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ON A PROPERTY OF PAIRS OF ALMOST PERIODIC ZERO SETS

It was proved in [1] that each quasipolynomial
\[ Q(z) = \sum_{n=1}^{N} a_n e^{\lambda_n z}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C}, \quad (1) \]
with a discrete set of differences of its zeros is periodic up to a multiplier without zeros, consequently it has the form
\[ Q(z) = C e^{\beta z} \prod_{k=1}^{N} \cosh(\omega z + b_k), \quad \beta, \omega \in \mathbb{R}, \quad C, b_k \in \mathbb{C}. \quad (2) \]
The result is also valid for infinite sums
\[ S(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C}, \quad (3) \]
under conditions
\[ \sum_{n=1}^{\infty} |a_n| < \infty, \quad \lambda_1 = \sup_{n} \lambda_n < \infty, \quad \lambda_2 = \inf_{n} \lambda_n > -\infty, \quad a_1 a_2 \neq 0. \quad (4) \]
Note that zeros of functions (1), (3) are located in a vertical strip of a finite width.

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In 1949 M. G. Krein and B. Ja. Levin (see [5], also [6], Ch. 6, p. 2 and Appendix 6) introduced and studied the class $\Delta$ of entire almost periodic functions of exponential growth with zeros in a horizontal strip of a finite width. In particular, if the sum $S(z)$ in (3) satisfies condition (4), then $S(iz)$ belongs to $\Delta$. In fact, representation (2) was obtained in [1] for functions from $\Delta$ with a discrete sets of differences of zeros (of course, one should replace $e^{\beta z}$ by $e^{i\beta z}$ and $\cosh(\omega z + b_k)$ by $\cos(\omega z + b_k)$).

The similar phenomenon takes place for a pair of functions when the set of differences of their zeros is discrete ([3]). For example, let $Q_1, Q_2$ be entire functions of form (3) under conditions (4). If the set $\{z - w, Q_1(z) = 0, Q_2(w) = 0\}$ is discrete, then the both functions have form (2) with the same $\omega$ and possibly different $C, \beta, N, b_k$. In particular, in the case $Q_2(-z) = Q_1(z) = Q(z)$ we get representation (2) for any function (3) with the discrete set $\{z + z', Q(z) = Q(z') = 0\}$.

The proof of the above results are based on the property of zeros $\{z_j\}$ of functions $f \in \Delta$ ([6], Appendix 6, p. 2) to be almost periodic in the sense of the following definition.

Definition 1. A zero set $\{z_j\}$ is almost periodic if for any $\varepsilon > 0$ there is a relatively dense set $E_\varepsilon \subset \mathbb{R}$ such that for each $\tau \in E_\varepsilon$ there exists a bijection $\sigma: \mathbb{N} \to \mathbb{N}$ with the property

$$\sup_j |z_j + i\tau - z_{\sigma(j)}| < \varepsilon. \quad (5)$$

Recall that a set $E \subset \mathbb{R}$ is relatively dense if there exists $L < \infty$ such that $E \cap [a, a + L] \neq \emptyset$ for any $a \in \mathbb{R}$.

In a sense, the above definition is applied to zero sets in a closed vertical strip. In the present paper we investigate holomorphic almost periodic functions and almost periodic sets in an open strip, in particular, in the complex plane and in half-planes.

Definition 2. A continuous function $f(z)$ in the strip $S = \{z: a < \text{Re} z < b\}$, $-\infty \leq a < b \leq \infty$ is almost periodic, if for any substrip $S_0, \overline{S_0} \subset S$, and any $\varepsilon > 0$ there is a relatively dense set $E_{\varepsilon, S_0} \subset \mathbb{R}$ such that for any $\tau \in E_{\varepsilon, S_0}$

$$\sup_{z \in S_0} |f(z + i\tau) - f(z)| < \varepsilon. \quad (6)$$

A typical example of an entire almost periodic function is sum (3) under conditions

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sup_n \lambda_n < \infty, \quad \inf_n \lambda_n > -\infty.$$ 

Note that a holomorphic function $f(z)$ in the strip $S = \{z: a < \text{Re} z < b\}$, $-\infty \leq a < b \leq \infty$ is almost periodic if and only if there exists a sequence of quasipolynomials $Q_n$ of form (1) such that for any substrip $S_0, \overline{S_0} \subset S$,

$$\sup_{z \in S_0} |f(z) - Q_n(z)| \to 0, \quad n \to \infty,$$

(see [4], item 8).

