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ON GROUP TOPOLOGIES DETERMINED BY FAMILIES OF SETS

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Let G be an abelian group, and F a downward directed family of subsets of G. In [6], I. Protasov and E. Zelenyuk describe the finest group topology $\mathcal{T}$ on G under which F converges to 0; in particular, their description yields a criterion for $\mathcal{T}$ to be Hausdorff. They then show that if F is the filter of cofinite subsets of a countable subset $X \subseteq G$ (the Fréchet filter on X), there is a simpler criterion: $\mathcal{T}$ is Hausdorff if and only if for every $g \in G - \{0\}$ and positive integer n, there is an $S \in F$ such that g does not lie in the n-fold sum $n\,(S \cup \{0\} \cup -S)$.

In this note, their proof is adapted to a larger class of families F. In particular, if X is any infinite subset of G, κ any regular infinite cardinal $\leq \operatorname{card}(X)$, and F the set of complements in X of subsets of cardinality $< \kappa$, then the above criterion holds.

We also give some negative examples, including a countable downward directed set F (not of the above sort) of subsets of $\mathbb{Z}$ which satisfies the "$g \notin n\,(S \cup \{0\} \cup -S)$" condition but does not induce a Hausdorff topology.

We end with a version of our main result for noncommutative G.

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Пусть G — абелева группа и F — семейство подмножеств группы G, образующее убывающую направленность. В [6], И. Протасов и Е. Зеленюк дали описание сильнейшей групповой топологии $\mathcal{T}$ на G, в которой семейство F сходится к 0; в частности, ими был установлен критерий хаусдорфовости этой топологии. Они доказали, что если F — фильтр Фреше на счётном бесконечном подмножестве $X \subseteq G$, то существует простой критерий хаусдорфовости: топология $\mathcal{T}$ хаусдорфова тогда и только тогда, когда для каждого елемента $g \neq 0$ и произвольного натурального n, существует такой елемент S семейства F, что $g \notin n(S \cup \{0\} \cup -S)$.

В этой статье мы переносим результаты из [6] на широкий класс семейств F. В частности, если X — произвольное бесконечное подмножество в G, κ — регулярный бесконечный кардинал $\leq \operatorname{card}(X)$, и F — множество дополнений до X всех подмножеств мощности $< \kappa$, тогда вышеуказанный критерий выполняется.

Также мы даём некоторые отрицательные примеры счётных направленностей семейств F (иного рода, чем вышеупомянутые) в $\mathbb{Z}$ которые удовлетворяют условию "$g \notin n(S \cup \{0\} \cup -S)$" но не порождают хаусдорфовую топологию.

В заключении мы рассматриваем некоммутативную версию главного результата статьи.

1. Introduction. Let G be a group, let F be a set of subsets of G which is downward directed, i.e., such that whenever S_1, $S_2 \in F$, there is an $S_3 \in F$ which is contained in

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$S_1 \cap S_2$, and let $\mathcal{T}$ be a *group topology* on G; that is, a (not necessarily Hausdorff) topology under which the group multiplication and inverse operation are continuous. We say that F converges to an element $x \in G$ under $\mathcal{T}$ if every $\mathcal{T}$-neighborhood of x contains a member of F.

Given G and F, it is not hard to show that there will exist a finest group topology $\mathcal{T}_F$ on G under which F converges to the identity element of G. The explicit description of $\mathcal{T}_F$ is simpler and easier to study for abelian G than for general G, so we shall assume, until §6, that

$$G \text{ is an abelian group, with operations written additively.} \tag{1}$$

To describe the topology $\mathcal{T}_F$, let us set up some notation. For any subset $S \subseteq G$, let

$$S^* \;=\; S \cup \{0\} \cup -S. \tag{2}$$

For any sequence of subsets $S_0,\, S_1,\, \dots \subseteq G$ indexed by the set ω of natural numbers, let

$$U(S_0, S_1, \dots) \;=\; \bigcup_{n\in\omega} \sum_{i<n} S_i^* \;=\; \{x_0 + \dots + x_{n-1} \mid n \in \omega,\ x_i \in S_i^* \text{ for } i \in \{0, \dots, n-1\}\}. \tag{3}$$

(The $n = 0$ term of the above union, i.e., the sum of the vacuous sequence of sets, is understood to be $\{0\}$.)

Then one has

[6, Lemma 2.1.1] The sets $U(S_0, S_1, \dots)$, as $(S_i)_{i\in\omega}$ runs over all sequences of elements of F, form a basis of open neighborhoods of 0 under $\mathcal{T}_F$, the finest group topology on G under which F converges to 0. (4)

Thus, as noted in [6, Theorem 2.1.3], the topology $\mathcal{T}_F$ is Hausdorff (equivalently, there exists a Hausdorff group topology under which F converges to 0) if and only if

$$\bigcap_{S_0,\, S_1,\, \dots \in F} U(S_0,\, S_1,\, \dots) \;=\; \{0\}. \tag{5}$$

(Our formulations of these statements are different from those in [6] because there, group topologies are by definition Hausdorff. Though Hausdorff topologies are what we are interested in, it will convenient, for making statements like (4), to allow non-Hausdorff topologies. Incidentally, a topological group is Hausdorff if and only if it is T_0 [5, p. 32, Proposition 4 and preceding Exercise].)

From the fact that (5) is necessary and sufficient for $\mathcal{T}_F$ to be Hausdorff, we get a weaker condition which is necessary.

Corollary 1. *A* necessary *condition for the topology $\mathcal{T}_F$ to be Hausdorff is*

$\bigcup_{n>0} \bigcap_{S\in F} n\,S^* \;=\; \{0\}$. *In other words, for every $g \in G - \{0\}$ and every $n > 0$, there exists $S \in F$ with $g \notin n\,S^*$.* (6)

Proof. Assuming (5), consider any $g \in G - \{0\}$ and any $n > 0$. By (5) we can choose $S_0, S_1, \dots$ such that $g \notin U(S_0,\, S_1,\, \dots)$. In particular, $g \notin S_0^* + \dots + S_{n-1}^*$. Letting S be a common lower bound for $S_0, \dots, S_{n-1}$ in the downward directed set F, we have $g \notin n\,S^*$, as required. □

As illustrated by the notation "$G - \{0\}$" in the above proof, a "$-$" sign between sets in this note indicates relative complement; thus, $X - Y$ never denotes $X + (-Y)$.

