УДК 517.55

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WIMAN'S INEQUALITY FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES WITH RAPIDLY OSCILLATING COEFFICIENTS

A. O. Kuryliak, L. O. Shapovalovska. *Wiman's inequality for entire functions of several complex variables with rapidly oscillating coefficients*, Mat. Stud. **43** (2015), 16–26.

Let $\mathcal{E}^p$ be a class of entire functions of the form $f(z)=\sum_{\|n\|=0}^{+\infty}a_nz^n$, $\|n\|=n_1+\ldots+n_p$ $(p\geq 2)$, $z=(z_1,\ldots,z_p)\in\mathbb{C}^p$, and $\mathcal{K}(f,\theta)=\{f(z,t)=\sum_{\|n\|=0}^{+\infty}a_n\exp\{2\pi i\theta_nt\}r^n\colon t\in\mathbb{R}\}$, where $\{\theta_n\}$ is a sequence of positive integers such that its arrangement $\{\theta_k^*\}$ by increasing, i.e. $\{\theta_n\colon n\in\mathbb{Z}_+^p\}=\{\theta_k^*\colon k\geq 0\}$, $\theta_{k+1}^*>\theta_k^*$, satisfies the condition $\theta_{k+1}^*/\theta_k^*\geq q>1$ $(k\geq 0)$.

In this paper it is established that for $f\in\mathcal{E}^p$ almost surely for $t\in\mathbb{R}$ there exists a set $E(t)\subset\mathbb{R}_+^p$, such that for all $r\in\mathbb{R}_+^p\setminus E(t)$ the inequality

$$\mathfrak{M}_f(r,t)=\max_{|z|\leq r}|f(z,t)|\leq\mu_f(r)(\Lambda_f(r))^{1/4}\ln^3\Lambda_f(r)$$

holds, where $E(t)$ is a set of finite asymptotically logarithmic measure and $M_f(r)=\max\{|f(z,t)|\colon |z_i|=r_i,\ i\in\{1,\ldots,p\}\}$, $\mu_f(r)=\max_{n\in\mathbb{Z}_+^p}\{|a_n|r^n\colon r=(r_1,\ldots,r_p)\in\mathbb{R}_+^p\}$, $\Lambda_f(r)=\ln^p\mu_f(r)\cdot\prod_{i=1}^p\ln^{p-1}r_i$.

А. О. Курыляк, Л. О. Шаповаловска. *Неравенство Вимана для целых функций многих комплексных переменных с быстро колеблющимися коэффициентами* // Мат. Студії. – 2015. – Т.43, №1. – C.16–26.

Пусть $\mathcal{E}^p$ — класс целых функций вида $f(z)=\sum_{\|n\|=0}^{+\infty}a_nz^n$, $\|n\|=n_1+\ldots+n_p$ $(p\geq 2)$, $z=(z_1,\ldots,z_p)\in\mathbb{C}^p$, и $\mathcal{K}(f,\theta)=\{f(z,t)=\sum_{\|n\|=0}^{+\infty}a_n\exp\{2\pi i\theta_nt\}r^n\colon t\in\mathbb{R}\}$, где $\{\theta_n\}$ — последовательность натуральных чисел, упорядочение которой по возрастанию $\{\theta_k^*\}$, т.е. $\{\theta_n\colon n\in\mathbb{Z}_+^p\}=\{\theta_k^*\colon k\geq 0\}$, $\theta_{k+1}^*>\theta_k^*$, удовлетворяет условию $\theta_{k+1}^*/\theta_k^*\geq q>1$ $(k\geq 0)$. В статье доказано, что для $f\in\mathcal{E}^p$ почти наверное по $t\in\mathbb{R}$ существует множество $E(t)\subset\mathbb{R}_+^p$ такое, что для всех $r\in\mathbb{R}_+^p\setminus E(t)$ имеет место неравенство

$$\mathfrak{M}_f(r,t)=\max_{|z|\leq r}|f(z,t)|\leq\mu_f(r)(\Lambda_f(r))^{1/4}\ln^3\Lambda_f(r),$$

где $E(t)$ — множество асимптотически конечной логарифмической меры и $M_f(r)=\max\{|f(z,t)|\colon |z_i|=r_i,\ i\in\{1,\ldots,p\}\}$, $\mu_f(r)=\max_{n\in\mathbb{Z}_+^p}\{|a_n|r^n\colon r=(r_1,\ldots,r_p)\in\mathbb{R}_+^p\}$, $\Lambda_f(r)=\ln^p\mu_f(r)\cdot\prod_{i=1}^p\ln^{p-1}r_i$.

1. Introduction. In this paper we consider the class $\mathcal{E}^p$ of entire functions f of the form

$$f(z)=f(z_1,\ldots,z_p)=\sum_{\|n\|=0}^{+\infty}a_nz^n,\quad z=(z_1,\ldots,z_p)\in\mathbb{C}^p,\tag{1}$$

2010 *Mathematics Subject Classification:* 26A21, 30B20, 32A15.
Keywords: Wiman-Valiron's inequality; random entire function of several variables.
doi:10.15330/ms.43.1.16-26

where $z^n = z_1^{n} \dots z_p^{n_p}$, $p \in \mathbb{N}$, $p \geq 2$, $n = (n_1, \dots, n_p) \in \mathbb{Z}_+^p := (\mathbb{N} \cup \{0\})^p$, $\|n\| = \sum_{j=1}^p n_j$. For $r = (r_1, \dots, r_p) \in \mathbb{R}_+^p := (0, +\infty)^p$ and $f \in \mathcal{E}^p$ we denote

$$\Pi(r) = \{t \in \mathbb{R}_+^p \colon t_j \geq r_j,\ j \in \{1, \dots, p\}\},\ |r| = \sqrt{r_1^2 + \dots + r_p^2},$$
$$M_f(r) = \max\{|f(z)| \colon |z_j| \leq r_j,\ j \in \{1, \dots, p\}\},$$
$$\mu_f(r) = \max\{|a_n| r^n \colon n \in \mathbb{Z}_+^p\},\quad \mathfrak{M}_f(r) = \sum_{\|n\|=0}^{+\infty} |a_n| r^n,\ \ln_2 x = \ln\ln x.$$

