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# THE RIESZ MEASURES AND A REPRESENTATION OF MULTIPLICATIVELY PERIODIC $\delta$-SUBHARMONIC FUNCTIONS IN A PUNCTURED EUCLIDEAN SPACE

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We describe the Riesz measures of multiplicatively periodic $\delta$-subharmonic functions in $\mathbb{R}^m\setminus\{0\}$, $m \geq 3$ and give their integral representations.

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Описываются меры Рисса мультипликативно периодических $\delta$-субгармонических в $\mathbb{R}^m\setminus\{0\}$, $m \geq 3$ функций. Найдены интегральные представления таких функций.

**1. Introduction.** Multiplicatively periodic (loxodromic) meromorphic functions in the punctured complex plane $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ are closely related to elliptic functions on $\mathbb{C}$ ([1]–[3]). Their natural extensions are multiplicatively periodic $\delta$-subharmonic functions in $\mathbb{C}^*$ which were studied in [4].

It was proved in [4] and [5] that each multiplicatively periodic subharmonic function in $\mathring{\mathbb{R}}^m = \mathbb{R}^m\setminus\{0\}$, $m \geq 2$, is constant.

In this paper we consider multiplicatively periodic $\delta$-subharmonic functions in $\mathring{\mathbb{R}}^m$, $m \geq 3$, that is, the differences $u = u_1 - u_2$ of two subharmonic functions $u_1$ and $u_2$ satisfying the condition $u(qx) = u(x)$ for some $q$, $0 < q < 1$, and all $x \in \mathring{\mathbb{R}}^m$.

The main problems are:

1. to describe the Riesz measures of multiplicatively periodic $\delta$-subharmonic in $\mathring{\mathbb{R}}^m$ functions;
2. to represent each multiplicatively periodic $\delta$-subharmonic in $\mathring{\mathbb{R}}^m$ function.

**2. The Riesz measures of multiplicatively periodic $\delta$-subharmonic functions.** Let $u$ be a subharmonic function in a domain. The positive measure

$$\mu_u = \frac{1}{c_m}\Delta u,$$

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where $c_m$ is the area of the unit sphere in $\mathbb{R}^m$, $\Delta$ is the Laplace operator in the sense of the Schwarz distributions, is called the Riesz measure of $u$ ([6]). For a $\delta$-subharmonic function $u$ the distribution $\frac{1}{c_m}\Delta u$ is the difference of positive measures.

Denote by $\mathcal{B}$ the class of bounded Borel sets in $\overset{\circ}{\mathbb{R}}{}^m$ whose closures are contained in $\overset{\circ}{\mathbb{R}}{}^m$. For $B \in \mathcal{B}$ put

$$qB = \{qx \colon x \in B\}, \quad 0 < q < 1.$$

**Definition 1.** Let $\mu$ be a measure on $\overset{\circ}{\mathbb{R}}{}^m$. Fix $t_0 > 0$ and a value $\nu(t_0)$. The function

$$\nu(t) = \begin{cases} \nu(t_0) + \mu\{x \colon t_0 < |x| \leq t\}, & t_0 < t, \\ \nu(t_0) - \mu\{x \colon 0 < t < |x| \leq t_0\}, & t < t_0, \end{cases}$$

is said to be the *distribution function of the measure* $\mu$ ([7]).

Such a function is right hand continuous, nondecreasing and determined up to a constant. The difference $\nu(t_2) - \nu(t_1)$ gives the measure of the ball layer $\{x \colon t_1 < |x| \leq t_2\}$.

For a $\delta$-subharmonic in $\overset{\circ}{\mathbb{R}}{}^m$ function $u$ denote

$$I(r) = \frac{1}{c_m r^{m-1}} \int_{S(0,r)} u(x) d\sigma(x),$$

where $S(0,r)$ is the sphere of radius $r$ centered at the origin.

**Lemma 1.** *Let $u$ be a $\delta$-subharmonic function in $\overset{\circ}{\mathbb{R}}{}^m$ and $\nu$ be the distribution function of $\mu_u$. Then $\nu(r) = \frac{r^{m-1}}{m-2} I'_+(r) + C$, where $I'_+(r)$ is the right hand derivative of $I(r)$, $C$ is a constant.*

*Proof.* It was proved in [7] that

$$\frac{m-2}{r_0^{2-m} - r^{2-m}} \int_{r_0}^{r} \frac{\nu(t)}{t^{m-1}} dt - \frac{m-2}{s^{2-m} - r_0^{2-m}} \int_{s}^{r_0} \frac{\nu(t)}{t^{m-1}} dt = \frac{I(r) - I(r_0)}{r_0^{2-m} - r^{2-m}} - \frac{I(r_0) - I(s)}{s^{2-m} - r_0^{2-m}}, \quad (1)$$

where $0 < s < r_0 < r < 1$.