In §4, we shall see by example that (6) is not in general *sufficient* for $\mathcal{T}_F$ to be Hausdorff. However, I. Protasov and E. Zelenyuk [6, Theorem 2.1.4] show that it *is* sufficient if F is the filter of cofinite subsets of a countable subset $X = \{x_0, x_1, \dots\} \subseteq G$; in other words, if $\mathcal{T}_F$ is the finest group topology on G making $\lim_{i\to\infty} x_i = 0$. Generalizing their argument, we shall obtain below the corresponding result for a wider class of F. In §6 we shall extend this result to nonabelian G.

2. Co-κ filters, and a peculiar condition that they satisfy. Here is our generalization of the class of filters considered in [6].

Definition 1 ([3, Example II.2.5])**.** Let X be an infinite set and κ an infinite cardinal $\leq \mathrm{card}(X)$. Then by the *co-κ filter* on X we shall mean the (downward directed) set of complements in X of subsets with cardinality $< \kappa$. For $\kappa = \aleph_0$, this will be called the *cofinite filter* on X.

(Remark: The cofinite filter on an infinite set X is often called the Fréchet filter on X. In some places, the co-card(X) filter on X has been called the "generalized Fréchet filter"; in [2, p. 197] the term "Fréchet filter" is used, instead, for the latter construction.)

To state the property of these filters that we will use, we make the following definition. It has the same form as the definition of convergence of a family of points under a group topology on G, but with the system of neighborhoods of 0 replaced by a more general family.

Definition 2. Suppose F is any downward directed family of subsets of the abelian group G, and $(x_i)_{i\in I}$ a family of elements of G indexed by a downward directed partially ordered set I. We shall say that $(x_i)_{i\in I}$ "converges strongly" to an element $x \in G$ with respect to F if for every $S \in F$, there exists $i \in I$ such that for all $j \leq i$, $x_j - x \in S^*$.

(Since F is not assumed to be a neighborhood basis of a group topology, this is not a very natural condition. I use the modifier "strongly" because the condition is stronger than convergence in the group topology *determined* by F as in (4). Note, incidentally, that the way in which the ordering on I is used in Definition 2 is the reverse of the usual. This is not essential; it will simply spare us reversing a certain natural ordering below. In any case, when an index set I is described as *downward* rather than upward directed, it is natural to adjust what one understands convergence of an I-indexed family to mean.)

We can now state the condition around which our main result will center.

Definition 3. A downward directed family F of subsets of the abelian group G will be called *self-indulgent* if for every $S \in F$, and every family $(x_T)_{T\in F'}$ of elements of S^* indexed by a downward cofinal subset $F' \subseteq F$, there exist an $x \in S^*$, and a downward cofinal subset $F'' \subseteq F'$, such that $(x_T)_{T\in F''}$ converges strongly to x with respect to F.

A strange feature of this condition (which motivates its name) is that it involves the family F in three unrelated ways: first, S is taken to be a member of F; second, the family of points $x_T \in S^*$ is indexed by a subfamily of F, and third, the convergence asked for is strong convergence with respect to F.

Lemma 1. *Let X be any infinite subset of the abelian group G, and κ any regular infinite cardinal $\leq$ card(X). Then the co-κ filter F on X is self-indulgent as a family of subsets of G.*

Proof. Let $S \in F$, and let $(x_T)_{T \in F'}$ be a family of elements of S^* indexed by a downward cofinal subset $F' \subseteq F$. If there exists an $x \in S^*$ which occurs "frequently" as a value of x_T, in the sense that $\{T \in F' \mid x_T = x\}$ is downward cofinal in F', then for this x, and $F'' = \{T \in F' \mid x_T = x\}$, the condition of Definition 3 is trivially satisfied: for $T \in F''$ we have $x - x_T = 0$, which belongs to R^* for every $R \in F$.

If there is no such "frequently occurring" value, then I claim we can use $F'' = F'$ and $x = 0$. Indeed, again writing R where the definition of strong convergence refers to a set $S \in F$ (since we already have a set we are calling S), note that for every $R \in F$ we have $\mathrm{card}(S^* - R^*) < \kappa$; and for each $s \in S^* - R^*$, the fact that s does not occur "frequently" among the x_T tells us that we can find $T_s \in F'$ such that no x_T with $T \subseteq T_s$ and $T \in F'$ is equal to s. If we let T_0 be the intersection of these T_s over all $s \in S^* - R^*$, then by regularity of the cardinal κ, we have $T_0 \in F$, hence by downward cofinality of F' in F, the set F' contains some $T_R \subseteq T_0$. For all $T \subseteq T_R$ in F', we have $x_T \in R^*$, completing the proof that $(x_T)_{T \in F'}$ converges strongly to 0 with respect to F. □

It is strange that the above proof makes essentially no use of the operation of G. The role played by F in the definition of strong convergence is used nontrivially only for convergence to 0, while the criterion by which we obtain convergence to other points is the fact that a system of elements with constant value x converges strongly to x. (This involves one small use of the operation of $G : x - x = 0 \in S^*$.) However, in our application of the above result, it will be combined with standard facts about how a group topology behaves with respect to the group operation.

3. Our main result. We shall now prove that for F a self-indulgent family, and $\mathcal{T}_F$ the topology it determines, we have (5) $\Longleftrightarrow$ (6). We know that (5) $\Longrightarrow$ (6) by Corollary 1. The plan of our proof of the converse will be to show that, given $g \in G - \{0\}$ which we want to exclude from the intersection in (5), we can build up, in a recursive manner, a sequence $S_0, S_1, \ldots$ with $g \notin U(S_0, S_1, \ldots)$. The recursive step is given by the next lemma. (The corresponding recursive step in the proof of [6, Theorem 2.1.4] uses an "either/or" argument at each substep. These were collapsed here into the single either/or argument in the above proof that co-κ filters on subsets of G are self-indulgent.)