Let $\mathcal{E}_0^p$ be the class of entire functions $f \in \mathcal{E}^p$ such that $\frac{\partial}{\partial z_j} f(z) \not\equiv 0$ in $\mathbb{C}^p$ for any $j \in \{1, \dots, p\}$. We say that a subset E of $\mathbb{R}_+^p$ is a *set of asymptotically finite logarithmic measure* ([1]) if E is Lebesque measurable in $\mathbb{R}_+^p$ and there exists an $R \in \mathbb{R}_+^p$ such that $E \cap \Pi(R)$ is a *set of finite logarithmic measure,* i.e.

$$\ln_p\text{–}\operatorname{meas}(E \cap \Pi(R)) := \int\limits_{E\cap\Pi(R)\setminus B_1} \cdots \int \prod_{j=1}^p \frac{dr_j}{r_j} < +\infty,\quad B_1 := \{r \in \mathbb{R}_+^p \colon |r| < 1\}.$$

For entire functions $f \in \mathcal{E}^p$ Wiman's type inequality can be found in [1]–[6], also analogues of this inequality without exceptional sets for entire functions $f \in \mathcal{E}^p$ can be found in [7]. A result of [4] proved for integrals implies the following statement.

Let e^K be the image of a set $K \subset \mathbb{R}^p$ by the mapping $r_1 = e^{\sigma_1}, \dots, r_p = e^{\sigma_p}$, and

$$\gamma(f) := \Big\{(\sigma_1, \dots, \sigma_p) \in \mathbb{R}^p \colon \lim_{t\to+\infty} \frac{1}{t} \ln M_f(e^{t\sigma_1}, \dots, e^{t\sigma_p}) = +\infty\Big\}.$$

Theorem A ([4]). *Let $f \in \mathcal{E}^p$. For every $\varepsilon > 0$ there exist a constant $C_0 = C_0(f, \varepsilon) > 0$ and a set $E \subset \mathbb{R}_+^p$ of finite logarithmic measure such that for an arbitrary cone $K \subset \mathbb{R}^p$ with vertex at the origin such that $\overline{K} \setminus \{0\} \subset \gamma(f)$, and for all $r \in e^K \setminus E$ the inequality*

$$M_f(r) \leq C_0 \mu_f(r) \prod_{i=1}^p \ln^{p-1} r_i \cdot (\ln \mu_f(r))^{p/2} (\ln_2 \mu_f(r))^{p+\varepsilon} := A_1(r) \tag{2}$$

holds.

In [3] the following assertion is proved: *for every $\varepsilon > 0$ there exist a positive constant C, independent of $f \in \mathcal{E}^p$, a set $E \subset \mathbb{R}_+^p$ of asymptotically finite logarithmic measure such that the inequality*

$$M_f(r) \leq C \mu_f(r) \big(\ln \mu_f(r) \cdot \ln(r_1 \cdots r_p)\big)^{p/2+\varepsilon} := A_2(r) \tag{3}$$

holds for all $r \notin E$.

In particular, if $\sum_{j=1}^p \ln_2^+ r_j = o(\ln_2 \mu_f(r))$ as $|r| \to +\infty$ then $A_1(r) = o(A_2(r))$ ($|r| \to +\infty$) and inequality (3) follows immediately from (2).

For $p = 2$ the exceptional set E in inequality (2) is "smaller" than the exceptional set in inequality (1.8) from Theorem 1 in [8]

$$M_f(r_1, r_2) \leq C \mu_f(r_1, r_2) \ln^+ \mu_f(r_1, r_2) \big(\ln_2^+ \mu_f(r_1, r_2) \cdots (\ln_k^+ \mu_f(r_1, r_2))^{1+\varepsilon}\big)^2 \quad (k \geq 3).$$

However, this inequality is the best possible ([8]) in the following sense: *there exist a function* $f \in \mathcal{E}^2$ *and a set* $E \subset \mathbb{R}^2_+$ *such that for all* $r \in E$

$$M_f(r) \geq \mu_f(r) \ln^2 \mu_f(r) \quad \text{and} \quad \int_{E\cap\Delta_R} \frac{dr_1 dr_2}{r_1 r_2} \geq c \cdot \ln R \quad (R \geq R_0 > 1),\ c > 0,$$

i.e. E *is a set of asymptotically infinite logarithmic measure, where* $\Delta_R = \{r = (r_1, r_2)\colon 1 \leq r_1 \leq R, 1 \leq r_2 \leq R\}$. This can be seen ([8]) for a function of the form $f(z_1, z_2) = f_0(z_1)$, where

$$f_0(z_1) = \sum_{k=1}^{+\infty} \frac{z_1^{n_k}}{n_k!}\Big(1 + \sum_{j=1}^{n_k^2} \frac{z_1^j}{(n_k)^j}\Big),$$

(n_k) is a sequence integer numbers n_k such that $n_{k+1} \geq n_k + n_k^2 + 1$ $(k \geq 1), n_1 = 1$. The function f_0 is an entire function of one variable for which (with $M_{f_0}(r_1)$ denoting the maximum modulus and $\mu_{f_0}(r_1)$ the maximum term) $M_{f_0}(r_1) > \mu_{f_0}(r_1)\big(\ln \mu_{f_0}(r_1)\big)^2$, for all r_1 in an unbounded set $E_0 = \bigsqcup_{k=1}^{+\infty}(t_k, T_k)$ of finite logarithmic measure, where $\big((t_k, T_k)\big)_{k=1}^{+\infty}$ is a disjoint system of nonempty open intervals $(t_k, T_k) \ni n_k$ $(k \geq 1)$. For $r = (r_1, r_2) \in E_0 \times [1, +\infty)$ we obtain $M_f(r) = M_{f_0}(r_1) > \mu_{f_0}(r_1)\big(\ln \mu_{f_0}(r_1)\big)^2 = \mu_f(r)\big(\ln \mu_f(r)\big)^2$.

