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FREE n-NILPOTENT TRIOIDS

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We introduce the notion of a nilpotent trioid, construct a free n-nilpotent trioid and describe its structure. We also characterize the least n-nilpotent congruence on a free trioid and give examples of nilpotent trioids of nilpotency index 2.

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Введено понятие нильпотентного триоида, построен свободный n-нильпотентный триоид и описана его структура. Также охарактеризована наименьшая n-нильпотентная конгруэнция на свободном триоиде и приведены примеры нильпотентных триоидов индекса нильпотентности 2.

1. Introduction. J.-L. Loday and M. O. Ronco ([1]) introduced a type of algebras, called trioids, which are sets endowed with three binary associative operations $\dashv, \vdash$ and $\perp$ satisfying eight axioms: $(x \dashv y) \dashv z = x \dashv (y \vdash z)\,(T1)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)\,(T2)$, $(x \dashv y) \vdash z = x \vdash (y \vdash z)\,(T3)$, $(x \dashv y) \dashv z = x \dashv (y \perp z)\,(T4)$, $(x \perp y) \dashv z = x \perp (y \dashv z)\,(T5)$, $(x \dashv y) \perp z = x \perp (y \vdash z)\,(T6)$, $(x \vdash y) \perp z = x \vdash (y \perp z)\,(T7)$, $(x \perp y) \vdash z = x \vdash (y \vdash z)\,(T8)$. Trioids have applications in the theory of trialgebras ([1]). Recall that trialgebras are linear analogs of trioids. This kind of algebras is closely related to ternary planar trees. It is well known that dialgebras (dimonoids) ([2, 3]) can be obtained from trialgebras (trioids). In the survey paper [4], numerous examples of trioids were presented. The problem of constructing free trioids was solved in [1, 4]. Free rectangular trioids were given in [5].

Nilpotency in different algebras has been extensively studied by many authors. So, the notion of a nilpotent semigroup was introduced by A. I. Malcev ([6]) and independently by B. H. Neuman and T. Taylor ([7]). The relationships between nilpotent semigroups and semigroup algebras were studied by E. Jespers and J. Okninski ([8]). Nilpotency in rings was considered in [9]. Papers [10, 11] are devoted to studying (di)nilpotent dimonoids.

This paper develops the variety theory of trioids. In Section 2 constructions of a free trioid and some other algebras are given. In Section 3 we introduce the notion of a nilpotent trioid, give examples of nilpotent trioids of nilpotency index 2 and construct a free n-nilpotent trioid. In Section 4 we introduce the notion of a 0-triband of subtrioids and in terms of 0-tribands of subtrioids describe the structure of free n-nilpotent trioids. In the final section the least n-nilpotent congruence on a free trioid is characterized.

2. Preliminaries. Consider free trioids (see [4]).

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Let Y be an arbitrary nonempty set, $\overline{Y} = \{\overline{x}\,|\,x \in Y\}$, $X = Y \cup \overline{Y}$ and $F[X]$ be the free semigroup on X. Let further $P \subset F[X]$ be a subsemigroup which contains words w with the element $\overline{x}$ $(x \in Y)$ occuring in w at least one time. It is easy to see that $F[X]$ is a band of semigroups P and $F[X]\backslash P$ [12].

Let $w \in P$. Denote by $\widetilde{w}$ the word obtained from w by the replacement of all letters $\overline{x}$ $(x \in Y)$ with x. For instance, if $w = x\overline{x}\,\overline{y}x\overline{z}$, then $\widetilde{w} = xxyxz$. Obviously, $\widetilde{w} \in F[X]\backslash P$.

Define operations $\dashv$, $\vdash$ and $\perp$ on P by

$$w \dashv u = w\widetilde{u}, \quad w \vdash u = \widetilde{w}u, \quad w \perp u = wu$$

for all $w, u \in P$. Denote the algebra $(P, \dashv, \vdash, \perp)$ by $\mathrm{Frt}(Y)$.

The proof of the following statement is similar to the proof of Proposition 1.9 from [1] obtained for the free trioid of rank 1.

Proposition 1. $\mathrm{Frt}(Y)$ *is the free trioid of an arbitrary rank.*

If $Y = \{x\}$, then $\mathrm{Frt}(Y)$ is the free trioid of rank 1 presented by J.-L. Loday and M. O. Ronco in [1]. In the latter paper it was shown that the free associative trialgebra over a vector space is completely determined by the free associative trialgebra on one generator and the description of that trialgebra is reduced to the description of the free trioid of rank 1. A trioid which is isomorphic to the free trioid of rank 1 can be found in [4].

