetilde{x^{e_1}}x^{e_2-w_1-w_2}]=[w]^{-1}$. Uniqueness of $[w]^{-1}$ is obvious.

Now define a mapping φ of $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ into $(E\times\mathbb{Z},\dashv,\vdash)$ by $[w]\varphi=(w_1,w_2)$ for all irreducible words $w=\widetilde{x^{w_1}}x^{w_2}\in\mathrm{Fd}(X)$. Taking into account Lemma 2, it is clear that φ is a bijective mapping.

Further for all $[u],[v]\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$, where $u=\widetilde{x^{u_1}}x^{u_2}, v=\widetilde{x^{v_1}}x^{v_2}$, we have

$$([u]\prec[v])\varphi=[\widetilde{x^{u_1}}x^{u_2}\prec\widetilde{x^{v_1}}x^{v_2}]\varphi=[\widetilde{x^{u_1}}x^{u_2+v_1+v_2}]\varphi=(u_1,u_2+v_1+v_2)=$$
$$=(u_1,u_2)\dashv(v_1,v_2)=[u]\varphi\dashv[v]\varphi.$$

Since digroups $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ and $(E\times\mathbb{Z},\dashv,\vdash)$ are abelian,

$$(A\succ B)\varphi=(B\prec A)\varphi=B\varphi\dashv A\varphi=A\varphi\vdash B\varphi$$

for all $A,B\in(\mathrm{Fd}(X),\prec,\succ)/\sigma$. $\square$

From this lemma it follows that for $(\mathrm{Fd}(X),\prec,\succ)/\sigma$ there exist only two distinct bar-units $e_1=[(\widetilde{x},x^{-1})]$ and $e_2=[(\widetilde{x^{-1}},x)]$.

Theorem 2. *The binary relation σ is the least abelian digroup congruence on the free dimonoid* $(\mathrm{Fd}(X), \prec, \succ)$ *with* $X = \{x, x^{-1}\}$.

Proof. The proof of this statement follows from Lemma 2 and Lemma 3. □

4. Endomorphisms of the free abelian digroup of rank 1. For an arbitrary digroup $\mathfrak{D} = (D, \dashv, \vdash)$ by $\mathrm{End}(\mathfrak{D})$ we denote the endomorphism monoid of $\mathfrak{D}$. First, we describe all endomorphisms of the free abelian monogenic digroup.

Lemma 4. *Let e be the fixed bar-unit of the free abelian digroup $(E\times\mathbb{Z}, \dashv, \vdash)$ and $t \in E\times\mathbb{Z}$. A transformation $\xi_{e,t}$ of $(E \times \mathbb{Z}, \dashv, \vdash)$ defined by*

$$(a, n)\xi_{e,t} = \begin{cases} (t_1, nt^+ + t_2), & \text{if } a = 1, \\ (e_1, (n-1)t^+ + e_2), & \text{if } a = -1 \end{cases}$$

is an endomorphism.

Proof. For all $(a, n), (a', n') \in E \times \mathbb{Z}$, we have the following cases:

1) $a = a' = 1$, then

$$((1, n) \dashv (1, n'))\xi_{e,t} = (1, n + 1 + n')\xi_{e,t} = (t_1, (n + 1 + n')t^+ + t_2) = \\ = (t_1, nt^+ + t_2) \dashv (t_1, n't^+ + t_2) = (1, n)\xi_{e,t} \dashv (1, n')\xi_{e,t};$$

2) $a = 1, a' = -1$, then

$$((1, n) \dashv (-1, n'))\xi_{e,t} = (1, n - 1 + n')\xi_{e,t} = (t_1, (n - 1 + n')t^+ + t_2) = \\ = (t_1, nt^+ + t_2) \dashv (e_1, (n' - 1)t^+ + e_2) = (1, n)\xi_{e,t} \dashv (-1, n')\xi_{e,t};$$

3) $a = -1, a' = 1$, then

$$((-1, n) \dashv (1, n'))\xi_{e,t} = (-1, n + 1 + n')\xi_{e,t} = (e_1, (n + n')t^+ + e_2) = \\ = (e_1, (n - 1)t^+ + e_2) \dashv (t_1, n't^+ + t_2) = (-1, n)\xi_{e,t} \dashv (1, n')\xi_{e,t};$$

4) $a = a' = -1$, then

$$((-1, n) \dashv (-1, n'))\xi_{e,t} = (-1, n - 1 + n')\xi_{e,t} = (e_1, (n + n' - 2)t^+ + e_2) = \\ = (e_1, (n - 1)t^+ + e_2) \dashv (e_1, (n' - 1)t^+ + e_2) = (-1, n)\xi_{e,t} \dashv (-1, n')\xi_{e,t}.$$

From 1)–4) it follows that $\xi_{e,t} \in \mathrm{End}(E\times\mathbb{Z}, \dashv)$. Since $(E\times\mathbb{Z}, \dashv, \vdash)$ is an abelian digroup, $\xi_{e,t} \in \mathrm{End}(E \times \mathbb{Z}, \dashv, \vdash)$ for all $e, t \in E \times \mathbb{Z}, e^2 = e$. □

Note that endomorphisms $\xi_{e,t}$, $e, t \in E \times \mathbb{Z}$, where $e^+ = 0$, are not injective in general. For example, if $e = t, e^2 = e$, we have $x\xi_{e,t} = e$ for all $x \in E \times \mathbb{Z}$.