Lemma 2. *Let F be a self-indulgent downward directed system of subsets of G satisfying* (6)*. Suppose $g \in G - \{0\}$, and that for some $n \geq 0$, $S_0, \ldots, S_{n-1}$ are members of F such that*

$$g \notin S_0^* + \cdots + S_{n-1}^*. \tag{7}$$

Then there exists $S_n \in F$ such that

$$g \notin S_0^* + \cdots + S_{n-1}^* + S_n^*. \tag{8}$$

Proof. Assume the contrary. Then for each $T \in F$, the fact that (8) does not hold with $S_n = T$ shows that we may choose $n+1$ elements,

$$g_{0,T} \in S_0^*, \quad \ldots, \quad g_{n-1,T} \in S_{n-1}^*, \quad g_{n,T} \in T^* \tag{9}$$

such that

$$g \;=\; g_{0,T} + \cdots + g_{n-1,T} + g_{n,T}. \tag{10}$$

Assuming for the moment that $n > 0$, let us focus on the first term on the right-hand side of (10), and apply the assumption that F is self-indulgent to the family of elements $g_{0,T} \in S_0^*$, as T ranges over F. This tells us that we can find a $g_0 \in S_0^*$ and a downward cofinal subset $F_0 \subseteq F$ such that

$$(g_{0,T})_{T\in F_0} \text{ converges strongly to } g_0 \text{ with respect to } F. \tag{11}$$

If $n > 1$, we then go through the same process for the values $g_{1,T} \in S_1^*$, as T ranges over the above downward cofinal subset $F_0 \subseteq F$. By the self-indulgence of F, we can find $g_1 \in S_1^*$ and a downward cofinal subset F_1 of F_0, such that

$$(g_{1,T})_{T\in F_1} \text{ converges strongly to } g_1 \text{ with respect to } F. \tag{12}$$

We continue this way, through the construction of g_{n-1} and F_{n-1}. At the next step, we simply set $F_n = F_{n-1}$ (or if $n = 0$, $F_n = F$), and $g_n = 0$, since the assumption $g_{n,T} \in T^*$ in (9) says that the family $(g_{n,T})_{T\in F}$ already converges strongly to 0, whence the same holds when we restrict the index T to the cofinal subset $F_{n-1} \subseteq F$.

Now since $g_i \in S_i^*$ for $i < n$, while $g_n = 0$, we have $g_0 + \cdots + g_n \in S_1^* + \cdots + S_{n-1}^*$, so by (7), $g \neq g_0 + \cdots + g_n$. Letting $g' = g - (g_0 + \cdots + g_n) \neq 0$, condition (10) becomes

$$g' \;=\; (g_{0,T} - g_0) + \cdots + (g_{n,T} - g_n) \text{ for all } T \in F_n. \tag{13}$$

We now apply our hypothesis that F satisfies (6). Since $g' \neq 0$, this says there is some $S \in F$ such that

$$g' \notin (n+1)S^*. \tag{14}$$

But since for each i, the system $(g_{i,T} - g_i)_{T\in F_n}$ converges strongly to 0, we can find $T \in F_n$ such that each element $g_{i,T} - g_i$ $(0 \leq i \leq n)$ lies in S^*. Thus, (13) contradicts (14), and this contradiction completes the proof of the lemma. □

We deduce

Theorem 1 (cf. [6, Theorem 2.1.4]). *Let F be a self-indulgent downward directed system of subsets of an abelian group G. (In particular, by Lemma 1, for any infinite $X \subseteq G$ and any $\kappa \leq \mathrm{card}(X)$, such an F is given by the co-κ filter on X.) Then the finest group topology on G under which F converges to 0 is Hausdorff if and only if F satisfies (6).*

Proof. By Corollary 1, (6) is necessary for our topology to be Hausdorff. Conversely, assuming (6), we can use Lemma 2 recursively to build up, for any $g \in G - \{0\}$, a sequence $S_0, S_1, \ldots$ of members of F, starting with the vacuous sequence, such that for all n, $g \notin \sum_{i<n} S_i^*$. Thus, $g \notin U(S_0, S_1, \ldots)$, giving (5), which is equivalent to our topology being Hausdorff. □

One may ask whether allowing co-κ filters with κ strictly less than $\mathrm{card}(X)$ provides any useful examples. Such a filter only "scratches the surface" of X, so it might seem implausible that it could converge to 0 in a group topology. But in fact, if G is the group $\mathbb{Z}^I$ for an uncountable set I, under the product topology, and X the set of elements of G which have value 1 at a single point, and 0 everywhere else, then we see that the cofinite (i.e., co-$\aleph_0$) filter determined by X does converge to 0 in G.

4. Some counterexamples. Before giving the rather complicated example showing that Theorem 1 fails if the assumption that F is self-indulgent is removed, let us note a couple of easier examples of things that go wrong in the absence of self-indulgence.

Example 1. An abelian group G with an element g, a downward directed family F of subsets, and a sequence $S_0, \ldots, S_{n-1} \in F$ satisfying (7), which cannot, as in Lemma 2, be extended so as to satisfy (8).

Construction and proof. Let G be the additive group of the real line, F the set of neighborhoods $(-\varepsilon, \varepsilon)$ of 0 ($\varepsilon > 0$), and $g = 1 \in G$. Then the 1-term sequence given by $S_0 = (-1, 1)$ satisfies $g \notin S_0^*$, but cannot be extended to a 2-term sequence with $g \notin S_0^* + S_1^*$. $\square$

Indeed, whenever, as in the above example, F consists of neighborhoods of the identity in the topology we are constructing, then the conclusion of Lemma 2 implies that $S_0^* + \cdots + S_{n-1}^*$ is closed in that topology. So if, starting with a topological group G, we take a basis F of neighborhoods of 0 not all of which are closed sets, the conclusion of that lemma must fail.

Getting closer to our main example, we give

Example 2. An abelian group G and a downward directed family F of subsets of G such that the union in (6) is a proper subgroup of G, but the intersection in (5) is all of G.