The notion of a normal form for elements of $\mathrm{Frt}(Y)$ of rank 1 (see [1], Lemma 1.10) can be naturally extended to the case of an arbitrary set Y. Namely, let Y be an arbitrary nonempty set and $w \in \mathrm{Frt}(Y)$. Then we obtain the normal form for w (see [13]):

$$\begin{aligned} w &= u_1^{(0)}u_2^{(0)}\ldots u_{k_0}^{(0)}\overline{u_1^{(1)}}u_2^{(1)}\ldots u_{k_1}^{(1)}\overline{u_1^{(2)}}u_2^{(2)}\ldots u_{k_2}^{(2)}\ldots u_{k_{j-1}}^{(j-1)}\overline{u_1^{(j)}}u_2^{(j)}\ldots u_{k_j}^{(j)} = \\ &= (\overline{u_1^{(0)}} \vdash \ldots \vdash \overline{u_{k_0}^{(0)}}) \vdash (\overline{u_1^{(1)}} \dashv \ldots \dashv \overline{u_{k_1}^{(1)}}) \perp \ldots \perp (\overline{u_1^{(j)}} \dashv \ldots \dashv \overline{u_{k_j}^{(j)}}), \end{aligned}$$

where $u_l^{(i)} \in Y$, $1 \le l \le k_i$ for all $i \in \{0, 1, \ldots, j\}$, or

$$\begin{aligned} w &= \overline{u_1^{(1)}}u_2^{(1)}\ldots u_{k_1}^{(1)}\overline{u_1^{(2)}}u_2^{(2)}\ldots u_{k_2}^{(2)}\ldots u_{k_{j-1}}^{(j-1)}\overline{u_1^{(j)}}u_2^{(j)}\ldots u_{k_j}^{(j)} = \\ &= (\overline{u_1^{(1)}} \dashv \ldots \dashv \overline{u_{k_1}^{(1)}}) \perp (\overline{u_1^{(2)}} \dashv \ldots \dashv \overline{u_{k_2}^{(2)}}) \perp \ldots \perp (\overline{u_1^{(j)}} \dashv \ldots \dashv \overline{u_{k_j}^{(j)}}), \end{aligned}$$

where $u_l^{(i)} \in Y$, $1 \le l \le k_i$ for all $i \in \{1, 2, \ldots, j\}$. Let further $(T, \dashv', \vdash', \perp')$ be an arbitrary trioid and $\varphi\colon \overline{Y} \to T$ be an arbitrary map. Since $\mathrm{Frt}(Y)$ is a free trioid, there exists a homomorphism $\Phi\colon\ \mathrm{Frt}(Y) \to (T, \dashv', \vdash', \perp')$. It is defined by the following rule (see [13]):

$$\begin{aligned} w\Phi = (\overline{u_1^{(0)}}\varphi \vdash' \overline{u_2^{(0)}}\varphi \vdash' \ldots \vdash' \overline{u_{k_0}^{(0)}}\varphi) \vdash' (\overline{u_1^{(1)}}\varphi \dashv' \overline{u_2^{(1)}}\varphi \dashv' \ldots \dashv' \overline{u_{k_1}^{(1)}}\varphi) \perp' \\ \perp' \ldots \perp' (\overline{u_1^{(j)}}\varphi \dashv' \overline{u_2^{(j)}}\varphi \dashv' \ldots \dashv' \overline{u_{k_j}^{(j)}}\varphi) \end{aligned}$$

or

$$\begin{aligned} w\Phi = (\overline{u_1^{(1)}}\varphi \dashv' \overline{u_2^{(1)}}\varphi \dashv' \ldots \dashv' \overline{u_{k_1}^{(1)}}\varphi) \perp' (\overline{u_1^{(2)}}\varphi \dashv' \overline{u_2^{(2)}}\varphi \dashv' \ldots \dashv' \overline{u_{k_2}^{(2)}}\varphi) \perp' \\ \perp' \ldots \perp' (\overline{u_1^{(j)}}\varphi \dashv' \overline{u_2^{(j)}}\varphi \dashv' \ldots \dashv' \overline{u_{k_j}^{(j)}}\varphi). \end{aligned}$$

We will call Φ the canonical homomorphism.

Now recall the definition of a dimonoid ([2, 3]).

A nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1)-(T3)$ is called a *dimonoid*. If $D=(D,\dashv,\vdash)$ is a dimonoid, then the trioid $(D,\dashv,\vdash,\dashv)$ (respectively, $(D,\dashv,\vdash,\vdash)$) will be denoted by $(D)^{\dashv}$ (respectively, $(D)^{\vdash}$). It is clear that $(D)^{\dashv}$ and $(D)^{\vdash}$ are distinct as trioids but they coincide as dimonoids.