Lemma 5. *Let $x = (x_1, x_2) \in E \times \mathbb{Z}$ and $m \in \mathbb{N}$. Then $x^m = (x_1, x_2 + (m - 1)x^+)$.*

Proof. The proof of this statement is obvious. □

Lemma 6. *Let ξ be an arbitrary endomorphism of $(E \times \mathbb{Z}, \dashv, \vdash)$ and $(1, 0)\xi = t$. Then $\xi = \xi_{e,t}$ for some bar-unit $e \in E \times \mathbb{Z}$.*

Proof. Assume that $(-1,1)\xi = e$. It is clear that $e^2 = e$, i.e. e is the bar-unit of $(E\times\mathbb{Z}, \dashv, \vdash)$. Then there exists a unique inverse element $t^{-1} = (e_1, e_2 - t^+)$ to t with respect to e. By Corollary 1, for all $(a,n) \in E\times\mathbb{Z}$ we have $(a,n) = (a,0) \dashv (j,0)^n$ for suitable $j \in E$. Using Lemma 5, we obtain the following cases:

1) $n \geq 0$, $a = 1$, then $(1,n)\xi = ((1,0)^{n+1})\xi = t^{n+1} = (t_1, t_2 + nt^+)$;
2) $-n < 0$, $a = 1$, then

$$(1,-n)\xi = ((1,0) \dashv (-1,0)^n)\xi = t \dashv t^{-n} =$$
$$= (t_1,t_2) \dashv (e_1, e_2 - t^+ + (n-1)(e^+ - t^+)) = (t_1, t_2 - nt^+);$$

3) $n \geq 0$, $a = -1$, then

$$(-1,n)\xi = ((-1,0) \dashv (1,0)^n)\xi = t^{-1} \dashv t^n =$$
$$= (e_1, e_2 - t^+) \dashv (t_1, t_2 + (n-1)t^+) = (e_1, e_2 + (n-1)t^+);$$

4) $-n < 0$, $a = -1$, then

$$(-1,-n)\xi = (-1,0)^{n+1})\xi = (e_1, e_2 - t^+)^{n+1} = (e_1, e_2 - t^+ + n(e^+ - t^+)) = (e_1, e_2 - (n+1)t^+).$$

From 1)–4) it follows that ξ coincides with $\xi_{e,t}$ (see Lemma 4), where $e = (-1,1)\xi$. $\square$

Let W be the set of all bar-units of $(E\times\mathbb{Z}, \dashv, \vdash)$, that is $W = \{(1,-1), (-1,1)\}$. Consider a binary operation $\circ$ on $W\times(E\times\mathbb{Z})$ defined as follows

$$(e,t)\circ(i,s) = \begin{cases} ((s_1,-s_1),(s_1, t_2s^+ + s_2)), & \text{if } e_1 = t_1 = 1, \\ ((s_1,-s_1),(i_1, t^+s^+ - i_1)), & \text{if } e_1 = 1, t_1 = -1, \\ ((i_1,-i_1),(s_1, t_2s^+ + s_2)), & \text{if } e_1 = -1, t_1 = 1, \\ ((i_1,-i_1),(i_1, t^+s^+ - i_1)), & \text{if } e_1 = t_1 = -1. \end{cases}$$

It is clear that the operation $\circ$ is completed on $W\times(E\times\mathbb{Z})$.

Lemma 7. *The algebra* $(W\times(E\times\mathbb{Z}), \circ)$ *is a monoid with the identity* $((-1,1),(1,0))$.

Proof. Take arbitrary $(e,t),(i,s),(j,r) \in W\times(E\times\mathbb{Z})$ and put $A = ((e,t)\circ(i,s))\circ(j,r)$, $B = (e,t)\circ((i,s)\circ(j,r))$.