Construction and proof. Let G be the countable direct product group $\prod_{n>0} \mathbb{Z}/n\mathbb{Z}$, and for each positive integer m, let $S(m) \subseteq G$ consist of all elements whose first through m-th coordinates lie in $\{1, 0, -1\}$, the remaining coordinates being unrestricted. Thus, $S(1) \supseteq S(2) \supseteq \ldots$, so $F = \{S(m)\}$ is downward directed. (These sets satisfy $S(m)^* = S(m)$, but I will write $S(m)^*$ below when the conditions we want to verify refer to sets S^*.)

To show that the intersection in (5) is all of G, we will in fact show that for any $m_0, m_1, \ldots$, we have

$$S(m_0)^* + \cdots + S(m_{m_0})^* \ = \ G. \tag{15}$$

Indeed, let $g \in G$. To describe the summands comprising an expression for g as a member of $S(m_0)^* + \cdots + S(m_{m_0})^*$, we shall begin by describing their first m_0 coordinates (in $\mathbb{Z}/1\mathbb{Z}, \ldots, \mathbb{Z}/m_0\mathbb{Z}$), then describe their remaining coordinates. We take the former coordinates all to lie in $\{1, 0, -1\}$, and to be chosen so that for each $i \leq m_0$, the i-th coordinates of these m_0 elements sum to the i-th coordinate of g. This is possible because the relevant coordinates of g are members of groups $\mathbb{Z}/n\mathbb{Z}$ with $n \leq m_0$.

We then choose the coordinates after the m_0-th by taking these coordinates of the summand in $S(m_0)$ to agree with those of g, and those in the other summands to be zero. It is easy to see that the elements we have constructed belong to the desired $S(m_i)^*$ and sum to g.

On the other hand, consider any g in the union in (6). Say it lies in the member of that union indexed by $n \in \omega$. Thus, for every m, g lies in $n\, S(m)^*$; i.e., for every m, the first m coordinates of g are sums of n terms in $\{1, 0, -1\}$; in other words, g is the image in $\prod_m \mathbb{Z}/m\mathbb{Z}$ of an element of $\prod_m \mathbb{Z}$ the absolute values of whose coordinates admit a common bound n. Such elements clearly form a proper subgroup of G. $\square$

Finally, here is our example showing that in the absence of self-indulgence, Theorem 1 fails. In the development below, where we use square roots of 7 modulo powers of 3, we could, more generally, replace 3 by any prime p, take any invertible irrational element α of the ring $\mathbb{Z}_p$ of p-adic integers, and look at the images of α, 0, $-\alpha \in \mathbb{Z}_p$ in the rings $\mathbb{Z}/p^k\mathbb{Z}$. But the choice of a quadratic irrationality makes the presentation a little simpler.

Example 3. A countable, downward directed family F of subsets of $\mathbb{Z}$ for which (6) holds, but (5) does not.

Construction and proof. For each integer $k>0$, let

$$S(k) \;=\; \{x\in\mathbb{Z} \mid \text{the image of } x \text{ in } \mathbb{Z}/3^k\mathbb{Z} \text{ is either } 0\text{, or a square root of 7 in that ring}\}. \tag{16}$$

Since $S(1)\supseteq S(2)\supseteq\dots$, the set $F=\{S(k)\}$ is downward directed. To show that (6) holds, let g be any nonzero member of $\mathbb{Z}$, and n any positive integer. Choose a positive integer k large enough so that

$$3^k \text{ does not divide any of the } n+1 \text{ nonzero integers } g^2-7\,m^2 \text{ with } 0\le m\le n. \tag{17}$$

(E.g., take any k such that $3^k>\max(g^2,\,7n^2)$.) Then I claim that $g\notin nS(k)$.

Indeed, suppose we had

$$g \;=\; g_0+\dots+g_{n-1}, \text{ with all } g_i\in S(k). \tag{18}$$

If we let c denote a square root of 7 in $\mathbb{Z}/3^k\,\mathbb{Z}$, (which exists, by Hensel's Lemma [4, Theorem 3.4.1], and is unique up to sign), then by (16), each of the g_i in (18) has residue modulo 3^k either 0, c, or $-c$. Hence (18) implies that the residue of g modulo 3^k has the form $m\,c$ for some integer m of absolute value $\le n$. Squaring, we conclude that $g^2\equiv 7\,m^2 \pmod{3^k}$, contradicting (17). So (18) fails for all $g\neq 0$, establishing (6).

To show that (5) does not hold, consider any sequence $S(m_0)$, $S(m_1)$, $\dots$ of elements of F, determined by nonnegative integers $m_0, m_1,\dots$. I claim that $U(S(m_0),S(m_1),\dots)=\mathbb{Z}$; in fact, that

$$S(m_0)^*+S(m_1)^*+\,\dots\,+S(m_{3^{m_0}})^* \;=\; \mathbb{Z}. \tag{19}$$

For let c be a square root of 7 in $\mathbb{Z}/3^{m_0}\,\mathbb{Z}$. I claim that every $S(m_i)$ contains an integer c_i whose residue modulo 3^{m_0} is c. For if $m_i\le m_0$, then $S(m_i)$ contains $S(m_0)$, and so contains every integer whose residue class modulo 3^{m_0} is c, while if $m_i\ge m_0$, then the residue class c in $\mathbb{Z}/3^{m_0}\mathbb{Z}$ can be lifted to a square root of 7 in $\mathbb{Z}/3^{m_i}\mathbb{Z}$ (cf. proof of Hensel's Lemma), a representative of which in $\mathbb{Z}$ will be the desired c_i.