Consider some algebras from [14] and [5] which will be used in Section 4.

For an arbitrary nonempty set Y let $Y_{\ell z}=(Y,\dashv)$, $Y_{rz}=(Y,\vdash)$, $Y_{rb}=Y_{\ell z}\times Y_{rz}$ be a left zero semigroup, a right zero semigroup and a rectangular band, respectively. By [14] $Y_{\ell z,rz}=(Y,\dashv,\vdash)$ is the free left zero and right zero dimonoid (or the free left and right diband).

Define operations $\dashv$ and $\vdash$ on Y^2 by $(x,y)\dashv(a,b)=(x,b),\quad (x,y)\vdash(a,b)=(a,b)$ for all $(x,y),(a,b)\in Y^2$. By [14] $(Y^2,\dashv,\vdash)$ is the free (rb,rz)-dimonoid. It is denoted by $Y_{rb,rz}$.

Define operations $\dashv$ and $\vdash$ on Y^2 by $(x,y)\dashv(a,b)=(x,y),\quad (x,y)\vdash(a,b)=(x,b)$ for all $(x,y),(a,b)\in Y^2$. By [14] $(Y^2,\dashv,\vdash)$ is the free $(\ell z,rb)$-dimonoid. It is denoted by $Y_{\ell z,rb}$.

A trioid (dimonoid) is called a *rectangular triband* ([5], rectangular diband [14]), if each of its semigroups is a rectangular band.

Define operations $\dashv$ and $\vdash$ on Y^3 by

$$(x_1,x_2,x_3)\dashv(y_1,y_2,y_3)=(x_1,x_2,y_3),\ (x_1,x_2,x_3)\vdash(y_1,y_2,y_3)=(x_1,y_2,y_3)$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3)\in Y^3$. The algebra $(Y^3,\dashv,\vdash)$ is denoted by $\mathrm{FRct}(Y)$. According to Theorem 1 from [14] $\mathrm{FRct}(Y)$ is a free rectangular diband.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^3 by

$$(a_1,b_1,c_1)\dashv(a_2,b_2,c_2)=(a_1,b_1,c_1),\ (a_1,b_1,c_1)\vdash(a_2,b_2,c_2)=(a_1,b_2,c_2),$$
$$(a_1,b_1,c_1,)\perp(a_2,b_2,c_2)=(a_1,b_1,c_2)$$

for all $(a_1,b_1,c_1),(a_2,b_2,c_2)\in Y^3$. By Lemma 1 from [5] $(Y^3,\dashv,\vdash,\perp)$ is a rectangular triband. It is denoted by $Y_{lz,rd}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^3 by

$$(a_1,b_1,c_1)\dashv(a_2,b_2,c_2)=(a_1,b_1,c_2),\ (a_1,b_1,c_1)\vdash(a_2,b_2,c_2)=(a_2,b_2,c_2),$$
$$(a_1,b_1,c_1,)\perp(a_2,b_2,c_2)=(a_1,b_2,c_2)$$

for all $(a_1,b_1,c_1),(a_2,b_2,c_2)\in Y^3$. By Lemma 2 from [5] $(Y^3,\dashv,\vdash,\perp)$ is a rectangular triband. It is denoted by $Y_{rd,rz}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^2 by

$$(a_1,b_1)\dashv(a_2,b_2)=(a_1,b_1),\quad (a_1,b_1)\vdash(a_2,b_2)=(a_2,b_2),(a_1,b_1)\perp(a_2,b_2)=(a_1,b_2)$$

for all $(a_1,b_1),(a_2,b_2)\in Y^2$. By Lemma 3 from [5] $(Y^2,\dashv,\vdash,\perp)$ is a rectangular triband. It is denoted by $Y^{rb}_{lz,rz}$. Note that the trioid $Y^{rb}_{lz,rz}$ was first constructed in [15].

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^4 by

$$(x_1,x_2,x_3,x_4)\dashv(y_1,y_2,y_3,y_4)=(x_1,x_2,x_3,y_4),$$
$$(x_1,x_2,x_3,x_4)\vdash(y_1,y_2,y_3,y_4)=(x_1,y_2,y_3,y_4),$$
$$(x_1,x_2,x_3,x_4)\perp(y_1,y_2,y_3,y_4)=(x_1,x_2,y_3,y_4)$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in Y^4$. The algebra $(Y^4, \dashv, \vdash, \perp)$ is denoted by $\mathrm{FRT}(Y)$. By Theorem 5 from [5] $\mathrm{FRT}(Y)$ is a free rectangular triband.