It is easy to see that $f \notin \mathcal{E}_0^2$.

It was proved in [1] that for every $\varepsilon > 0$ there exist $R \in \mathbb{R}^p_+$ and a subset E of $\Pi(R)$ of asymptotically finite logarithmic measure such that the inequality

$$M_f(r) \leq \mu_f(r)\Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/2+\varepsilon} \tag{4}$$

holds for all $r \in \Pi(R)\backslash E$. Using methods of [1] we can prove the following “sharper” analogue of this inequality.

Theorem 1. *Let* $f \in \mathcal{E}_0^p$ *and* $\delta > 0$. *There exist* $R \in \mathbb{R}^p_+$ *and a set* $E \subset \Pi(R)$ *of asymptotically finite logarithmic measure such that the inequality*

$$M_f(r) \leq \mu_f(r)\Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/2} \cdot \ln^{5/2+\delta}\Big(\ln^p \mu_f(r) \cdot \prod_{i=1}^{p} \ln^{p-1} r_i\Big) \tag{5}$$

holds for all $r \in \Pi(R) \setminus E$.

We remark if $f \in \mathcal{E}_0^p$ and $f_j(z_j) := f(\ldots, 0, z_j, 0, \ldots)$ is the transcendental function of $z_j \in \mathbb{C}$ for any $j, 1 \leq j \leq p$ then $\Pi(R) \subset e^{\gamma(f)}$ for all $R \in (1, +\infty)^p$ because in this case $e_j := (0, \ldots, 0, \varepsilon_j, 0, \ldots, 0) \in \gamma(f)$ for arbitrary $\varepsilon_j > 0$ and any $j, 1 \leq j \leq p$.

Remark 1. There exist a set E of asymptotically infinite logarithmic measure such that for the entire function $g(z) = \exp\{\sum_{j=1}^{p} z_i\}$, each $\varepsilon > 0$ and $r \in E$ we have

$$M_g(r) \geq \mu_g(r)\Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_g(r)\Big)^{1/2-\varepsilon}$$

Therefore exponent 1/2 in inequality (4) cannot be replaced with a number smaller than 1/2 ([5, 6]). In this connection the following question arrives naturally: *how can one describe the "quantity" of those entire functions, for which inequality* (4) *can be improved?*

Let $Z = (Z_n(t))$ be a sequence of complex-valued random variables $Z_n(t) = X_n(t)+iY_n(t)$ such that both $X = (X_n(t))$ and $Y = (Y_n(t))$ are real multiplicative systems (MS) uniformly bounded by the number 1 ([5, 6]), which is defined on Steinhaus's probability space. In [5, 6] it was proved that for the class $\mathcal{K}(f, Z) = \{f(z, t) = \sum_{\|n\|=0}^{+\infty} a_n Z_n(t) z^n : t \in \mathbb{R}\}$ exponent 1/2 can be replaced with exponent 1/4 in inequality (5) almost surely (Levy's phenomenon).

In this paper we consider the class $\mathcal{K}(f, \theta)$ of entire functions with $\theta = (e^{2\pi i \theta_n t})$. Here (θ_n) is a sequence of positive integers such that its arrangement (θ_k^*) by increasing $\{\theta_n : n \in \mathbb{Z}_+^p\} = \{\theta_k^* : k \in \mathbb{Z}_+\}$, $\theta_{k+1}^* > \theta_k^*$ $(k > 0)$, satisfies the condition

$$\theta_{k+1}^*/\theta_k^* \geq q > 1, \ k \geq 0. \tag{6}$$

Remark, that in the case $q \geq 2$ entire functions of the form

$$f(z, t) = \sum_{\|n\|=0}^{+\infty} a_n e^{i\theta_n t} z^n, \ t \in \mathbb{R} \tag{7}$$

satisfy the assumptions of Theorem 2 from [5], because $(\cos \theta_n t)$, $(\sin \theta_n t)$ are MS. But in the case $q > 1$ the sequence of random variables $(\cos \theta_n t)_{n \in \mathbb{Z}_+^p}$ need not be a MS (see [9]).

Remark that in [9] Fenton's inequality ([8]) was improved for random entire functions $f \in \mathcal{E}^2$ and in [10, 11] for entire function $f \in \mathcal{E}^2$ of form (7). In papers [5, 6] inequality (4) was improved for random entire functions $f \in \mathcal{E}^p$, $p \geq 2$. A result from [5, 6] implies the same result for the class $\mathcal{K}(f, \theta)$ and $f \in \mathcal{E}_0^p$ by condition (6) with $q \geq 2$.

We consider the case where condition (6) is satisfied for $q > 1$ and improve inequality (5) also in this case, i.e. we consider the posed above question for the class $\mathcal{K}(f, \theta)$ with $f \in \mathcal{E}_0^p$.

Theorem 2. *Let $f(z, t) \in \mathcal{K}(f, \theta)$ be an entire function of the form (7), $f(z) \in \mathcal{E}_0^p$ and a sequence of a positive integers $(\theta_n)_{n \in \mathbb{Z}_+^p}$ satisfies condition* (6). *Then almost surely for $t \in \mathbb{R}$ there exist $R \in \mathbb{R}_+^p$ and a subset E of $\Pi(R)$ of finite logarithmic measure such that for all $r \in \Pi(R) \setminus E$ the inequality*

$$\mathfrak{M}_f(r, t) = \max\{f(z, t) : |z| \leq r\} \leq \mu_f(r)(\Lambda_f(r))^{1/4} \ln^3 \Lambda_f(r) \tag{8}$$

holds, where $\Lambda_f(r) = \ln^p \mu_f(r) \cdot \prod_{i=1}^p \ln^{p-1} r_i$, in particular

$$\mathfrak{M}_f(r, t) \leq \mu_f(r)(\Lambda_f(r))^{1/4+\varepsilon}, \tag{9}$$

for each $\varepsilon > 0$.