In order to take into account multiplicities of zeros, we use the term divisor instead of zero set. Namely, a divisor $Z$ in a domain $D$ is the mapping $Z: D \to \mathbb{N} \cup \{0\}$ such that $|Z| = \text{supp}Z$ is a set without limit points in $D$. In other words, a divisor in $D$ is the sequence
of points \( \{z_j\} \subset D \) without limit points in \( D \) such that every point of \( D \) appears at most finite times in the sequence. If \( Z(z) \leq 1 \) for all \( z \in D \), we identify \( Z \) and \( |Z| \). The divisor of zeros of a holomorphic function \( f \) in \( D \) is the map \( Z_f \) such that \( Z_f(a) \) equals the multiplicity of zero of the function \( f \) at the point \( a \).

The following definition appeared at first in \([8]\).

**Definition 3** ([2], [8]). A divisor \( Z = \{z_j\} \) in a strip \( S = \{z: a < \text{Re}z < b \} \), \(-\infty \leq a < b \leq \infty \), is called *almost periodic* if for any \( \varepsilon > 0 \) and any substrip \( S_0, S_0^c \subset S \), there is a relatively dense set

\[
E_{\varepsilon,S_0} = \{ \tau \in \mathbb{R}: z_j \in S_0 \lor z_{\sigma(j)} \in S_0 \implies |z_j + i\tau - z_{\sigma(j)}| < \varepsilon \},
\]  

(7)

where \( \sigma = \sigma_\tau \) is a suitable bijection \( \mathbb{N} \rightarrow \mathbb{N} \).

**Remark.** If we take \( \sigma^{-1}(j) \) instead of \( j \) in (7), we get

\[
z_j \in S_0 \lor z_{\sigma^{-1}(j)} \in S_0 \implies |z_j - i\tau - z_{\sigma^{-1}(j)}| < \varepsilon.
\]  

(8)

**Theorem 1** ([2]).

a) If \( f \) is a holomorphic function in a strip \( S \) with the almost periodic modulus, then the divisor of zeros of \( f \) is almost periodic.

b) For any almost periodic divisor \( Z \) in a strip \( S \) there is the holomorphic function \( f \) in \( S \) with the almost periodic modulus such that \( Z = Z_f \).

   c) For any almost periodic divisor \( Z \) in a strip \( S \) such that \( |Z| \cap S' = \emptyset \) for some open substrip \( S' \subset S \) there is the holomorphic almost periodic function \( f \) in \( S \) such that \( Z = Z_f \).

There exists an almost periodic divisor in the plane with a discrete set of differences of their points that is not periodic.

**Example 1.** Let

\[
Z = \{z_{n,k} = i2^{nk} + 2^k, \ n \in \mathbb{Z}, \ k \in \mathbb{N}\}
\]

be a discrete set in \( \mathbb{C} \). It is easy to see that the set is almost periodic, differences of its points form a discrete set, but \( Z \) is a countable union of periodic sets with different commensurable periods.

In our article we prove the following theorem:

**Theorem 2.** Let \( Z, W \) be almost periodic divisors in the strip \( S = \{z: a < \text{Re}z < b \} \), \(-\infty \leq a < b \leq \infty \). If

a) \( |Z| \cap S' = \emptyset \) for some substrip \( S' \subset S \),

b) for any substrip \( S_0 \) of a finite width, \( S_0^c \subset S \), the set \( \{z - w, \ z \in |Z| \cap S_0, \ w \in |W| \cap S_0\} \) is discrete,

then \( Z, W \) are at most countable sums of periodic divisors with commensurable periods.

Show that condition a) is essential.

**Example 2.** Let \( S = \{z: |\text{Re}z| < 1\} \), \( Z = \{z_{m,n} = (m + in)e^{i\alpha} \in S, \ m, n \in \mathbb{Z}\} \), where \( \alpha \in \mathbb{R} \) such that \( \cot \alpha \) is an irrational number. Clearly, the set of differences of elements of \( Z \) is discrete. Let us prove that \( Z \) is an almost periodic divisor without any periods.