For any $g\in\mathbb{Z}$, the element $g/c\in\mathbb{Z}/3^{m_0}\mathbb{Z}$ is the residue of an integer h satisfying

$$0\le h<3^{m_0} \text{ and } h\,c \text{ is the residue of } g \text{ in } \mathbb{Z}/3^{m_0}\mathbb{Z}. \tag{20}$$

Given such h, let us choose elements $g_i\in S(m_i)$ for $i\in\{1,\dots,3^{m_0}\}$ such that for exactly h values of i, g_i is the element c_i chosen in the preceding paragraph, while for the remaining values, $g_i=0$. Then the sum $g_1+\dots+g_{3^{m_0}}$ has residue $h\,c$ in $\mathbb{Z}/3^{m_0}\mathbb{Z}$, which by (20) is the residue of g. On the other hand, $S(m_0)$ contains all multiples of 3^{m_0} (see (16)), so by choosing $g_0\in S(m_0)$ to be an appropriate one of these, we can get exact equality,

$$g \;=\; g_0+g_1+\dots+g_{3^{m_0}}, \tag{21}$$

as required to establish (19), and hence falsify (5). □

One can get similar examples by replacing the group of 3-adic integers implicit in the above construction with other examples of a topological group K containing a subgroup G and a dense cyclic subgroup H having trivial intersection. (In the above example, $K = \mathbb{Z}_3$ (the group of 3-adic integers), $G = \mathbb{Z}$, and $H = \sqrt{7}\,\mathbb{Z}$.) For instance, one can take $K = \mathbb{R}/\mathbb{Z}$, let G be its dense subgroup $\mathbb{Q}/\mathbb{Z}$, and let H be the subgroup generated by the image β of an irrational $b \in \mathbb{R}$. Letting F consist of the intersections of G with a family of neighborhoods of $\{\beta, 0, -\beta\} \subseteq K$ under the usual topology, one gets the same sort of behavior as in Example 3.

5. Remarks on self-indulgent sets. Though the concept of a self-indulgent set of subsets of G has proved useful, it is not clear that we have formulated the best version of it. Originally, I thought it would be enough to require that for every family $(x_T)_{T\in F}$ there should exist a cofinal subset $F' \subseteq F$ making $(x_T)_{T\in F'}$ converge strongly to x: I thought this would imply the condition now used, that for every such family indexed by a cofinal subset $F' \subseteq F$, one can get strong convergence on a smaller cofinal subset $F'' \subseteq F'$. But I was unable to prove this.

Before settling on the present fix for that problem, I considered other possibilities. For instance, instead of looking at cofinal subsets of F, one might look at isotone maps f of arbitrary downward directed posets I into F, having downward cofinal images. Convergence of the system $(c_{f(i)})_{i\in I}$ with respect to the ordering on I would be a weaker condition than convergence with respect to the ordering on the image set $f(I)$. But if we require this for all such I, we have, in particular, the case $I = F$, giving the condition we have used.

One may also ask whether examples can be found of self-indulgent families essentially different from our co-κ filters. The answer is, "Yes, but …". The lemma below gives such examples, but they require knowing in advance the topology one is aiming at, so they are of no evident use in getting new applications of Theorem 1.

Lemma 3. *Let G be a locally compact Hausdorff topological abelian group, and let F be the set of all compact neighborhoods of* 0 *in G (or any downward cofinal subset thereof). Then F is self-indulgent.*

Proof. Because G is locally compact, F is a neighborhood basis of 0 in G, so strong convergence with respect to F is equivalent to convergence.

Now for all $S \in F$, compactness of S^* implies that every system of points indexed by a directed set has a cofinal subsystem which converges to a point of S^*; so in particular, we have the cases of this condition required by the definition of self-indulgence. □

One may ask whether for F a self-indulgent family that yields a Hausdorff topology on a group G, the members of F must become compact under that topology. The difficulty, when one tries to prove this, is that the self-indulgence condition only applies to families of points indexed by cofinal subsets of F, while compactness would require a like condition for families indexed by arbitrary directed sets. In a similar vein, I. V. Protasov (personal communication) has asked whether under a topology so induced, the group G must be complete. I do not know the answer.

6. The nonabelian case. Let us now drop the assumption that G is abelian, and see how the statement and proof of Theorem 1 can be adapted to this situation. Thus, in this section,

$$G \text{ is a not necessarily abelian group, written multiplicatively.} \tag{22}$$

In particular, we shall denote the identity element of G by e, and for $S \subseteq G$ write

$$S^* \ = \ S \cup \{e\} \cup S^{-1}. \tag{23}$$

In [6, §3.1-§3.2] I. Protasov and E. Zelenyuk likewise generalize their results to noncommutative groups. (Cf. also [7, §1.3].) As the analog of the sums $\sum_{i<n} S_i^*$ of (3), they use the union, over all permutations of the index set n, of the corresponding permuted product of the S_i^* (and then, as in (3), take the union of this over all n).

We will take a different approach here. Let us first note that it will not work to simply replicate the definition (3) with sums $S_0^* + S_1^* + \dots + S_{n-1}^*$ replaced by (unpermuted) products $S_0^*\, S_1^* \, \dots \, S_{n-1}^*$. The trouble is that we cannot say that a set of the form $\bigcup_{n \in \omega} S_0^*\, S_1^* \, \dots \, S_{n-1}^*$ will contain the product of two sets of that same form; essentially because ω does not contain a union of two successive copies of itself as an ordered set.

So let us use an index set which does. Let

$$Q \ = \text{a totally ordered set of the order-type of the rational numbers.} \tag{24}$$

(We do not call this $\mathbb{Q}$ because we are not interested in its algebraic structure, but only in its order-type. In fact, in our one explicit calculation, in the proof of Lemma 5, a different realization of this order-type will be used.)

Given any Q-tuple $(S_q)_{q \in Q}$ of subsets of G, let

$$U((S_q)_{q \in Q}) \ = \ \bigcup_{q_0 < \dots < q_n \in Q} S_{q_0}^* \dots S_{q_n}^*, \tag{25}$$

where the union is over all finite increasing sequences in Q. The sets (25) have the property which we just noted that ω-indexed products lack; namely, it is easy to see

Lemma 4. *Let $(S_q)_{q \in Q}$ be a family of subsets of G, and let $\sigma, \tau : Q \to Q$ be two order-embeddings such that $\sigma(q) < \tau(q')$ for all $q, q' \in Q$. Then*

$$U((S_{\sigma(q)})_{q \in Q})\, U((S_{\tau(q)})_{q \in Q}) \ \subseteq \ U((S_q)_{q \in Q}). \qquad \square \tag{26}$$

The next result shows that sets of the form $U((S_q)_{q \in Q})$ can be made small enough to do what we will need.