In the class $\mathcal{K}(f, Z)$ of random entire functions, where $Z = (Z_n)$ is a uniformly bounded complex-valued MS and $f \in \mathcal{E}_0^p$, inequality (9) was proved in [5, 6].

In [12] the "quantity" of those entire functions of complex variable for which classical Wiman-Valiron's inequality can be improved, is described in the sense of the Baire category. A similar question for analytic functions in the unit disc and for entire functions from the class $\mathcal{K}(f, \theta)$ with $f \in \mathcal{E}^1$ and a sequence (θ_k) that satisfies the condition (6) was considered in [13, 16], respectively.

2. Auxiliary lemmas.

Lemma 1 ([16]). *Let* $(\theta_k^*)_{k=1}^N$ *be a sequence of integers such that condition* (6) *holds. Then there exist constants* A_q *and* B_q *(depending only on* q*) such that for any* $\{b_k\colon 1\le k\le N\}\subset\mathbb{C}$ *and* $\lambda>0$ *we have*

$$P\Big\{t\colon \Big|\sum_{k=1}^N b_k e^{2\pi i\theta_k^* t}\Big|\ge A_q\lambda S_N\Big\}\le B_q e^{-\lambda^2},$$

where $S_N^2=\sum_{k=1}^n|b_k|^2$, P *is the Lebesque measure on* $[0;1]$.

Lemma 2. *Let* $\theta=(\theta_n)_{n\in\mathbb{Z}_+^p}$ *be a sequence of integers, which satisfies* (6)*. Then for any* $\beta>0, p\ge 1$, $l\in\mathbb{N}$, $l\ge p$ *and* $\{c_n\colon n\in\mathbb{Z}_+^p\}\subset\mathbb{C}$ *we get*

$$P\Big\{t\colon \max\Big\{\Big|\sum_{\|n\|=0}^l c_n\exp\Big\{\sum_{s=1}^p in_s\psi_s+2\pi i\theta_n t\Big\}\Big|\colon \psi\in[0,2\pi]^p\Big\}\ge A_{\beta p}S_l\ln^{1/2}l\Big\}\le\frac{(5\pi+1)^pB}{l^\beta},$$

where $S_l^2=\sum_{\|n\|=0}^l|c_n|^2$, $A=\sqrt{\beta+\frac{p}{2}(3+p)}A_q+1$ *and* $B=B_q$ *(*A_q,B_q *are constants from Lemma 1).*

Proof. Let $M=[5\pi l^{3p/2+1}]+1$, $\psi_{k,j_k}=\frac{2\pi j_k}{M}$, $j_k\in\{1,2,\dots,M\}$, $k\in\{1,\dots,M\}$.

$$q(\psi,t)=\Big|\sum_{\|n\|=0}^l c_n\exp\Big\{\sum_{s=1}^p in_s\psi_s+2\pi i\theta_n t\Big\}\Big|\le\Big|\sum_{\|n\|=0}^l c_n\exp\Big\{\sum_{s=1}^p in_s\psi_{s,j_s}+2\pi i\theta_n t\Big\}\Big|+$$
$$+\Big|\sum_{\|n\|=0}^l c_n\exp\Big\{\sum_{s=1}^p in_s\psi_s\Big\}-\exp\Big\{\sum_{s=1}^p in_s\psi_{s,j_s}e^{2\pi i\theta_n t}\Big|\Big\}.$$

Then using Cauchy-Bunyakovsky's inequality and $|e^{ia}-e^{ib}|\le a-b$ $(a,b\in\mathbb{R})$ we get for $\psi_s\in[\psi_{s,j_s},\psi_{s,j_s+1}]$, $s\in\{1,\dots,p\}$

$$\Big|\sum_{\|n\|=0}^l c_n\Big(\exp\Big\{\sum_{s=1}^p in_s\psi_s\Big\}-\exp\Big\{\sum_{s=1}^p in_s\psi_{s,j_s}\Big\}\Big)e^{2\pi i\theta_n t}\Big|\le S_l\Big(\sum_{\|n\|=0}^l\sum_{s=1}^p\Big|n_s(\psi_s-\psi_{s,j_s})\Big|^2\Big)^{1/2}\le$$
$$\le S_l\Big(\sum_{\|n\|=0}^l\sum_{s=1}^p\Big|n_s\frac{2\pi}{M}\Big|^2\Big)^{1/2}=\frac{2\pi}{M}S_l\Big(\sum_{\|n\|=0}^l\sum_{s=1}^p n_s^2\Big)^{1/2}=\frac{2\pi}{M}S_l\Big(\sum_{s=1}^p\sum_{\|n\|=0}^l n_s^2\Big)^{1/2}\le$$
$$\le\frac{2\pi}{M}S_l\sqrt{\frac{p}{(p-1)!}}\cdot l\cdot\Big(\sum_{s=0}^l(l+p-1)^{p-1}\Big)^{1/2}=\frac{2\pi}{M}S_l\sqrt{\frac{p}{(p-1)!}}\cdot l^{3/2}(l+p-1)^{\frac{p-1}{2}}\le$$
$$\le\frac{2\pi}{M}S_l\sqrt{\frac{p}{(p-1)!}}\cdot l^{3/2}(2l)^{\frac{p-1}{2}}=\frac{2\pi}{M}S_l\sqrt{\frac{p}{(p-1)!}}(\sqrt{2})^{p-1}\cdot l^{p/2+1}\le\frac{2\pi}{M}S_l\sqrt{6}\cdot l^{p/2+1}<$$
$$<\frac{5\pi}{M}S_l\cdot l^{p/2+1}<S_l.$$