By Kronecker Lemma (see, for example, [7], Ch.2, §2), for any \( \varepsilon > 0 \) the inequalities

\[
|\exp(2\pi it \cot \alpha) - 1| < \varepsilon, \quad |\exp(2\pi it) - 1| < \varepsilon,
\]

has a relatively dense set of common solutions. Therefore, the inequality
\[|\exp(2\pi im \cot \alpha) - 1| < \varepsilon(1 + |\cot \alpha|)\]
has a relatively dense set of integer solutions. The latter means that for any \(\delta\) there exist pairs of integers \((m, n) \in \mathbb{Z}^2\) such that
\[|m \cos \alpha - n \sin \alpha| < \delta, \quad (9)\]
and the set of \(m\) with this property is relatively dense. Put
\[E = \{m \sin \alpha + n \cos \alpha : |m \cos \alpha - n \sin \alpha| < \delta\}.\]

Let \(\tau = \bar{m} \sin \alpha + \bar{n} \cos \alpha \in E\). For any \(z_{m,n} \in \mathbb{Z}\) and \(m' = m + \bar{m}, \ n' = n + \bar{n}\) we have
\[z_{m,n} + i\tau - z_{m',n'} = (m + in)e^{i\alpha} + i \Im[(\bar{m} + i\bar{n})e^{i\alpha}] - (m' + in')e^{i\alpha} = -\Re[(\bar{m} + i\bar{n})e^{i\alpha}].\]

By (9), \(|z_{m,n} + i\tau - z_{m',n'}| < \delta\). Since
\[|\tau - (\cos^2 \alpha/\sin \alpha + \sin \alpha)\bar{m}| = |\bar{n} \cos \alpha - (\cos^2 \alpha/\sin \alpha)\bar{m}| < \delta|\cot \alpha|,\]
we see that the set \(E\) is relatively dense. So, \(Z\) is an almost periodic divisor.

Since \(Z\) sites in the vertical strip of width 2, we see that any period of \(Z\) must have the form \(iT, \ T \in \mathbb{R}\). Hence, for some \((n, m), (n', m') \in \mathbb{Z}^2, (n, m) \neq (n', m')\),
\[(n + im)e^{i\alpha} - (n' + im')e^{i\alpha} = iT, \ (n - n') + i(m - m') = iT(\cos \alpha - i \sin \alpha).\]

This equality contradicts our choice of \(\alpha\). Hence any part of \(Z\) has no periods.

Our proof of Theorem 2 uses the following lemmas.

**Lemma 1.** Let \(Z = \{z_n\}\) be an almost periodic divisor in a strip \(S, S_0\) be a substrip, \(S_0 \subset S, \ \varepsilon > 0, \ \tau_1, \ \tau_2 \in E_{\varepsilon,S_0}, \) where \(E_{\varepsilon,S_0}\) is from Definition 3. Then there is a bijection \(\sigma : \mathbb{N} \rightarrow \mathbb{N}\) such that if \(z_j \in S_0, \ \text{dist}(z_j, \partial S_0) > \varepsilon\), then
\[|z_j + i(\tau_1 - \tau_2) - z_{\sigma(j)}| < 2\varepsilon, \quad |z_j - i(\tau_1 - \tau_2) - z_{\sigma^{-1}(j)}| < 2\varepsilon. \quad (10)\]

It can be proved easily that the similar assertion is valid for \(\tau_1 + \tau_2\) as well.

**Proof.** If \(z_j\) satisfies conditions of the lemma, then for some bijection \(\sigma_1 : \mathbb{N} \rightarrow \mathbb{N}\) we have \(|z_j + i\tau_1 - z_{\sigma_1(j)}| < \varepsilon\), therefore, \(|\Re(z_{\sigma_1(j)} - z_j)| < \varepsilon\) and \(z_{\sigma_1(j)} \in S_0\). By (8), we also get the inequality \(|\sigma_1^{-1}(\tau_1) - i\tau_2 - z_{\sigma_1(j)}| < \varepsilon\) for some bijection \(\sigma_2 : \mathbb{N} \rightarrow \mathbb{N}\). Hence we obtain the first part in (10) with \(\sigma = \sigma_1^{-1} \circ \sigma_1\). If we change places of \(\tau_1\) and \(\tau_2\), we obtain the second inequality in (10).