Assume that $e_1 = t_1 = i_1 = s_1 = 1$. Then

$$A = ((s_1,-s_1),(s_1,t_2s^+ + s_2))\circ(j,r) = ((r_1,-r_1),(r_1, t_2s^+r^+ + s_2r^+ + r_2)) =$$
$$= ((r_1,-r_1),(r_1, t_2(r^+ + s_2r^+) + s_2r^+ + r_2)) = (e,t)\circ((r_1,-r_1),(r_1, s_2r^+ + r_2)) = B.$$

For $e_1 = t_1 = i_1 = s_1 = -1$ we have

$$A = ((i_1,-i_1),(i_1, t^+s^+ - i_1))\circ(j,r) =$$
$$= ((j_1,-j_1),(j_1, t^+s^+r^+ - j_1)) = (e,t)\circ((j_1,-j_1),(j_1, s^+r^+ - j_1)) = B.$$

Let $e_1 = i_1 = 1, t_1 = s_1 = -1$. Then

$$A = ((s_1,-s_1),(i_1, t^+s^+ - i_1))\circ(j,r) = ((j_1,-j_1),(r_1, r_2 + (t^+s^+ - i_1)r^+)) =$$
$$= ((j_1,-j_1),(r_1, t^+s^+r^+ - r_1)) = (e,t)\circ((r_1,-r_1),(j_1, s^+r^+ - j_1)) = B.$$

If $e_1 = i_1 = -1, t_1 = s_1 = 1$, then

$$A = ((i_1, -i_1), (s_1, t_2s^+ + s_2)) \circ (j, r) = ((j_1, -j_1), (r_1, t_2s^+r^+ + s_2r^+ + r_2)) =$$
$$= ((j_1, -j_1), (r_1, t_2(r^+ + s_2r^+) + s_2r^+ + r_2)) = (e, t) \circ ((j_1, -j_1), (r_1, s_2r^+ + r_2)) = B.$$

In similar way all other cases are proved. Thus, $(W \times (E \times \mathbb{Z}), \circ)$ is a semigroup. A direct verification shows that an identity of $(W \times (E \times \mathbb{Z}), \circ)$ is $((-1, 1), (1, 0))$. □

The main result of this paper is the following theorem.

Theorem 3. *(i) For any $(e, t) \in W \times (E \times \mathbb{Z})$ a transformation $\xi_{e,t}$ of the free abelian monogenic digroup $(E \times \mathbb{Z}, \dashv, \vdash)$ defined by*

$$(a, n)\xi_{e,t} = \begin{cases} (t_1, nt^+ + t_2), & \text{if } a = 1, \\ (e_1, (n-1)t^+ + e_2), & \text{if } a = -1 \end{cases}$$

is an endomorphism. And every endomorphism of $(E \times \mathbb{Z}, \dashv, \vdash)$ has the above form.

(ii) The endomorphism monoid $\mathrm{End}(E \times \mathbb{Z}, \dashv, \vdash)$ *is isomorphic to* $(W \times (E \times \mathbb{Z}), \circ)$.

Proof. The proof of (i) immediately follows from Lemmas 4 and 6. Show that the statement (ii) holds. Define a bijection Υ of $\mathrm{End}(E \times \mathbb{Z}, \dashv, \vdash)$ into $(W \times (E \times \mathbb{Z}), \circ)$ by

$$\xi_{e,t}\Upsilon = (e, t) \ \textit{for all} \ \xi_{e,t} \in \mathrm{End}(E \times \mathbb{Z}, \dashv, \vdash).$$

Let $\xi_{e,t}, \xi_{i,s} \in \mathrm{End}(E \times \mathbb{Z}, \dashv, \vdash)$ and $(a, n) \in E \times \mathbb{Z}$. We have the following four cases.

1) $e_1 = t_1 = 1$. Then

$$(1, n)\xi_{e,t}\xi_{i,s} = (1, nt^+ + t_2)\xi_{i,s} = (s_1, s_2 + (nt^+ + t_2)s^+) =$$
$$= (s_1, nt^+s^+ + (t_2s^+ + s_2)) = (1, n)\xi_{(s_1,-s_1),(s_1,s_2+t_2s^+)},$$
$$(-1, n)\xi_{e,t}\xi_{i,s} = (1, (n-1)t^+ + e_2)\xi_{i,s} = (s_1, s_2 + ((n-1)t^+ + e_2)s^+) =$$
$$= (s_1, (n-1)t^+s^+ + (s_2 - s^+)) = (-1, n)\xi_{(s_1,-s_1),(s_1,s_2+t_2s^+)}.$$

Thus, $\xi_{(1,-1),(1,t_2)}\xi_{i,s} = \xi_{(s_1,-s_1),(s_1,s_2+t_2s^+)}$.