Therefore,

$$\max\{q(\psi,t)\colon\psi\in[0,2\pi]^p\}\le\max\Big\{\Big|\sum_{\|n\|=0}^l c_n\exp\Big\{\sum_{s=1}^p in_s\psi_{s,j_s}+2\pi i\theta_n t\Big\}\Big|\Big\}\colon$$

$$: j_s \in \{1, 2, \ldots, M\},\ s \in \{1, 2, \ldots, p\}\Big\} + S_l.$$

For $j \in \mathbb{N}$ and $n \in \mathbb{Z}_+^p$ such that $\theta_j^* = \theta_n$ we denote $b_j = c_n \exp\Big\{\sum_{s=1}^p in_s\psi_s\Big\}$, where $\varphi \in [0, 2\pi]^p$ is fixed. By Lemma 1 with $\lambda = \sqrt{\beta + \frac{p}{2}(3+p)}$ and

$$N = \sum_{\|n\|=0}^{l} 1 = \sum_{s=0}^{l} C_{p+s-1}^s \le \frac{l}{(p-1)!}(l+p-1)^{(p-1)} \le \frac{l^p}{(p-1)!}2^{p-1} \le 2l^p$$

we get

$$P\Big\{t\colon \Big|\sum_{\|n\|=0}^{l} c_n \exp\Big\{\sum_{s=1}^{p} in_s\psi_s + 2\pi i\theta_n t\Big\}\Big| \ge \sqrt{\beta + \frac{p}{2}(3+p)} AS_l \ln^{1/2} l\Big\} \le$$
$$\le B\exp\Big\{-\Big(\beta + \frac{p}{2}(3+p)\Big)\ln l\Big\} \le B\exp\Big\{-\Big(\beta + \frac{p}{2}(3+p)\Big)\ln l\Big\} \le \frac{B}{l^{3p/2+p^2/2+\beta}}.$$

Finally we get

$$P\left\{t\colon\ \max\{q(\psi, t)\colon \psi \in [0, 2\pi]^p\} \ge \Big(\sqrt{\beta + \frac{p}{2}(3+p)}A + 1\Big)S_l \ln^{1/2} l\right\} \le$$
$$\le P\left\{t\colon\ \max\{q(\psi_{1,j_1}, \psi_{2,j_2}, \ldots, \psi_{p,j_p}, t)\colon j_s \in \{1, 2, \ldots, M\},\ s \in \{1, 2, \ldots, p\}\} + S_l \ge\right.$$
$$\left.\ge \Big(\sqrt{\beta + \frac{p}{2}(3+p)}A + 1\Big)S_l \ln^{1/2} l\right\} \le P\Big\{t\colon\ \max\{q(\psi_{1,j_1}, \psi_{2,j_2}, \ldots, \psi_{p,j_p}, t)\colon$$
$$j_s \in \{1, 2, \ldots, M\},\ S \in \{1, 2, \ldots, p\}\} \ge \sqrt{\beta + \frac{p}{2}(3+p)} \cdot AS_l \ln^{1/2} l\Big\} \le$$
$$\le \sum_{s=1}^{p}\sum_{j_s=1}^{M} P\Big\{t\colon q(\psi_{1,j_1}, \psi_{2,j_2}, \ldots, \psi_{p,j_p}, t) \ge \sqrt{\beta + \frac{p}{2}(3+p)} \cdot AS_l \ln^{1/2} l\Big\} \le$$
$$\le \frac{M^p B}{l^{3p/2+p^2/2+\beta}} \le \frac{(5\pi+1)^p l^{3p/2+p^2/2} B}{l^{p+p^2/2+\beta}} = \frac{(5\pi+1)^p B}{l^\beta}. \qquad \square$$

Lemma 3 ([1])**.** *Let $h\colon \mathbb{R}_+^p \to \mathbb{R}_+$ satisfy*

$$\int_1^{+\infty} \ldots \int_1^{+\infty} \frac{du_1 \ldots du_p}{h(u)} < +\infty.$$

Then there exist $R \in \mathbb{R}_+^p$ and a subset E of $\Pi(R)$ of finite logarithmic measure such that for all $r \in \Pi(R) \setminus E$ and $s \in \{1, \ldots, p\}$ we have

$$\sum_{\|n\|=0}^{+\infty} n_s|a_n|r^n \le h(\ln r_1, \ldots, \ln r_{s-1}, \ln \mathfrak{M}_f(r), \ln r_{s+1}, \ldots, \ln r_p).$$

Lemma 4. *There exist $R\in\mathbb{R}^p_+$ and a subset E of $\Pi(R)$ of finite logarithmic measure such that for all $r\in\Pi(R)\setminus E$ we have*

$$\sum_{\|n\|=0}^{+\infty}\|n\|a_nr^n\le\mu_f(r)\cdot\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)^{1/2}\ln^{\frac{5}{2}+\delta}\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)\times$$
$$\times\ln\mu_f(r)\prod_{i=1}^p\ln r_i\cdot\ln_2^{1+\delta}\mu_f(r)\prod_{i=1}^p\ln_2^{1+\delta}r_i,\ \ \delta>0.$$

Proof. Let $h(r)=\prod_{i=1}^p r_i\ln^{1+\delta_1}r_i$, $\delta_1>0$. By Lemma 3, there exist $R\in\mathbb{R}^p_+$ and subsets E_j of $B(R_j)$ of finite logarithmic measure such that for all $r\in B(R_j)\setminus E_j$ $(j\in\{1,\dots,p\})$ we have

$$\sum_{\|n\|=0}^{+\infty}n_j|a_n|r^n\le\mathfrak{M}_f(r)\ln\mathfrak{M}_f(r)\ln_2^{1+\delta}\mathfrak{M}_f(r)\prod_{i=1,i\ne j}^p\ln r_i\ln^{1+\delta}r_i.$$