**Lemma 2.** Let \(Z = \{z_n\}, \ W = \{w_m\}\) be almost periodic divisors in a strip \(S\). Then for any \(\varepsilon > 0\) and any substrip \(S_0 \subset S\) such that \(\text{dist}(S_0, \partial S) > \varepsilon\), there is a relatively dense set \(E \subset \mathbb{R}\) with the property: for any \(\tau \in E\) there are bijections \(\sigma_Z, \ \sigma_W\) from \(\mathbb{N}\) to \(\mathbb{N}\) such that for all \(z_j \in |Z| \cap S_0, \ w_r \in |W| \cap S_0\)
\[|z_j + i\tau - z_{\sigma_Z(j)}| < \varepsilon, \quad |w_r + i\tau - w_{\sigma_W(r)}| < \varepsilon, \quad (11)\]
\[|z_j - i\tau - z_{\sigma_Z^{-1}(j)}| < \varepsilon, \quad |w_r - i\tau - w_{\sigma_W^{-1}(r)}| < \varepsilon. \quad (12)\]
Proof. Put \( \tilde{S} = \{ z \in S : \text{dist}(z, S_0) < \varepsilon/2 \} \). By (7) and (8), there is a number \( L \) such that any interval of length \( L \) contains \( \tau_Z \) and \( \tau_W \) with the properties

\[
|z_j + i\tau_Z - z_{\sigma_Z(j)}| < \varepsilon/4, \quad |z_j - i\tau_Z - z_{\sigma_Z^{-1}(j)}| < \varepsilon/4
\]

(13)

for all \( z_j \in |Z| \cap \tilde{S} \), and

\[
|w_r + i\tau_W - w_{\sigma_W(r)}| < \varepsilon/4, \quad |w_r - i\tau_W - w_{\sigma_W^{-1}(r)}| < \varepsilon/4
\]

(14)

for all \( w_r \in |W| \cap \tilde{S} \). Here \( \sigma_Z, \sigma_W \) are some bijections \( \mathbb{N} \to \mathbb{N} \).

We may suppose that \( N = 2L/\varepsilon \) is an integer. Hence for any \( k \in \mathbb{Z} \) there are integers \( n(k), m(k), 0 \leq n(k), m(k) \leq N \), such that

\[
|kL + n(k)\varepsilon/2 - \tau_Z| < \varepsilon/4, \quad |kL + m(k)\varepsilon/2 - \tau_W| < \varepsilon/4.
\]

The differences \( n(k) - m(k) \) take at most \( 2N + 1 \) values. Choose \( k_1, \ldots, k_r, r \leq 2N + 1 \), such that for any \( k \in \mathbb{Z} \) there is \( k_s \) with the property \( n(k) - m(k) = n(k_s) - m(k_s) \). It follows easily that any interval of length \( L(\max_s(|k_s| + 2)) \) contains a point of the form

\[
\tau = Lk + n(k)\varepsilon/2 - Lk_s - n(k_s)\varepsilon/2 = Lk + m(k)\varepsilon/2 - Lk_s - m(k_s)\varepsilon/2.
\]

If we replace \( \tau_Z \) by \( Lk + n(k)\varepsilon/2 \) or \( Lk_s + n(k_s)\varepsilon/2 \) in (13), we obtain that this inequality satisfies with \( \varepsilon/2 \) instead of \( \varepsilon/4 \). The same is true if we replace \( \tau_W \) by \( Lk + m(k)\varepsilon/2 \) or \( Lk_s + m(k_s)\varepsilon/2 \) in (14). Applying Lemma 1 with \( S_0 = \tilde{S} \) and \( \varepsilon/2 \) instead of \( \varepsilon \), we see that \( \tau \) satisfies (11) and (12). \( \Box \)

Proof of Theorem 2. Set a sequence of vertical open substrips \( S_k, \overline{S_k} \subset S_{k+1} \), such that

\[
|Z| \cap S_1 \neq \emptyset, \quad |W| \cap S_1 \neq \emptyset, \quad S_1 \supset \overline{S}, \quad \bigcup_k S_k = S.
\]

Take any points \( z \in |Z| \cap S_1 \), \( w \in |W| \cap S_1 \).

