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A. Savchenko, M. Zarichnyi

OPEN PROBLEMS IN THE THEORY OF FUZZY METRIC SPACES

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We formulate some open problems concerning the space of compact fuzzy metric (measure) spaces.

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Сформулированы некоторые открытые проблемы, касающиеся пространства компактных нечетких метрических пространств (с мерой).

1. Introduction. The notion of fuzzy metric space, in one of its forms, was introduced by Kramosil and Michalek [1]. In this note we use the version of this concept given in the paper [2] by George and Veeramani.

Definition 1. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is a *continuous t-norm* if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a*1=a$ for all $a\in[0,1]$;

(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, and $a,b,c,d\in[0,1]$.

The following are examples of t-norms:

$$a*b=ab,\ a*b=\min\{a,b\},\ a*b=\max\{a+b-1,0\}$$

(Łukasiewicz t-norm).

Definition 2. A 3-tuple $(X,M,*)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$:

(i) $M(x,y,t)>0$;

(ii) $M(x,y,t)=1$ if and only if $x=y$;

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(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(v) the function $M(x, y, -): (0, \infty) \to [0, 1]$ is continuous.

The fuzzy Gromov-Hausdorff metric is introduced in [5]. Given a fuzzy metric M, by M_H we denote the Hausdorff fuzzy metric defined in [7]. Let $(X_i, M_i, *)$, $i \in \{1, 2\}$, be compact fuzzy metric spaces. The number

$$M_{GH}((X_1, M_1, *), (X_2, M_2, *), t) = \sup\{M_H(F_1(X_1), F_2(X_2), t) \mid F_i\colon X_i \to Z \text{ is an isometric embedding into a fuzzy metric space } Z\}$$

is called the *fuzzy Gromov-Hausdorff distance between* $(X_1, M_1, *)$ *and* $(X_2, M_2, *)$ *at* t.

2. Questions. Denote by **CFM** the set of isometry classes of compact fuzzy metric spaces endowed with the fuzzy Gromov-Hausdorff metric.

Question 1. *Is the space* **CFM** *separable?*

One can also conjecture that the fuzzy metric space **CFM** is completable (note that there are noncompletable fuzzy metric spaces [3]) and the elements of its completion can be naturally described as complete separable fuzzy metric spaces.

Question 2. *Is the space of finite fuzzy metric spaces dense in* **CFM**?

Similar questions can be formulated for some classes of fuzzy metric spaces, e.g., for stationary fuzzy metric spaces. Recall that a fuzzy metric space is called stationary if its fuzzy metric does not depend on t.

One can define a fuzzy counterpart of the notion of metric measure space (mm-space). A compact fuzzy mm-space is a quadruple $(X, M, *, \mu)$, where $(X, M, *)$ is a compact fuzzy metric space and μ is a probability measure on X. The Prokhorov fuzzy metric M_P on the set of probability measures $P(X)$ on a fuzzy metric space $(X, M, *)$ was defined in [8].

Given two compact fuzzy mm-spaces $\mathcal{X}_i = (X_i, M_i, *, \mu_i)$, $i \in \{1, 2\}$, define

$$M_{GP}(\mathcal{X}_1, \mathcal{X}_2, t) = \sup\{M_P((P(F_1)(\mu_1), P(F_2)(\mu_2), t) \mid F_i\colon X_i \to Z \text{ is an isometric embedding into a fuzzy metric space } Z\}$$

(the Gromov-Prokhorov distance).

Question 3. *Is the obtained fuzzy metric space of all compact fuzzy mm-spaces separable? Is it completable?*

Nothing is known on the geometry of the fuzzy metric space of all compact fuzzy mm-spaces.

In the theory of mm-spaces, various metrics on the set of probability measures on a metric space are considered. Here, we mention only two examples: the Kantorovich metric (and its generalizations) [6] and the Gromov $\underline{\square}_\lambda$-metric [4, Chapter $3\frac{1}{2}$]. It is an open question to define counterparts of these metrics in the fuzzy setting.

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Kherson Agrarian University
Ivan Franko National University of Lviv
savchenko1960@rambler.ru
mzar@litech.lviv.ua

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