Therefore for $r\in\Pi(R)\setminus\Big(\bigcup_{i=1}^p E_i\Big)$ we obtain

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n\le\mathfrak{M}_f(r)\ln\mathfrak{M}_f(r)\ln_2^{1+\delta}\mathfrak{M}_f(r)\cdot\sum_{j=1}^p\Big(\prod_{i=1,i\ne j}^p\ln r_i\ln_2^{1+\delta}r_i\Big)\le$$
$$\le p\mathfrak{M}_f(r)\ln\mathfrak{M}_f(r)\ln_2^{1+\delta}\mathfrak{M}_f(r)\prod_{i=1}^p\ln r_i\ln_2^{1+\delta}r_i,$$

where $\Pi(R)\subset\Big(\bigcap_{j=1}^p B(R_j)\Big)\cap[e;+\infty)^p$. By Theorem 1 we get for $r\in\Pi(R)\setminus\Big(\bigcup_{i=1}^{p+1}E_i\Big)$

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n\le\mu_f(r)\cdot\Big(\prod_{i=1}^p\ln^{p-1}r_i\cdot\ln^p\mu_f(r)\Big)^{1/2}\ln^{\frac{5p}{2}+\delta}\Big(\ln^p\mu_f(r)\cdot\prod_{i=1}^p\ln^{p-1}r_i\Big)\times$$
$$\times\Big\{\ln\mu_f(r)+\frac{p-1}{2}\sum_{i=1}^p\ln_2 r_i+\frac{p}{2}\ln_2\mu_f(r)+\Big(\frac{5p}{2}+\delta\Big)\ln_2\Big(\ln^p\mu_f(r)\cdot\prod_{i=1}^p\ln^{p-1}r_i\Big)\Big\}\times$$
$$\times\ln^{1+\delta}\Big\{2\ln\mu_f(r)+\frac{p}{2}\sum_{i=1}^p\ln_2 r_i\Big\}\le\mu_f(r)\cdot\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)^{1/2}\times$$
$$\times\ln^{\frac{5p}{2}+2\delta}\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)\ln\mu_f(r)\cdot\ln_2^{1+2\delta}\mu_f(r)\cdot\prod_{i=1}^p\ln r_i\ln_2^{1+2\delta}r_i=$$
$$=\mu_f(r)\cdot\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)^{1/2}\ln^{\frac{5p}{2}+\delta}\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)\times$$
$$\times\ln\mu_f(r)\cdot\prod_{i=1}^p\ln r_i\cdot\ln_2\mu_f(r)\cdot\prod_{i=1}^p\ln_2^{1+\delta}r_i,$$

where E_{p+1} is the exceptional set from Theorem 1 and $\delta_1>2\delta$. □

3. Proofs.

Proof of Remark 1. For function $g_0(z) = e^z$ we have (see, for example [17])

$$\lim_{r\to\infty} \frac{M_{g_0}(r)}{\mu_{g_0}(r)\ln^{1/2}\mu_{g_0}(r)} = \sqrt{2\pi}.$$

Then for $r \in (r_0, +\infty)^p$ we get $M_g(r) > \mu_g(r)\prod_{i=1}^{p}\ln^{1/2}\mu_{g_0}(r_i)$. Denote $\psi(r) = \ln\mu_g(r)$. Remark that

$$A_t = \Big\{r\colon r_1 = t; r_i \in (t_1, t_2) = \Big(\psi^{-1}(\psi(r_1)/2), \psi^{-1}(2\psi(r_1))\Big)\Big\} \subset$$
$$\subset \Big\{r\colon \prod_{i=1}^{p}\psi(r_i) \geq \frac{1}{2^{p-1}(2p-1)}\Big(\sum_{i=1}^{p}\psi(r_i)\Big)^p\Big\}.$$

Indeed, if $r \in A_t$ then

$$\prod_{i=1}^{p}\psi(r_i) = \psi(r_1)\prod_{i=2}^{p}\psi(r_i) > \psi(r_1)\prod_{i=2}^{p}\frac{\psi(r_1)}{2} = \frac{\psi^p(r_1)}{2^{p-1}} =$$
$$= \frac{1}{2^{p-1}(2p-1)}(\psi(r_1) + 2\psi(r_1) + \ldots + 2\psi(r_1)) > \frac{1}{2^{p-1}(2p-1)}\Big(\sum_{i=1}^{p}\psi(r_i)\Big)^p.$$

For $r \in A = \bigcup_{r=r_0}^{+\infty} A_t$ we obtain

$$M_g(r,t) > \mu_g(r)\prod_{i=1}^{p}\ln^{1/2}\mu_{g_0}(r_i) > \mu_g(r)\frac{1}{2^{p-1}(2p-1)}\Big(\sum_{i=1}^{p}\ln\mu_{g_0}(r_i)\Big)^{p/2} >$$
$$> \mu_g(r)\ln^{1/2}\mu_g(r)\cdot\frac{1}{2^{p-1}(2p-1)} > \mu_g(r)\Big(\prod_{i=1}^{p}\ln^{p-1}r_i\cdot\ln^p\mu_f(r)\Big)^{1/2-\varepsilon}.$$

It remains to remark that the set A has infinite asymptotically logarithmic measure ([5, 6]). □

Proof Theorem 2. As in [5, 6] for $k \in \mathbb{N}$ we denote

$$G_k = \{r = (r_1, \ldots, r_p) \in \mathbb{R}_+^p\colon\ k \leq \ln\mu_f(r) < k+1\} \cap [1; +\infty)^p.$$

Then $G_k \neq 0$ for $k \geq k_0$. From

$$\lim_{r_j\to+\infty}\mu_f(r_1^0, \ldots, r_{j-1}^0, r_j, r_{j+1}^0, \ldots, r_p^0) = +\infty,\ j \in \{1, \ldots, p\}$$

(see [5, 6]) we deduce that G_k is a bounded set for all $k \in \mathbb{N}$. Let $G_k^* = \bigcup_{j=k}^{+\infty} G_j$. Remark that $\ln_p$-meas$\bigcup_{j=1}^{k_0-1} G_j < +\infty$.