2) $e_1 = 1, t_1 = -1$. Then

$$(1, n)\xi_{e,t}\xi_{i,s} = (-1, nt^+ + t_2)\xi_{i,s} = (i_1, i_2 + (nt^+ + t_2 - 1)s^+) =$$
$$= (i_1, nt^+s^+ + (t^+s^+ - i_1)) = (1, n)\xi_{(s_1,-s_1),(i_1,t^+s^+-i_1)},$$
$$(-1, n)\xi_{e,t}\xi_{i,s} = (1, (n-1)t^+ + e_2)\xi_{i,s} = (s_1, ((n-1)t^+ + e_2)s^+ + s_2) =$$
$$= (s_1, (n-1)t^+s^+ - s_1) = (-1, n)\xi_{(s_1,-s_1),(i_1,t^+s^+-i_1)}.$$

So, $\xi_{(1,-1),(-1,t_2)}\xi_{i,s} = \xi_{(s_1,-s_1),(i_1,t^+s^+-i_1)}$.

3) $e_1 = -1, t_1 = 1$. Then

$$(1, n)\xi_{e,t}\xi_{i,s} = (1, nt^+ + t_2)\xi_{i,s} = (s_1, nt^+s^+ + (t_2s^+ + s_2)) = (1, n)\xi_{(i_1,-i_1),(s_1,t_2s^++s_2)},$$
$$(-1, n)\xi_{e,t}\xi_{i,s} = (-1, (n-1)t^+ + e_2)\xi_{i,s} = (i_1, i_2 + ((n-1)t^+ + e_2 - 1)s^+) =$$
$$= (i_1, (n-1)t^+s^+ - i_1) = (-1, n)\xi_{(i_1,-i_1),(s_1,t_2s^++s_2)}.$$

Therefore, $\xi_{(-1,1),(1,t_2)}\xi_{i,s} = \xi_{(i_1,-i_1),(s_1,t_2s^++s_2)}$.

4) $e_1 = t_1 = -1$. Then

$$\begin{gathered}(1,n)\xi_{e,t}\xi_{i,s} = (-1, nt^+ + t_2)\xi_{i,s} = (i_1, (nt^+ + t_2 - 1)s^+ + i_2)) = \\ = (i_1, nt^+s^+ + (t^+s^+ - i_1)) = (1,n)\xi_{(i_1,-i_1),(i_1,t^+s^+-i_1)}, \\ (-1,n)\xi_{e,t}\xi_{i,s} = (-1,(n-1)t^+ + e_2)\xi_{i,s} = (i_1, i_2 + ((n-1)t^+ + e_2 - 1)s^+) = \\ = (i_1,(n-1)t^+s^+ - i_1) = (-1,n)\xi_{(i_1,-i_1),(i_1,t^+s^+-i_1)}.\end{gathered}$$

Thus, $\xi_{(-1,1),(-1,t_2)}\xi_{i,s} = \xi_{(i_1,-i_1),(i_1,t^+s^+-i_1)}$.

Finally, using equalities from cases 1)–4), we obtain

$$(\xi_{e,t}\xi_{i,s})\Upsilon = \begin{cases} ((s_1,-s_1),(s_1,t_2s^+ + s_2)), & \text{if } e_1 = t_1 = 1, \\ ((s_1,-s_1),(i_1,t^+s^+ - i_1)), & \text{if } e_1 = 1, t_1 = -1, \\ ((i_1,-i_1),(s_1,t_2s^+ + s_2)), & \text{if } e_1 = -1, t_1 = 1, \\ ((i_1,-i_1),(i_1,t^+s^+ - i_1)), & \text{if } e_1 = t_1 = -1 \end{cases}$$

for all $\xi_{e,t}, \xi_{i,s} \in \mathrm{End}(E \times \mathbb{Z}, \dashv, \vdash)$.

On the other hand,

$$\xi_{e,t}\Upsilon \circ \xi_{i,s}\Upsilon = \begin{cases} ((1,-1),(1,t_2)) \circ (i,s) = ((s_1,-s_1),(s_1,t_2s^+ + s_2)), & \text{if } e_1 = t_1 = 1, \\ ((1,-1),(-1,t_2)) \circ (i,s) = ((s_1,-s_1),(i_1,t^+s^+ - i_1)), & \text{if } e_1 = 1, t_1 = -1, \\ ((-1,1),(1,t_2)) \circ (i,s) = ((i_1,-i_1),(s_1,t_2s^+ + s_2)), & \text{if } e_1 = -1, t_1 = 1, \\ ((-1,1),(-1,t_2)) \circ (i,s) = ((i_1,-i_1),(i_1,t^+s^+ - i_1)), & \text{if } e_1 = t_1 = -1, \end{cases}$$

which completes the proof of this theorem. □

Observe that the automorphism group of the free abelian digroup $(E \times \mathbb{Z}, \dashv, \vdash)$ is two-element, that is, $\mathrm{Aut}(E \times \mathbb{Z}, \dashv, \vdash) = \{\xi_{(-1,1),(1,0)}, \xi_{(1,-1),(-1,0)}\}$.

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