A nonempty subset A of a trioid $(T, \dashv, \vdash, \perp)$ is called a subtrioid, if for any $a, b \in T$, $a,\ b \in A$ implies $a \dashv b,\ a \vdash b, a \perp b \in A$.

As usual, $\mathbb{N}$ denotes the set of all positive integers.

3. Nilpotency in trioids. In this section we introduce the notion of a nilpotent trioid, give examples of nilpotent trioids of nilpotency index 2 and construct a free n-nilpotent trioid of an arbitrary rank.

An element 0 of a trioid $(T, \dashv, \vdash, \perp)$ is called zero ([16]), if $x * 0 = 0 * x = 0 * 0 = 0$ for all $x \in T$ and $* \in \{\dashv, \vdash, \perp\}$.

A trioid $(T, \dashv, \vdash, \perp)$ with zero will be called nilpotent, if for some $n \in \mathbb{N}$ and any $x_i \in T$, $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash, \perp\}$, $1 \leq j \leq n$, any parenthesizing of $x_1 *_1 x_2 *_2 \ldots *_n x_{n+1}$ gives $0 \in T$. The least such n we shall call the nilpotency index of $(T, \dashv, \vdash, \perp)$. For $k \in \mathbb{N}$ a nilpotent trioid of nilpotency index $\leq k$ is said to be k-nilpotent.

It is clear that operations of any 1-nilpotent trioid coincide and it is a zero semigroup.

Now we give examples of nilpotent trioids of nilpotency index 2.

Let X_1 and X_2 be arbitrary disjoint sets, $0 \in X_1$, and let

$$\varphi_1 \colon X_2 \times X_2 \to X_1, \quad \varphi_2 \colon X_2 \times X_2 \to X_1, \quad \varphi_3 \colon X_2 \times X_2 \to X_1$$

be arbitrary distinct maps. Define operations $\dashv$, $\vdash$ and $\perp$ on $X_1 \cup X_2$ by

$$x \dashv y = \begin{cases} (x, y)\varphi_1, & x, y \in X_2, \\ 0, & \text{otherwise}, \end{cases} \quad x \vdash y = \begin{cases} (x, y)\varphi_2, & x, y \in X_2, \\ 0, & \text{otherwise}, \end{cases}$$

$$x \perp y = \begin{cases} (x, y)\varphi_3, & x, y \in X_2, \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y \in X_1 \cup X_2$.

The proof of the following statement is similar to the proof of Proposition 2 from [10].

Proposition 2. *$(X_1 \cup X_2, \dashv, \vdash, \perp)$ is a nilpotent trioid of nilpotency index 2.*

Recall that a trioid is called commutative, if its three operations are commutative.

Let Y be an arbitrary set such that $0,\ a,\ b,\ c,\ d, e, f \in Y$ and $a \neq b$, $b \neq c$, $c \neq d$, $d \neq a$, $b \neq e$, $d \neq e$, $f \neq e$, $a \neq f$, $c \neq f$. Define operations $\dashv$, $\vdash$ and $\perp$ on Y, assuming

$$x \dashv y = \begin{cases} b, & x = y = a, \\ 0, & \text{otherwise}, \end{cases} \quad x \vdash y = \begin{cases} d, & x = y = c, \\ 0, & \text{otherwise}, \end{cases} \quad x \perp y = \begin{cases} f, & x = y = e, \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y \in Y$.

The proof of the following statement is similar to that of Proposition 3 from [10].

Proposition 3. *If $b \neq 0$ or $d \neq 0$, or $f \neq 0$, then $(Y, \dashv, \vdash, \perp)$ is a nilpotent commutative trioid of nilpotency index 2.*

Note that the trioid $(Y, \dashv, \vdash, \perp)$ was first constructed in [15].

It is not difficult to see that the class of all n-nilpotent trioids is a subvariety of the variety of all trioids. A trioid which is free in the variety of n-nilpotent trioids will be called a *free n-nilpotent trioid.*

See [4, 17, 18] for more information about trioids.

For every $w \in F[X]$ denote the length of w by l_w. Let $n \in \mathbb{N}$ and $P_n \subset P$ be a set which contains words w with the length no more than n (see Section 2). Define operations $\prec, \succ$ and $\uparrow$ on the set $P_n \cup \{0\}$ by

$$w{\prec}u = \begin{cases} w\widetilde{u}, & l_{wu}{\le}n, \\ 0, & l_{wu} > n, \end{cases} \quad w{\succ}u = \begin{cases} \widetilde{w}u, & l_{wu}{\le}n, \\ 0, & l_{wu} > n, \end{cases}$$

$$w{\uparrow}u = \begin{cases} wu, & l_{wu}{\le}n, \\ 0, & l_{wu} > n, \end{cases} \quad w * 0 = 0 * w = 0 * 0 = 0$$

for all $w, u \in P_n$ and $* \in \{\prec, \succ, \uparrow\}$. Denote the algebra $(P_n \cup \{0\}, \prec, \succ, \uparrow)$ by $P_n^0(Y)$.