Denote $E_3 = E_2 \cup E_1 \cup \Big(\bigcup_{j=1}^{k_0-1} G_j\Big)$, where E_1 and E_2 are the exceptional sets from Theorem 1 and Lemma 3, respectively. So, for

$$d = d(r) = \Big(\ln\mu_f(r)\Big)^{\frac{p+4}{4}}\cdot\Big(\prod_{i=1}^{p}\ln r_i\Big)^{\frac{p+3}{4}}$$

and for $r\in\Pi(R)\setminus E_3$ we get

$$\sum_{\|n\|\ge d}|a_n|r^n\le\sum_{\|n\|\ge d}\frac{\|n\|}{d}|a_n|r^n=\frac{1}{d}\sum_{\|n\|\ge d}\|n\||a_n|r^n\le$$

$$\le\mu_f(r)\ln^{\frac{p}{2}+1-\frac{p+4}{4}}\mu_f(r)\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p+1}{2}-\frac{p+3}{4}}\ln^{\frac{5}{2}+\delta}\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big)\cdot\ln_2^{1+\delta}r_i=$$

$$=\mu_f(r)(\ln\mu_f(r))^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p}{4}}\ln^{\frac{5}{2}+\delta}\Big(\ln^p\mu_f(r)\prod_{i=1}^p\ln^{p-1}r_i\Big).$$

Let $G_k^*=G_k\setminus E_3$. By I we denote the set $\{k\colon k\ge k_0,\ G_k^*\ne 0\}$. Then $\#I=+\infty$. For $k\in I$ we may choose a sequence $r^{(k)}\in G_k^*$. So, for all $r\in G_k^*$ we obtain

$$\mu_f(r^{(k)})<e^{k+1}\le e\mu_f(r),\ \ \mu_f(r)<e^{k+1}<e\mu_f(r^{(k)}) \tag{10}$$

and also $[1;+\infty)^p\setminus E_3=\bigcup_{k\in I}G_k^*$. For $k\in I$ we denote $N_k=[d_1(r^{(k)})]$, where

$$d_1(r)=\ln(e\mu_f(r))^{\frac{p+4}{4}}\cdot\Big(\prod_{i=1}^p\ln^{p-1}r_i\Big)^{\frac{p+3}{4}}$$

and for $r\in G_k^*$

$$W_{N_k}(r,t)=\Big\{\Big|\sum_{\|n\|=0}^{N_k}a_nr^n\exp\Big\{\sum_{s=1}^p in_s\psi_s+2\pi i\theta_nt\Big\}\Big|\colon\psi\in[0,2\pi]^p\Big\}.$$

For a Lebesgue measurable set $G\subset G_k^*$ and for $k\in I$ we denote

$$\nu_k(G)=\frac{\mathrm{meas}_p(G)}{\mathrm{meas}_p(G_k^*)},$$

where meas_p denotes Lebesgue measure on $\mathbb{R}^p$.

Note that ν_k is a probability measure defined on the family of Lebesgue measurable subsets of G_k^*. Let $\Omega=\bigcup_{k\in I}G_k^*$ and $I=\{k_j\colon j\ge1\}\subset\mathbb{N}$, where $k_j<k_j+1$, $j\ge1$.

For a Lebesgue measurable subset G of Ω we denote

$$\nu(G)=\sum_{j=0}^{\infty}\frac{1}{\nu_{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big)\nu_{k_j+1}\Big(G\cap G_{k_j+1}^*\Big), \tag{11}$$

where $k_0=0$. Therefore ν is a probability measure, which is defined on measurable subsets of Ω. On $[1;0]\times\Omega$ we define the probability measure $P_0=P\times\nu$, which is the direct product of the probability measures P and ν. Now for $k\in I$ we define

$$F_k=\Big\{(t,r)\in[0,1]\times\Omega\colon W_{N_k}(r,t)>A_p\varphi^{1/2}(2N_k^p)S_{N_k}(r)\ln^{1/2}N_k\Big\},$$

where $S_{N_k}^2(r)=\sum_{\|n\|=0}^{N_k}|a_n|^2r^{2n}$ and A_p is the constant from Lemma 2 with $\beta=3$. Using Fubini's theorem and Lemma 2 with $c_n=a_nr^n$ and $\beta=3$, we get for $k\in I$

$$P_0(F_k)=\int_\Omega\int_{F_k(r)}dPd\nu=\int_\Omega P(F_k(r))d\nu\le\frac{(5\pi+1)^pB}{N_k}\nu(\Omega)=\frac{(5\pi+1)^pB}{N_k}.$$

Note that $N_k^3 > \ln^2 \mu_f(r^{(k)}) \geq k^2$. Therefore, $\sum_{k\in I} P_0(F_k) \leq (5\pi+1)^p B\cdot\sum_{k\in I}\frac{1}{N_k^3} < +\infty$. By Borel-Cantelli's lemma the infinite quantity of the events $\{F_k\colon k\in I\}$ may occur with probability zero. Then for any point $(t,r)\in F$ there exists $k_0=k(t,r)$ such that for all $k\geq k_0$ we obtain $W_{N_k}(r,t)\leq A_p S_{N_k}(r)\ln^{1/2} N_k$.