Lemma 5. *If $\mathcal{T}$ is a group topology on G, and S a neighborhood of e under $\mathcal{T}$, then one can choose for each $q \in Q$ a neighborhood S_q of e under $\mathcal{T}$ so that $U((S_q)_{q \in Q}) \subseteq S$.*

Proof. (Cf. [6, proof of Lemma 3.1.1].) Let $T_0 = S$, and choose recursively for each $i > 0$ a neighborhood T_i of e in $\mathcal{T}$ so that $T_i\, T_i\, T_i \subseteq T_{i-1}$. Identify Q as an ordered set with the set of those rational numbers in the unit interval $(0, 1)$ of the form $m/2^i$, and

$$\text{for each } q = m/2^i, \text{ written in lowest terms, let } S_q = T_i. \tag{27}$$

Then I claim that $U((S_q)_{q \in Q}) \subseteq S$.

To show this, it suffices to show that for all finite sequences $q_0 < \dots < q_n \in Q$ we have $S_{q_0}^* \dots S_{q_n}^* \subseteq S$. If we take a common denominator 2^j for all members of such a finite sequence, then by enlarging the sequence we can assume without loss of generality that $\{q_0, \dots, q_n\}$ is the whole set

$$\{m/2^j \ \mid \ 0 < m < 2^j\}. \tag{28}$$

Let us now enlarge the finite product of sets S_q determined by (27) and (28) still further, by changing those factors whose index q has the largest possible denominator, 2^j, from $S_q = T_j$ to the larger set T_{j-1}. (This will help in an induction to come.)

If we now classify the elements of (28) into those which, expressed in lowest terms, have denominator 2^j, those having denominator 2^{j-1}, and those with smaller denominators, we see that each term with denominator 2^{j-1} is flanked on each side by terms with denominator 2^j, and that the resulting 3-term strings of indices with denominators 2^j, 2^{j-1}, 2^j are disjoint. In the modified product of subsets of G that we have described, the factors corresponding to these strings of three terms have the form $T_{j-1}\,T_{j-1}\,T_{j-1}$. By assumption, this product is contained in T_{j-2}. Replacing each product $T_{j-1}\,T_{j-1}\,T_{j-1}$ with the possibly larger set T_{j-2}, we conclude that our product of subsets is contained in a product of the same form, but with subscripts now running not over (28) but over the elements of Q with denominator $\leq 2^{j-1}$. Here the qualifier "of the same form" includes the condition that elements q with largest possible denominator, now 2^{j-1}, are assigned the set T_{j-2} rather than T_{j-1}.

Iterating this reduction, we conclude that our product is contained in one with the single index element $1/2^1$, which is assigned the set $T_{1-1} = T_0 = S$, giving the desired inclusion. $\square$

(Tangential observation: The set Q used in the above proof has a natural order-isomorphism with the set of intervals deleted in the "middle third" construction of the Cantor set (arranged from left to right); and if we think of the relation $T_i\,T_i\,T_i \subseteq T_{i-1}$ in the above proof intuitively as saying that T_i has one-third the "weight" of T_{i-1}, then the weights of these sets can be taken to agree with the lengths of those deleted intervals. Thus, the above proof is related to the fact that the total length of those deleted intervals is 1.)

In studying the finest group topology under which a given downward directed set F converges to e, it will be convenient to require that F be closed under conjugation by elements of G; i.e., that for every $S \in F$ and $g \in G$ we have $g\,S\,g^{-1} \in F$. If, given an F not satisfying this condition, we simply replaced it with $\{g\,S\,g^{-1} \mid S \in F,\ g \in G\}$, we could lose downward directedness. On the other hand, if we passed to the sets $\bigcup_{g\in G} S^g$ $(S \in F)$, these could be much too large. (A group topology on G need not be generated by G-invariant neighborhoods of e.) The construction of the next lemma gives what we really need.

Lemma 6. *Let F be a downward directed family of nonempty subsets of G, and (following [6, Definition 3.1.6]) let us write F^G for the set of all subsets of G of the form $\bigcup_{g\in G} g\,S_g\,g^{-1}$, for G-tuples $(S_g)_{g\in G}$ of members of F.*

Then F^G is again a downward directed family of nonempty subsets of G, it is invariant under conjugation by elements of G, and for every group topology $\mathcal{T}$ on G, the family F^G converges to e under $\mathcal{T}$ if and only if F does.

Proof. That F^G is downward directed is easily seen to follow from the fact that F is; and the set F^G is conjugation invariant by construction. From the fact that each set $\bigcup_{g\in G} g\,S_g\,g^{-1} \in F^G$ contains a member of F, namely S_e, it follows that if F^G converges to e under $\mathcal{T}$ (i.e., if it has members contained in every $\mathcal{T}$-neighborhood of e), then so does F.

Now suppose, conversely, that F converges to e under $\mathcal{T}$, and let S be any neighborhood of e in $\mathcal{T}$. For each $g \in G$, the set $g^{-1}S\,g$ is also a neighborhood of e, hence contains some member of F, which we may denote S_g; thus S contains $g\,S_g\,g^{-1}$. Hence S will contain $\bigcup_{g\in G} g\,S_g\,g^{-1} \in F^G$; so F^G also converges to e, as required. $\square$

Restricting attention to conjugation-invariant families F, we can now get the analog of (4).

Proposition 1 (cf. [6, Theorem 3.1.4], [7, Theorem 1.17])**.** *Let F be a downward directed family of nonempty subsets of G, which is closed under conjugation by members of G. Then the sets $U((S_q)_{q\in Q})$ defined by (25), where $(S_q)_{q\in Q}$ ranges over all Q-tuples of members of F, form a basis of open neighborhoods of e in a group topology $\mathcal{T}_F$ on G, which is the finest group topology under which F converges to e.*

Proof. It is easy to see that the family of sets $U((S_q)_{q\in Q})$ is downward directed and closed under conjugation by elements of G (because F is), is closed under inverses (since for each $q \in Q$, $(S_q^*)^{-1} = S_q^*$, hence if we let $\sigma : Q \to Q$ be an order-antiautomorphism, we get $U((S_q)_{q\in Q})^{-1} = U((S_{\sigma(q)})_{q\in Q}),$) and has the property that each member of the family contains a product of two other members (by Lemma 4).