It follows from (11) with \( \varepsilon < \min\{\text{dist}(z, \partial S_1), \text{dist}(w, \partial S_1)\} \) that there is \( R < \infty \) such that any horizontal strip of the width \( R \) contains at least one point \( |Z| \cap S_1 \) and at least one point \( |W| \cap S_1 \). Let \( \delta \) be less than the width of the strip \( S' \). By condition b) of the theorem, for each \( k \) the set \( \{ z - w, z \in |Z| \cap S_k, w \in |W| \cap S_k \} \) is discrete. Hence there exists \( \gamma_k < \min\{1/2, \delta, \text{dist}(S_k, \partial S_{k+1})\} \) such that whenever

\[
z_n, z_n' \in |Z| \cap S_{k+1}, \quad w_m, w_m' \in |Z| \cap S_{k+1}, \quad z_n - w_m \neq z_{n'} - w_{m'}, \quad |\text{Im}(z_n - w_m)| < 2R + 3, \quad |\text{Im}(z_{n'} - w_{m'})| < 2R + 3,
\]

we get \( \gamma_k < |(z_{n'} - w_{m'}) - (z_n - w_m)| \). In particular, if we put \( w_m = w_{m'} \), then we get \( \gamma_k < |z_n - z_{n'}| \) for any \( z_n, z_{n'} \in |Z| \cap S_{k+1}, z_n \neq z_{n'} \).

Fix \( z_n \in |Z| \cap S_k \), and let a number \( \tau > 1 \) satisfies (11) for \( Z \) and \( W \) with \( \varepsilon = \gamma_k/2 \).

Then there is a unique \( z_{n'} \in |Z| \cap S_{k+1} \) such that \( |z_n + i\tau - z_{n'}| < \gamma_k/2 \). Indeed, otherwise we obtain

\[
|z_{n'} - z_{n''}| \leq |z_n + i\tau - z_{n'}| + |z_n + i\tau - z_{n''}| < \gamma_k.
\]

Set \( T_k = (z_{n'} - z_n)/i \). First suppose that \( T_k \) is real, hence, \( z_{n'} \in |Z| \cap S_k \).
Let \( w_m \in |W| \cap S_k \) be such that \(|\text{Im} (w_m - z_n)| < 2R + 2\). By (11), there is a point \( w_{m'} \in |W| \cap S_{k+1} \) such that \(|w_m + i\tau - w_{m'}| < \gamma_k/2\). Therefore,

\[
|(z_n - w_m) - (z_{n'} - w_{m'})| \leq |w_m + i\tau - w_{m'}| + |z_{n'} - z_n - i\tau| < \gamma_k.
\]

Since

\[
|\text{Im}(z_{n'} - w_{m'})| \leq |\text{Im}(z_n - w_m)| + |z_n - z_{n'} + i\tau| + |w_m - w_{m'} + i\tau| < 2R + 3,
\]

we get \( z_n - w_m = z_{n'} - w_{m'} \) due to the choice of \( \gamma_k \). Therefore, \( w_{m'} = w_m + iT_k, \ w_{m'} \in S_k \).

The latter equality takes place for all points \( |W| \cap \{w \in S_k: \text{Im} z_n - 2R < \text{Im} w < \text{Im} z_n + 2R\} \), in particular, for some \( w_1 \) such that \( \text{Im} z_n + R < \text{Im} w_1 < \text{Im} z_n + 2R \). Namely, there is \( w_1 \in |W| \cap S_k \) such that \( w_1 = w + iT_k \). Let \( \zeta \in |Z| \cap S_k \) be any point from the set \( \{z \in S_k: \text{Im} z_n \leq \text{Im} z < \text{Im} z_n + 3R\} \subset \{z \in S_k: \text{Im} w_1 - 2R < \text{Im} z < \text{Im} w_1 + 2R\} \). (15)