By F_Ω we denote the projection of F on Ω, i.e. $F_\Omega=\{r\in\Omega\colon(\exists t)[(t,r)\in F]\}$. Then $\nu(F_\Omega\cap G_k^*)=1$ for any $k\in I$ ([9]). Similarly, for the projection of F on $[0,1]$, $F_{[0,1]}=\cup_{r\in\Omega}F(r)$, we have $P(F_{[0,1]})=1$. Let $F^\wedge(t)=\{r\in\Omega\colon(t,r)\in F\}$. Then also $\nu(F^\wedge(t))=1$ ([4]).

There exists a subset $F_1\subset F_{[0,1]}$ such that $P(F_1)=1$ and $\nu(F^\wedge(t))=1$ for all $t\in F_1$. Observe that $(\forall t\in F_1)\colon \nu(F^\wedge(t))=1$ implies that $(\forall k\in I)\colon \nu(F^\wedge(t)\cap G_k^*)=1$ ([9]).

For any $t\in F_1$ and $k\in I$ we can choose $r_0^{(k)}(t)\in G_k^*$ such that $W_{N_k}(r_0^{(k)}(t),t)\geq\frac{3}{4}M_k(t),\ M_k(t)=\sup\{W_{N_k}(r,t)\colon r\in G_k^*\}$.

Then from $\nu_k(F^\wedge(t)\cap G_k^*)=1$ for all $k\in I$ we have that there exists a point $r^{(k)}(t)\in G_k^*\cap F^\wedge(t)$ such that $|W_{N_k}(r_0^{(k)}(t),t)-W_{N_k}(r^{(k)}(t),t)|<M_k(t)/4$ or $\frac{3}{4}M_k(t)\leq W_{N_k}(r_0^{(k)}(t),t)\leq W_{N_k}(r^{(k)}(t),t)+\frac{1}{4}M_k(t)$.

Since $(t,r^{(k)}(t))\in F$, from inequality (8) we get

$$\frac{1}{2}M_k(t)\leq W_{N_k}(r^{(k)}(t),t)\leq A_p\varphi^{1/2}(2N_k^p)S_{N_k}(r^{(k)}(t))\ln^{1/2}N_k. \tag{12}$$

So for $r^{(k)}=r^{(k)}(t)$ by Theorem 1 we have $S_N^2(r^{(k)})\leq\mu_f^2(r^{(k)})\Lambda_f^{1/2}(r^{(k)})\ln^{\frac{5}{2}+\delta}\Lambda_f(r^{(k)})$. Then for $t\in F_1$ and all $k\geq k_0(t)$, $k\in I$ we obtain $S_N(r^{(k)})\leq\mu_f(r^{(k)})\Lambda_f^{1/4}(r^{(k)})\ln^{\frac{5}{4}+\delta}\Lambda_f(r^{(k)})$.

It follows from (7) that $d_1(r^{(k)})\geq d(r)$ for $r\in G_k^*$. Then from $t\in F_1$, $r\in F^\wedge(t)\cap G_k^*$, $k\in I$ and $k\geq k_0(t)$ we get

$$M_f(r)\leq\sum_{\|n\|>d_1(r^{(k)})}|a_n|r^n+W_{N_k}(r,t)\leq\sum_{\|n\|\geq d(r)}|a_n|r^n+M_k(t).$$

Finally, for $t\in F_1$, we have

$$M_f(r,t)\leq\mu_f(r)\Big(\ln\mu_f(r)\Big)^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p}{4}}\ln^{\frac{5}{2}+\delta}\Lambda_f(r)+AS_{N_k}(r^{(k)})\ln^{1/2}N_k\leq$$
$$\leq\mu_f(r)\Big(\ln\mu_f(r)\Big)^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p}{4}}\cdot\ln^{\frac{5}{2}}\Lambda_f(r)+A_p\mu_f(r^{(k)})\ln^{\frac{p}{4}}\mu_f(r^{(k)})\times$$
$$\times\Big(\prod_{i=1}^p\ln^{p-1}r_i^{(k)}\Big)^{\frac{1}{4}}\cdot\ln^{\frac{5}{2}+\delta}\Lambda_f(r^{(k)})\Big(\frac{p+4}{4}\ln_2(e\mu_f(r^{(k)}))+\frac{p+3}{4}\sum_{i=1}^p\ln_2 r_i^{(k)}\Big)^{\frac{1}{2}+\delta}\leq$$
$$\leq\mu_f(r)\Big(\ln\mu_f(r)\Big)^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p-1}{4}}\cdot\ln^{\frac{5}{2}}\Lambda_f(r)+\mu_f(r)\Big(\ln\mu_f(r)\Big)^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p-1}{4}}\times$$
$$\times\ln^{3-\delta}\Lambda_f(r)=\mu_f(r)\Big(\ln\mu_f(r)\Big)^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p-1}{4}}\Big(\ln^{\frac{5}{2}}\Lambda_f(r)+\ln^{3-\delta}\Lambda_f(r)\Big)\leq$$
$$\leq\mu_f(r)\Big(\ln\mu_f(r)\Big)^{\frac{p}{4}}\cdot\Big(\prod_{i=1}^p\ln r_i\Big)^{\frac{p-1}{4}}\cdot\ln^3\Lambda_f(r)=\mu_f(r)\Lambda_f^{1/4}(r)\cdot\ln^3\Lambda_f(r).$$

It remains to remark that $\nu(G^*)$ defined in (11) satisfies $\nu(G^*)=\nu(\bigcup_{k\in I}(G_k^*\setminus F^\wedge(t)))=\sum_{k\in I}(\nu(G_k^*)-\nu(F^\wedge(t)))=0$.

Then for all $k \in i$ we obtain

$$\nu_k(G_k^* \setminus F^\wedge(t)) = \frac{\operatorname{meas}(G_k^* \setminus F^\wedge(t))}{\operatorname{meas} G_k^*} = 0,\ \operatorname*{meas}_p(G_k^* \setminus F^\wedge(t)) = 0,\ \operatorname*{meas}_p(G^* \setminus F^\wedge(t)) = 0. \square$$

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Received 15.07.2014