To conclude that these sets give a basis of *open* neighborhoods of e in a group topology on G, it remains to show that for every set $U((S_q)_{q\in Q})$ and element $x \in U((S_q)_{q\in Q})$, there exists a set $U((T_q)_{q\in Q})$ with

$$x\,U((T_q)_{q\in Q}) \ \subseteq\ U((S_q)_{q\in Q}). \tag{29}$$

To see that this holds, note that by (25), $x \in U((S_q)_{q\in Q})$ lies in a finite product $S_{q_0}^* \dots S_{q_n}^*$ with $q_0 < \dots < q_n \in Q$. Now $\{q \in Q \mid q > q_n\}$ is order-isomorphic to Q; let us write it $\tau(Q)$ where $\tau : Q \to Q$ is an order embedding. Then letting $T_q = S_{\tau(q)}$, we get (29).

So our sets give a basis of open sets for a group topology $\mathcal{T}_F$. Moreover, F converges to e in this topology, since each $U((S_q)_{q\in Q})$ contains members of F; indeed, contains each of the S_q.

To show that $\mathcal{T}_F$ is the finest group topology on G under which F converges to e, suppose $\mathcal{T}$ is any such topology. For every open neighborhood S of e in $\mathcal{T}$, Lemma 5 gives us a set of the form $U((S'_q)_{q\in Q})$ contained in S, with each S'_q an open neighborhood of e under $\mathcal{T}$. By the assumption that F converges to e under $\mathcal{T}$, each S'_q contains some $S_q \in F$, hence $U((S_q)_{q\in Q}) \subseteq U((S'_q)_{q\in Q}) \subseteq S$ is a neighborhood of e under $\mathcal{T}_F$ contained in S; so $\mathcal{T}_F$ is at least as fine as $\mathcal{T}$. □

We have thus generalized to nonabelian groups G the concepts and results on abelian G quoted in §1 as (1)–(4). The definitions and results that immediately followed these (the remaining material in §§1–2) go over to the nonabelian case with minimal change. Indeed, the argument that gave us Corollary 1, applied to Proposition 1, gives

Corollary 2. *If F is a conjugation-invariant downward directed family of subsets of G, then* a necessary *condition for the topology $\mathcal{T}_F$ to be Hausdorff is*

$\bigcup_{n>0} \bigcap_{S\in F} (S^*)^n \ =\ \{e\}$. *In other words, for every $g \in G - \{e\}$ and every $n > 0$, there exists $S \in F$ with $g \notin (S^*)^n$.* □ (30)

The analogs of Definitions 2 and 3 are

Definition 4. If F is a downward directed family of subsets of G, and $(x_i)_{i\in I}$ a family of elements of G indexed by a downward directed partially ordered set I, we shall say that (x_i) *converges strongly* to an element $x \in G$ with respect to F if for every $S \in F$, there exists $i \in I$ such that for all $j \leq i$, $x_j\, x^{-1} \in S^*$.

A downward directed family F of subsets of G will be called *self-indulgent* if for every $S \in F$, and every family $(x_T)_{T\in F'}$ of elements of S^* indexed by a downward cofinal subset $F' \subseteq F$, there exist an $x \in S^*$ and a downward cofinal subset $F'' \subseteq F'$ such that $(x_T)_{T\in F''}$ converges strongly to x with respect to F.

(The above definition of strong convergence is not right-left symmetric, since it uses $x_j\,x^{-1}$ rather than $x^{-1}x_j$. However, the family $(x_j\,x^{-1})_{j\in I}$ is conjugate, by x, to $(x^{-1}\,x_j)_{j\in I}$, hence when F is closed under conjugation by members of G, the condition becomes symmetric.)

The proof that co-κ filters are self-indulgent also goes over with no change. (Recall that the proof made essentially no use of the group operation.) We state this below, along with another fact, immediate to verify, that we will need.

Lemma 7. *Let X be any infinite subset of G, and κ any regular cardinal $\leq \mathrm{card}(X)$. Then the co-κ filter F on X is self-indulgent.*

Moreover, if X is invariant under conjugation by elements of G, then that filter F is likewise closed under conjugation by elements of G. □

We now come to the analogs of the material of §3. A little care is needed in generalizing Lemma 2, though the ideas are the same.

Lemma 8. *Let F be a self-indulgent downward directed system of subsets of G which is closed under conjugation by members of G, and satisfies* (30). *Let $g\in G$, and suppose that for some $n\geq 0$ and $0\leq m\leq n$, $S_0,\dots,S_{m-1},\ S_{m+1},\dots,S_n$ are members of F such that*

$$g\notin S_0^*\dots S_{m-1}^*\,S_{m+1}^*\dots S_n^*. \tag{31}$$

Then there exists $S_m\in F$ such that

$$g\notin S_0^*\dots S_{m-1}^*\,S_m^*\ S_{m+1}^*\dots S_n^*. \tag{32}$$

Proof. As before, the contrary assumption says that for each $T\in F$, we can choose $n+1$ elements

$$g_{0,T}\in S_0^*,\quad \dots,\quad g_{m-1,T}\in S_{m-1}^*,\quad g_{m,T}\in T^*,\quad g_{m+1,T}\in S_{m+1}^*,\quad \dots,\quad g_{n,T}\ \in\ S_n^* \tag{33}$$

(note how the m-th condition differs from the others), such that

$$g\ =\ g_{0,T}\ \cdots\ g_{m-1,T}\ \ g_{m,T}\ \ g_{m+1,T}\ \cdots\ g_{n,T}. \tag{34}$$

(However, from this point on, in writing expressions like the above we will omit the terms indexed by $m-1$ and $m+1$, and only show those indexed by 0, m and n.)