By (11), there is a point \( \zeta' \in |Z| \cap S_{k+1} \) such that \(|\zeta + i\tau - \zeta'| < \gamma_k/2\). Therefore,

\[
|(\zeta - w_1) - (\zeta' - w_{1'})| \leq |\zeta + i\tau - \zeta'| + |iT_k - i\tau| < \gamma_k.
\]

Since \(|\text{Im} \zeta - \text{Im} w_1| < 2R\) and

\[
|\text{Im}(\zeta' - w_{1'})| \leq |\text{Im}(\zeta - w_1)| + |\zeta + i\tau - \zeta'| + |iT_k - i\tau| < 2R + 3,
\]

we get \( \zeta - w_1 = \zeta' - w_{1'} \) due to the choice of \( \gamma_k \). Therefore, \( \zeta' = \zeta + iT_k \).

In particular, there is a point \( z_n \in |Z| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_1 < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \), we also can find \( w_{1'} \in |W| \cap S_k \) such that \( \text{Im} z_n - 2R < \text{Im} w_{1'} < \text{Im} z_n - R \).

Next, by (12), take for any \( z \in |Z| \cap S_k \) a point \( z_{1''} \in |Z| \cap S_{k+1} \) such that \(|z_{1''} + i\tau - z| < \gamma_k/2\). Then \( z_{1''} + iT_k \in |Z| \cap S_k \) and \(|z_{1''} + iT_k - z| \leq |z_{1''} + i\tau - z| + |i\tau - iT_k| < \gamma_k\). Therefore, \( z_{1''} + iT_k = z \) and \( z - iT_k \in |Z| \cap S_k \) for all \( z \in |Z| \cap S_k \). By the same arguments, \( w - iT_k \in |W| \) for all \( w \in |W| \cap S_k \).

Now suppose that \( \text{Im} T_k \not= 0 \). Clearly, either \( \text{dist}\{z_n, S'\} > \text{dist}\{z_n + iT_k, S'\} \), or \( \text{dist}\{z_n, S'\} > \text{dist}\{z_n - iT_k, S'\} \). In the first case the points \( z_{1''}, w_1, w_{1'} \), \( \zeta' \), \( \zeta'' \) belong to \( S_k \). Hence, we get \( z_n + iT_k \in S' \) for some integer \( M > 0 \), that is impossible. In the second case the same arguments show that \( z_n + iT_k \in S' \) for some integer \( M < 0 \). We get a contradiction in the both cases. Therefore, \( T_k \) is real and for any \( M \in \mathbb{Z} \) we get \(|Z| \cap S_k + iT_k \in |Z| \cap S_k \). Thus the restrictions \( Z \mid S_k, W \mid S_k \) are periodic divisors with period \( iT_k \).

The same arguments work for every \( k \in \{1, 2, \ldots\} \).

Let \( iT^0_k \) be the minimal common period of \( Z \mid S_k \) and \( W \mid S_k \). Clearly, \( T_k/T^0_k \in \mathbb{N} \). Besides, since \( |Z| \cap S_k \subset |Z| \cap S_m \) for \( m > k \), we have \( T_m/T^0_k \in \mathbb{N} \) as well.

Finally, let \( Z = Z \mid S_1, Z_k = Z \mid S_k \backslash S_{k-1} \), \( W = W \mid S_1, W_k = W \mid S_k \backslash S_{k-1} \). Then we obtain

\[
Z = Z_1 + Z_2 + Z_3 + \ldots, \quad W = W_1 + W_2 + W_3 + \ldots.
\]
Theorem 2 implies the corresponding result for almost periodic holomorphic functions.

**Theorem 3.** Let \( f, g \) be almost periodic functions in a strip \( S = \{ z : a < \text{Re} \, z < b \} \), \(-\infty \leq a < b \leq \infty \). If

- a) either \( f \), or \( g \) has no zeros in an open substrip \( S' \subset S \),
- b) for any substrip \( S_0, \overline{S}_0 \subset S \), the set \( \{ z - w, \, z, w \in S_0, \, f(z) = g(w) = 0 \} \) is discrete, then

\[
f(z) = f_0(z) \prod_{k=1}^{\infty} f_k(z), \quad g(z) = g_0(z) \prod_{k=1}^{\infty} g_k(z),
\]

where \( f_0, g_0 \) are holomorphic almost periodic functions in \( S \) without zeros, \( f_k, g_k, \, k \in \{1, 2, \ldots \} \), are periodic holomorphic in \( S \) with commensurable periods \( it_k \).