Multiplying (1) by $(s^{2-m} - r_0^{2-m})$ and taking the right hand side derivative with respect to $s^{2-m}$, we obtain

$$\frac{m-2}{r_0^{2-m} - r^{2-m}} \int_{r_0}^{r} \frac{\nu(t)}{t^{m-1}} dt - \nu(s) = \frac{I(r) - I(r_0)}{r_0^{2-m} - r^{2-m}} - \frac{s^{m-1}}{m-2} I'_+(s). \quad (2)$$

Multiplying both sides of equality (2) by $(r_0^{2-m} - r^{2-m})$ and proceeding similarly, we deduce

$$\nu(s) - \nu(r) = \frac{s^{m-1}}{m-2} I'_+(s) - \frac{r^{m-1}}{m-2} I'_+(r),$$

which completes the proof. □

The next theorem solves the first problem.

**Theorem 1.** *A measure $\mu$ in $\overset{\circ}{\mathbb{R}}{}^m$ is the Riesz measure of a multiplicatively periodic $\delta$-subharmonic functions of multiplicator $q$ if and only if* *(i)* $\mu(qB) = q^{m-2}\mu(B)$ *for each* $B \in \mathcal{B}$*;*

*(ii)* $\int_{qr}^{r} \frac{d\nu(t)}{t^{m-2}} = 0$ *for all* $r > 0$*, where* $\nu(t)$ *is a distribution function of* $\mu$*.*

*Proof.* Let $u$ be a multiplicatively periodic $\delta$-subharmonic function of multiplicator $q$. Put $\varphi_q(x) = \varphi(qx)$. If $\varphi \in C_0^\infty(\overset{\circ}{\mathbb{R}}{}^m)$, then $\Delta\varphi_q(x) = q^2\Delta\varphi(qx)$. Substituting $x = \frac{y}{q}$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, we obtain

$$\int_{\overset{\circ}{\mathbb{R}}{}^m} u(x)\Delta\varphi_q(x)\, dx_1 \dots dx_m = \int_{\overset{\circ}{\mathbb{R}}{}^m} q^2 u\left(\frac{y}{q}\right) \Delta\varphi(y) \frac{dy_1 \dots dy_m}{q^m} =$$
$$= q^{2-m} \int_{\overset{\circ}{\mathbb{R}}{}^m} u(y)\Delta\varphi(y)\, dy_1 \dots dy_m$$

That is the distribution $T_u = \frac{1}{c_m}\Delta u$ has the property

$$T_u\varphi = q^{m-2}T_u\varphi_q. \tag{3}$$

If $\varphi(x) \neq 0$, $x \in B$, then $\varphi(qx) \neq 0$ , $x \in \frac{1}{q}B$. That is, if $\operatorname{supp}\varphi = K$, then $\operatorname{supp}\varphi_q = \frac{1}{q}K$. By the process of extension [6] of $T_u$ to the measure $\mu_u$ we have $\mu_u(\frac{1}{q}B) = q^{2-m}\mu_u(B)$ for each $B \in \mathcal{B}$. Taking $qB$ instead of $B$ in this equality we obtain *(i)*.

Now we are going to prove property *(ii)*. Let $\nu$ be the distribution function of $\mu_u$. Integrating by parts, we obtain

$$\int_s^r \frac{d\nu(t)}{t^{m-2}} = \nu(r)r^{2-m} - \nu(s)s^{2-m} + (m-2)\int_s^r \frac{\nu(t)}{t^{m-1}}dt. \tag{4}$$

Since the function $\nu(t)$ is determined up to a constant, the integral $\int_s^r \frac{d\nu(t)}{t^{m-2}}$ does not depend on this constant. Therefore, we can put $C = 0$ in Lemma 1. Then it implies

$$\nu(t)t^{2-m} = \frac{t}{m-2}I'_+(t), \tag{5}$$

$$(m-2)\int_s^r \frac{\nu(t)}{t^{m-1}}dt = I(r) - I(s). \tag{6}$$

Using equalities (5) and (6), we can rewrite (4) as follows

$$\int_s^r \frac{d\nu(t)}{t^{m-2}} = \frac{1}{m-2}\left(rI'_+(r) - sI'_+(s)\right) + I(r) - I(s).$$