Making n successive applications of our self-indulgence assumption on F (we did these from left to right in proving Lemma 2; but the order makes no difference), we can get elements g_i $(0\leq i\leq n)$ such that

$$\text{for } i\neq m,\ g_i\in S_i^*,\ \text{while } g_m=e, \tag{35}$$

and a cofinal subfamily $F'\subseteq F$, such that for each i, the family $(g_{i,T})_{T\in F'}$ converges strongly to g_i with respect to F. Defining

$$g'_{i,T}\ =\ g_{i,T}\,g_i^{-1}, \tag{36}$$

we conclude that

$$\text{for each } i\in\{0,\dots,n\}, \text{ the family of elements } (g'_{i,T})_{T\in F'} \text{ converges strongly to } e \text{ with respect to } F. \tag{37}$$

Now (31) and (35) imply that $g \neq g_0 \ldots g_m \ldots g_n$, so let us write

$$g' \;=\; g \cdot (g_0 \,\ldots\, g_m \,\ldots\, g_n)^{-1} \;\neq\; e. \tag{38}$$

Since F satisfies (30), we can find $S \in F$ such that

$$g' \;\notin\; (S^*)^{n+1}. \tag{39}$$

On the other hand, note that if in the right-hand side of (38) we expand the initial factor g using (34), and then use (36) to rewrite each of the factors $g_{i,T}$ from (34) as $g'_{i,T}\, g_i$, we get

$$g' \;=\; (g'_{0,T}\, g_0) \,\ldots\, (g'_{m,T}\, g_m) \,\ldots\, (g'_{n,T}\, g_n)\, (g_0 \,\ldots\, g_m \,\ldots\, g_n)^{-1} \text{ for all } T \in F'. \tag{40}$$

Letting $h_i = g_0 \ldots g_{i-1}$ for $0 \leq i \leq n$, this becomes

$$g' \;=\; (h_0\, g'_{0,T}\, h_0^{-1}) \,\ldots\, (h_m\, g'_{m,T}\, h_m^{-1}) \,\ldots\, (h_n\, g'_{n,T}\, h_n^{-1}) \text{ for all } T \in F'. \tag{41}$$

From the facts that the $g'_{i,T}$ all converge strongly to e with respect to F, and that F is closed under conjugation by members of G, it follows that in (41), each of the factors $h_i\, g'_{i,T}\, h_i^{-1}$ converges strongly to e. Hence for some $T \in F'$, all the factors of (41) lie in the S^* of (39). That instance of (41) therefore contradicts (39), proving the lemma. □

Given F as in the above lemma, and any $g \in G - \{e\}$, we can use that lemma to build up, by recursion with respect to any enumeration of Q by the natural numbers, a system $(S_q)_{q \in Q}$ such that $g \notin U(S_q)_{q \in Q}$. We deduce

Theorem 2. *Let F be a downward directed system of subsets of G which is self-indulgent, and closed under conjugation by all elements of G.* (*In particular, by Lemma 7 this is true if for some conjugation-invariant $X \subseteq G$ and some $\kappa \leq \operatorname{card}(X)$, F is the co-κ filter on X.*) *Then the finest group topology on G under which F converges to e is Hausdorff if and only if F satisfies* (30). □

It is not clear to me how closely related this is to the nearest result in [6], Theorem 3.2.1. That result is restricted to countable groups G, but concerns the finest group topology under which a general sequence (equivalently, the cofinite (i.e., co-$\aleph_0$) filter on a general subset, *not* necessarily conjugation invariant) converges. The criterion given for that topology to be Hausdorff uses, in place of the n-fold products implicit in (30), arbitrary group words $f(x_0, \ldots, x_n)$ in $n+1$ variables, and constants from G, which satisfy $f(e, \ldots, e) = e$. These two sorts of expressions ultimately reduce to the same thing; but the quantification of the conditions is subtly different. Perhaps this is not surprising: (5) and (6) can also be looked at as similar conditions which involve different quantifications, but which become equivalent in the case of self-indulgent F.

In [6, §§3.3, 3.4], topologies on *rings* determined by families of subsets are similarly studied.

7. A Fibonacci connection. Many interesting applications are given in [6] of the criterion obtained there for the cofinite filter on a countable subset of an abelian group to converge to 0 in a Hausdorff group topology. In particular, it is shown that there exist such topologies on $\mathbb{Z}$ under which various integer sequences — for instance the Fibonacci sequence [6, Corollary 2.2.8] — converge to 0.

Note that in the free nonabelian group $G = \langle x, y \rangle$ on two generators, one can define a Fibonacci-like sequence by

$$f_0 = x, \quad f_1 = y, \quad f_{n+1} = f_{n-1} f_n \quad (n \in \mathbb{Z}). \tag{42}$$

I had hopes of proving that there was a Hausdorff group topology on $\langle x, y \rangle$ under which this sequence converged to e. However, if we define an automorphism φ of $\langle x, y \rangle$ by $\varphi(x) = y$, $\varphi(y) = xy$, then we see that in (42), $f_n = \varphi^n(x)$; so the result I hoped for would imply that every $g \in \langle x, y \rangle$ satisfied $\lim_{n\to\infty} \varphi^n(g) = e$. But calculation shows that the commutator $x\,y\,x^{-1}y^{-1}$ is fixed by φ^2; so this cannot be true. Indeed, there cannot even exist a Hausdorff group topology under which the sequence f_n approaches some fixed element c of G, or of a topological overgroup of G, since then we would have

$$\varphi^n(x\,y\,x^{-1}y^{-1}) = \varphi^n(x)\,\varphi^{n+1}(x)\,\varphi^n(x)^{-1}\varphi^{n+1}(x)^{-1} \to \ c\,c\,c^{-1}c^{-1} = e, \tag{43}$$

though as noted, the left-hand side has, for every even n, the value $x\,y\,x^{-1}y^{-1}$. However, I don't see any obstruction to there being a topological overgroup of G under which the values of f_{2n} and f_{2n+1} each approach constant values.

For another context in which the "Fibonacci automorphism" φ of $\langle x, y \rangle$ (there called $\sigma^{1/2}$) comes up, see [1].

8. Final remark, and acknowledgements. I do not know of interesting applications of the results of this note. My motivation has been structural: "What ideas underlie the arguments of [6]; and in what more general contexts are those ideas applicable?" Perhaps group theorists will find such applications.

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