Theorem 1. *$P_n^0(Y)$ is a free n-nilpotent trioid of an arbitrary rank.*

Proof. By Proposition 1 from [16] $P_n^0(Y)$ is a trioid with zero. For any $w_i \in P_n^0(Y) \backslash \{0\}$, $1 \le i \le n+1$, and $*_j \in \{\prec, \succ, \uparrow\}$, $1 \le j \le n$, any parenthesizing of $w_1 *_1 w_2 *_2 \ldots *_n w_{n+1}$ gives 0. Thus, $P_n^0(Y)$ is nilpotent. On the other hand, for $\overline{x} \in \overline{Y}$,

$$\underbrace{\overline{x} \prec \overline{x} \prec \ldots \prec \overline{x}}_{n} = \underbrace{\overline{x}x \ldots x}_{n} \neq 0.$$

It means that $P_n^0(Y)$ has nilpotency index n.

Let us show that $P_n^0(Y)$ is free in the variety of n-nilpotent trioids.

Let $(T, \dashv', \vdash', \perp')$ be an arbitrary n-nilpotent trioid, $\rho\colon \overline{Y} \to T$ be an arbitrary map and $\mu\colon \mathrm{Frt}(Y) \to (T, \dashv', \vdash', \perp')$ be the canonical homomorphism which is defined by ρ (see Section 2). Define a map $\delta\colon P_n^0(Y) \to (T, \dashv', \vdash', \perp') : w \mapsto w\delta$, assuming

$$w\delta = \begin{cases} w\mu, & w \in P_n^0(Y) \backslash \{0\}, \\ 0, & w = 0. \end{cases}$$

Show that δ is a homomorphism.

Let w_1, $w_2 \in P_n^0(Y) \backslash \{0\}$ and $l_{w_1} + l_{w_2} \le n$. As $w_1 \prec w_2 \in P_n^0(Y) \backslash \{0\}$, then

$$(w_1 \prec w_2)\delta = (w_1 \prec w_2)\mu = (w_1 \dashv w_2)\mu = w_1\mu \dashv' w_2\mu = w_1\delta \dashv' w_2\delta.$$

Analogously, $(w_1 \succ w_2)\delta = w_1\delta \vdash' w_2\delta$, $(w_1 \uparrow w_2)\delta = w_1\delta \perp' w_2\delta$. The map μ sends an arbitrary element w to the product of some l_w elements from T. Hence, in the remaining cases the equalities

$$(w_1 \prec w_2)\delta = (w_1 \succ w_2)\delta = (w_1 \uparrow w_2)\delta = 0 = w_1\delta{\dashv'}w_2\delta = w_1\delta{\vdash'}w_2\delta = w_1\delta \perp' w_2\delta$$

hold. Thus, δ is a homomorhism. □

4. 0-triband decompositions of $P_n^0(Y)$. In this section we introduce the notion of a 0-triband of subtrioids and in terms of 0-tribands of subtrioids describe the structure of free n-nilpotent trioids.

For trioids with zero there exists a natural analog of the notion of a triband of subtrioids (see [15]).

A trioid S with zero 0 (see Section 3) will be called a 0-triband of subtrioids S_β, $\beta \in B$, where B is an idempotent trioid [15], if $S = \bigcup_{\beta\in B} S_\beta$, $S_\beta \cap S_\gamma = \{0\}$ for $\beta \neq \gamma$ and $S_\beta \dashv S_\gamma \subseteq S_{\beta\dashv\gamma}$, $S_\beta \vdash S_\gamma \subseteq S_{\beta\vdash\gamma}$, $S_\beta \perp S_\gamma \subseteq S_{\beta\perp\gamma}$ for any $\beta, \gamma \in B$. If B is an idempotent semigroup (band), then we say that S is a 0-band of subtrioids S_β, $\beta \in B$.

Observe that the notion of a 0-triband of subtrioids generalizes the notion of a 0-diband of subdimonoids ([3]) and the notion of a 0-band of semigroups ([19]).