If we put $s = qr$, then the previous equality can be rewritten in the form

$$\int_{qr}^r \frac{d\nu(t)}{t^{m-2}} = \frac{1}{m-2}(rI'_+(r) - qrI'_+(qr)) + I(r) - I(qr). \tag{7}$$

Since the function $u$ is multiplicatively periodic of multiplicator $q$, we have $I(qr) = I(r)$. Using also the equality $qI'_+(qr) = I'_+(r)$, we see that (7) implies *(ii)*.

Now let $\mu$ be a Borel measure in $\overset{\circ}{\mathbb{R}}{}^m$ satisfying properties *(i)* and *(ii)*, where $\nu$ is its distribution function. We are going to construct a multiplicatively periodic $\delta$-subharmonic function of multiplicator $q$ such that $\mu_u = \mu$.

Consider the function

$$K(x,a)=\sum_{n=0}^{+\infty}\Big(\frac{1}{|a|^{m-2}}-\frac{1}{|q^n x-a|^{m-2}}\Big)-\sum_{n=1}^{+\infty}\frac{1}{|\frac{x}{q^n}-a|^{m-2}},$$

where $x\in\overset{\circ}{\mathbb{R}}{}^m$, $q<|a|\le 1$.

It is easy to verify ([5]) that

$$K(qx,a)=K(x,a)-\frac{1}{|a|^{m-2}}. \tag{8}$$

We will show that

$$v(x)=\int_{q<|a|\le 1}K(x,a)d\mu_a \tag{9}$$

is multiplicatively periodic $\delta$-subharmonic function of multiplicator $q$.

The function $v(x)$ can be represented as follows

$$v(x)=\sum_{n=0}^{+\infty}\int_{q^{1-n}<|a|\le q^{-n}}\left(\frac{1}{|a|^{m-2}}-\frac{1}{|x-a|^{m-2}}\right)d\mu_a-\sum_{n=1}^{+\infty}\int_{q^{n+1}<|a|\le q^n}\frac{d\mu_a}{|x-a|^{m-2}} \tag{10}$$

due to property *(i)*. The function $v(x)$ is $\delta$-subharmonic in $\overset{\circ}{\mathbb{R}}{}^m$ as the sum of the Riesz potentials.

Using equality (8), we obtain $v(qx)=v(x)-\int_{q<|a|\le 1}\frac{d\mu_a}{|a|^{m-2}}$.

Then property *(ii)* implies

$$\int_{q<|a|\le 1}\frac{d\mu_a}{|a|^{m-2}}=\int_q^1\frac{d\nu(t)}{t^{m-2}}=0.$$

Thus, $v(qx)=v(x)$, $x\in\overset{\circ}{\mathbb{R}}{}^m$. □

**3. Representation of multiplicatively periodic $\delta$-subharmonic functions.** The following theorem solves the second problem.

**Theorem 2.** *Each multiplicatively periodic $\delta$-subharmonic in $\overset{\circ}{\mathbb{R}}{}^m$ function $u$ of multiplicator $q$ has the representation*

$$u(x)=C+\int_{q<|a|\le 1}K(x,a)d\mu_u(a),$$

*where $C$ is a constant.*

*Proof.* Let $u$ be a multiplicatively periodic $\delta$-subharmonic in $\overset{\circ}{\mathbb{R}}{}^m$ function of multiplicator $q$. Theorem 1 shows that $\mu_u$ satisfies conditions *(i)* and *(ii)*. Consider the function $v(x)$ given by (9) with $\mu=\mu_u$. It follows from representation (10) that $\mu_v=\mu_u$ since $v$ is the sum of uniformly convergent potentials of measure $\mu_u$. So the difference $h=u-v$ is a harmonic function. Since both $u$ and $v$ are multiplicatively periodic, the function $h$ is as well. Therefore, ([5]) the function $h$ is a constant. Hence, $u(x)=C+v(x)$, $x\in\overset{\circ}{\mathbb{R}}{}^m$, where $C$ is a constant. □

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