**Proof.** By Theorem 1a), the divisors \( Z, W \) of zeros of \( f, g \), respectively, are almost periodic and satisfy other conditions of the previous theorem. Let \( S_k, Z_k, W_k \), be the same as in the proof of Theorem 2. Suppose that

\[
S_k = \{ z : \eta_k < \text{Re} \, z < \eta_k' \}, \quad \eta_k < \eta_{k-1}, \quad \eta_k' > \eta_{k-1}', \quad \forall k.
\]

For any \( k \) there is only a finite number of points \( a_1^k, \ldots, a_m^k \in |Z_k| \cap \{ z : 0 \leq \text{Im} \, z < T_k \} \). Therefore, \( |Z_k| = \{ a_1^k, \ldots, a_m^k \} + iT_k \mathbb{Z} \). Take \( \epsilon_k^j > 0, \, 1 \leq j \leq m_k, \, 3 \leq k < \infty \), such that

\[
\sum_{j,k} \epsilon_k^j < \infty.
\]

Set

\[
h_j(w) = 1 - w \exp(-2\pi a_j^k/T_k), \quad k \in \{1, 2, \ldots\}, \quad j \in \{1, \ldots, m_k\}.
\]

Clearly, \( h_j(\exp(2\pi z/T_k)) \) has the divisor \( a_j^k + iT_k \mathbb{Z} \). Now let \( k > 2 \). Since \( a_j^k \in S_k \setminus S_{k-1} \), we see that either \( \eta_{k-1}' \leq \text{Re} \, a_j^k \), or \( \eta_{k-1} \geq \text{Re} \, a_j^k \). In the first case, \( \log h_j(w) \) is holomorphic on the disc \( |w| < \exp(2\pi \text{Re} \, a_j^k/T_k) \), and there is a polynomial \( P_j^k \) such that

\[
|\log h_j(w) - P_j^k(w)| < \epsilon_k^j \quad \text{for} \quad |w| \leq \exp(2\pi \eta_{k-2}'/T_k).
\]

In the second one, \( \log h_j(w) \) is holomorphic on the set \( |w| > \exp(2\pi \text{Re} \, a_j^k/T_k) \), and there is a polynomial \( P_j^k \) such that

\[
|\log h_j(w) - P_j^k(w^{-1})| < \epsilon_k^j \quad \text{for} \quad |w| \geq \exp(2\pi \eta_{k-2}'/T_k).
\]

Put \( Q_j(z) = P_j^k(\exp(2\pi z/T_k)) \) in the first case, and \( Q_j(z) = P_j^k(\exp(-2\pi z/T_k)) \) in the second one. We obtain

\[
|h_j(e^{2\pi z/T_k})e^{-Q_j(z)}| < e^{\epsilon_k^j} \quad \text{for} \quad z \in S_{k-2}.
\]

Put

\[
f_k(z) = \prod_{j=1}^{m_k} h_j(e^{2\pi z/T_k}), \quad k = 1, 2, \quad f_k(z) = \prod_{j=1}^{m_k} h_j(e^{2\pi z/T_k})e^{-Q_j(z)}, \quad k > 2.
\]

Clearly, \( Z_f = Z_k \). By (17) and (18), the first product in (16) converges in every substrip \( S_k \). The function \( f(z)/\prod_{k=1}^{\infty} f_k(z) \) is holomorphic in \( S \), and, by [8], almost periodic in \( S \). In the same way, we obtain the representation of \( g \).

**Remark.** It follows easily that for entire almost periodic functions \( f, \, g \) with zeros in a strip \( \tilde{S} \) of a finite width one can take \( S_1 = \tilde{S} \). Therefore, \( f, \, g \) are periodic functions with the same period up to almost periodic multiplies without zeros. Hence some results of [1], [3] follow from Theorem 3.
REFERENCES


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