Let $\omega \in F[X]$ and $w \in P_n^0(Y)\backslash\{0\}$. Denote the first (respectively, last) letter of ω by $\omega^{(0)}$ (respectively, $\omega^{(1)}$). Suppose that u is the initial (respectively, terminal) subword of w with the minimal length such that $u^{(1)} \in \overline{Y}$ (respectively, $u^{(0)} \in \overline{Y}$). In this case $\widetilde{u^{(1)}}$ (respectively, $\widetilde{u^{(0)}}$) will be denoted by $w^{[0]}$ (respectively, $w^{[1]}$).

Let

$$Q_{(i,j)} = \{w \in P_n^0(Y)\backslash\{0\} \mid (\widetilde{w}^{(0)}, \widetilde{w}^{(1)}) = (i,j)\} \cup \{0\},$$
$$Q_{(i)} = \{w \in P_n^0(Y)\backslash\{0\} \mid \widetilde{w}^{(0)} = i\} \cup \{0\}, \ Q_{[i]} = \{w \in P_n^0(Y)\backslash\{0\} \mid \widetilde{w}^{(1)} = i\} \cup \{0\}$$

for $i, j \in Y$, $n > 1$ and $|Y| > 1$.

The following structural theorem gives decompositions of $P_n^0(Y)$ into 0-bands of subtrioids.

Theorem 2. *The free n-nilpotent trioid $P_n^0(Y)$ is a 0-band of subtrioids*

(i) $Q_{(i,j)}$, $(i,j) \in Y_{rb}$, *if* $n > 1$ *and* $|Y| > 1$*;*

(ii) $Q_{(i)}$, $i \in Y_{lz}$, *if* $n > 1$ *and* $|Y| > 1$*;*

(iii) $Q_{[i]}$, $i \in Y_{rz}$, *if* $n > 1$ *and* $|Y| > 1$.

Proof. We prove (i). It is obvious that in the case where $n > 1$ and $|Y| > 1$ one has $Q_{(i,j)}\backslash\{0\} \neq \varnothing$ for all $(i,j) \in Y_{rb}$. Moreover, $Q_{(i,j)}$, $(i,j) \in Y_{rb}$, is a subtrioid of $P_n^0(Y)$. Clearly,

$$P_n^0(Y) = \bigcup_{(i,j)\in Y_{rb}} Q_{(i,j)}, \ Q_{(i,j)} \cap Q_{(i',j')} = \{0\}$$

for $(i,j) \neq (i',j')$. It is immediate to verify that

$$Q_{(i,j)} \dashv Q_{(i',j')} \subseteq Q_{(i,j')}, \quad Q_{(i,j)} \vdash Q_{(i',j')} \subseteq Q_{(i,j')}, \quad Q_{(i,j)} \perp Q_{(i',j')} \subseteq Q_{(i,j')}$$

for any $(i,j), (i',j') \in Y_{rb}$. So, $P_n^0(Y)$ is a 0-band of subtrioids $Q_{(i,j)}$, $(i,j) \in Y_{rb}$.

The proofs of the remaining cases are similar. □

Assume

$$Q_{(i,j,k,s)} = \{w \in P_n^0(Y)\backslash\{0\} \mid (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) = (i,j,k,s)\} \cup \{0\}$$

for $i, j, k, s \in Y$, $n > 3$ and $|Y| > 1$;

$$Q_{(i,j,k)} = \{w \in P_n^0(Y)\backslash\{0\} \mid (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}) = (i,j,k)\} \cup \{0\},$$
$$Q_{[i,j,k]} = \{w \in P_n^0(Y)\backslash\{0\} \mid (w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) = (i,j,k)\} \cup \{0\}$$

for $i, j, k \in Y$, $n > 2$ and $|Y| > 1$; $Q_{[i,j]} = \{w \in P_n^0(Y)\backslash\{0\} \mid (w^{[0]}, w^{[1]}) = (i,j)\} \cup \{0\}$ for $i, j \in Y$, $n > 1$ and $|Y| > 1$.

The following two structural theorems give decompositions of $P_n^0(Y)$ into 0-tribands of subtrioids.

Theorem 3. *The free n-nilpotent trioid $P_n^0(Y)$ is a 0-triband of subtrioids*

(i) $Q_{(i,j,k,s)}$, $(i,j,k,s) \in \mathrm{FRT}(Y)$, *if* $n > 3$ *and* $|Y| > 1$;

(ii) $Q_{(i,j,k)}$, $(i,j,k) \in Y_{lz,rd}$, *if* $n > 2$ *and* $|Y| > 1$;

(ii) $Q_{[i,j,k]}$, $(i,j,k) \in Y_{rd,rz}$, *if* $n > 2$ *and* $|Y| > 1$;

(iv) $Q_{[i,j]}$, $(i,j) \in Y_{lz,rz}^{rb}$, *if* $n > 1$ *and* $|Y| > 1$.

Proof. We prove (i). It is easy to see that in the case where $n > 3$ and $|Y| > 1$ we get $Q_{(i,j,k,s)} \backslash \{0\} \neq \varnothing$ for all $(i,j,k,s) \in \mathrm{FRT}(Y)$. Furthermore, $Q_{(i,j,k,s)}$, $(i,j,k,s) \in \mathrm{FRT}(Y)$, is a subtrioid of $P_n^0(Y)$. Evidently,

$$P_n^0(Y) = \bigcup_{(i,j,k,s)\in \mathrm{FRT}(Y)} Q_{(i,j,k,s)}, \ Q_{(i,j,k,s)} \cap Q_{(i',j',k',s')} = \{0\}$$

for $(i,j,k,s) \neq (i^{'}, j^{'}, k^{'}, s^{'})$. It can be shown that

$$Q_{(i,j,k,s)} \dashv Q_{(i',j',k',s')} \subseteq Q_{(i,j,k,s')}, \quad Q_{(i,j,k,s)} \vdash Q_{(i',j',k',s')} \subseteq Q_{(i,j',k',s')},$$
$$Q_{(i,j,k,s)} \perp Q_{(i',j',k',s')} \subseteq Q_{(i,j,k',s')}$$

for any $(i,j,k,s), (i^{'}, j^{'}, k^{'}, s^{'}) \in \mathrm{FRT}(Y)$. Thus, $P_n^0(Y)$ is a 0-triband of subtrioids $Q_{(i,j,k,s)}$, $(i,j,k,s) \in \mathrm{FRT}(Y)$.

The proofs of the remaining cases are similar. □

Let

$$W_{(i,j,k)} = \{w \in P_n^0(Y) \backslash \{0\} \mid (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}) = (i,j,k)\} \cup \{0\},$$
$$W_{[i,j,k]} = \{w \in P_n^0(Y) \backslash \{0\} \mid (\widetilde{w}^{(0)}, w^{[1]}, \widetilde{w}^{(1)}) = (i,j,k)\} \cup \{0\}$$

for $i,j,k \in Y$, $n > 2$ and $|Y| > 1$;

$$W_{(i,j)} = \{w \in P_n^0(Y) \backslash \{0\} \mid (\widetilde{w}^{(0)}, w^{[0]}) = (i,j)\} \cup \{0\},$$
$$W_{[i,j]} = \{w \in P_n^0(Y) \backslash \{0\} \mid (\widetilde{w}^{(0)}, w^{[1]}) = (i,j)\} \cup \{0\},$$
$$W_{(i,j]} = \{w \in P_n^0(Y) \backslash \{0\} \mid (w^{[0]}, \widetilde{w}^{(1)}) = (i,j)\} \cup \{0\},$$
$$W_{[i,j)} = \{w \in P_n^0(Y) \backslash \{0\} \mid (w^{[1]}, \widetilde{w}^{(1)}) = (i,j)\} \cup \{0\},$$
$$W_{(i)} = \{w \in P_n^0(Y) \backslash \{0\} \mid w^{[0]} = i\} \cup \{0\}, \ W_{[i]} = \{w \in P_n^0(Y) \backslash \{0\} \mid w^{[1]} = i\} \cup \{0\}$$

for $i,j \in Y$, $n > 1$ and $|Y| > 1$.

Theorem 4. *The free n-nilpotent trioid $P_n^0(Y)$ is a 0-triband of subtrioids*

(i) $W_{(i,j,k)}$, $(i,j,k) \in (\mathrm{FRct}(Y))^{\dashv}$, *if* $n > 2$ *and* $|Y| > 1$;

(ii) $W_{[i,j,k]}$, $(i,j,k) \in (\mathrm{FRct}(Y))^{\vdash}$, *if* $n > 2$ *and* $|Y| > 1$;

(iii) $W_{(i,j)}$, $(i,j) \in (Y_{lz,rb})^{\dashv}$, *if* $n > 1$ *and* $|Y| > 1$;

(iv) $W_{[i,j]}$, $(i,j) \in (Y_{lz,rb})^{\vdash}$, *if* $n > 1$ *and* $|Y| > 1$;

(v) $W_{(i,j]}$, $(i,j) \in (Y_{rb,rz})^{\dashv}$, *if* $n > 1$ *and* $|Y| > 1$;

(vi) $W_{[i,j)}$, $(i,j) \in (Y_{rb,rz})^{\vdash}$, *if* $n > 1$ *and* $|Y| > 1$;

(vii) $W_{(i)}$, $i \in (Y_{lz,rz})^{\dashv}$, *if* $n > 1$ *and* $|Y| > 1$;

(viii) $W_{[i]}$, $i \in (Y_{lz,rz})^{\vdash}$, *if* $n > 1$ *and* $|Y| > 1$.

Proof. We prove (i). It is readily seen that in the case where $n > 2$ and $|Y| > 1$ one has $W_{(i,j,k)} \backslash \{0\} \neq \varnothing$ for all $(i,j,k) \in (\mathrm{FRct}(Y))^{\dashv}$. In addition, $W_{(i,j,k)}$, $(i,j,k) \in (\mathrm{FRct}(Y))^{\dashv}$, is a subtrioid of $P_n^0(Y)$. It is clear that

$$P_n^0(Y) = \bigcup_{(i,j,k)\in(\mathrm{FRct}(Y))^{\dashv}} W_{(i,j,k)}, \ W_{(i,j,k)} \cap W_{(i',j',k')} = \{0\}$$

for $(i,j,k) \neq (i',j',k')$. One can check that

$$W_{(i,j,k)} \dashv W_{(i',j',k')} \subseteq W_{(i,j,k')}, \ W_{(i,j,k)} \vdash W_{(i',j',k')} \subseteq W_{(i,j',k')}, \ W_{(i,j,k)} \perp W_{(i',j',k')} \subseteq W_{(i,j,k')}$$

for any $(i,j,k), (i',j',k') \in (\mathrm{FRct}(Y))^{\dashv}$. Therefore, $P_n^0(Y)$ is a 0-triband of subtrioids $W_{(i,j,k)}$, $(i,j,k) \in (\mathrm{FRct}(Y))^{\dashv}$.

The proofs of the remaining cases are similar. □

5. The least n-nilpotent congruence on a free trioid. In this section we present the least n-nilpotent congruence on a free trioid.

If $f \colon T_1 \to T_2$ is a homomorphism of trioids, then the corresponding congruence on T_1 will be denoted by Δ_f. If α is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that $(T, \dashv, \vdash, \perp)/\alpha$ is an n-nilpotent trioid (see Section 3), then we say that α is an n-nilpotent congruence.

Let $\mathrm{Frt}(Y)$ be a free trioid of an arbitrary rank (see Section 2). Fix $n \in \mathbb{N}$ and define a relation ν_n on $\mathrm{Frt}(Y)$ by

$$w_1 \nu_n w_2 \text{ if and only if } w_1 = w_2 \text{ or } l_{w_1} > n, \, l_{w_2} > n.$$

Theorem 5. *The relation* ν_n *is the least* n*-nilpotent congruence on the free trioid* $\mathrm{Frt}(Y)$.

Proof. Define a map $\xi \colon \mathrm{Frt}(Y) \to P_n^0(Y)$ by

$$w\xi = \begin{cases} w, & l_w \leq n, \\ 0, & l_w > n, \end{cases} \quad w \in \mathrm{Frt}(Y).$$

Take $w_1, w_2 \in \mathrm{Frt}(Y)$ and assume $l_{w_1 w_2} \leq n$. From $l_{w_1 w_2} \leq n$ it follows that $l_{w_1} < n$ and $l_{w_2} < n$. Then

$$(w_1 \dashv w_2)\xi = (w_1 \widetilde{w_2})\xi = w_1 \widetilde{w_2} = w_1 \prec w_2 = w_1\xi \prec w_2\xi,$$
$$(w_1 \vdash w_2)\xi = (\widetilde{w_1} w_2)\xi = \widetilde{w_1} w_2 = w_1 \succ w_2 = w_1\xi \succ w_2\xi,$$
$$(w_1 \perp w_2)\xi = (w_1 w_2)\xi = w_1 w_2 = w_1 \uparrow w_2 = w_1\xi \uparrow w_2\xi.$$

If $l_{w_1 w_2} > n$, then

$$(w_1 \dashv w_2)\xi = (w_1 \widetilde{w_2})\xi = 0 = w_1\xi \prec w_2\xi,$$
$$(w_1 \vdash w_2)\xi = (\widetilde{w_1} w_2)\xi = 0 = w_1\xi \succ w_2\xi, \ (w_1 \perp w_2)\xi = (w_1 w_2)\xi = 0 = w_1\xi \uparrow w_2\xi.$$

Consequently, ξ is a surjective homomorphism. According to Theorem 1 $P_n^0(Y)$ is a free n-nilpotent trioid of an arbitrary rank. Then Δ_ξ is the least n-nilpotent congruence on $\mathrm{Frt}(Y)$. From the definition of ξ it follows that $\Delta_\xi = \nu_n$. □

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