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OPEN PROBLEMS FOR ENTIRE FUNCTIONS OF BOUNDED INDEX IN DIRECTION

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This paper is devoted to some unsolved problems in the theory of entire functions of several variables in connection with investigation of functions of bounded L-index in direction.

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Статья посвящена некоторым нерешенным проблемам в теории целых функций нескольких переменных в связи с исследованием функций ограниченного L-индекса по направлению.

Let $L(z)$ be a positive continuous function on $\mathbb{C}^n$, and let $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$. An entire function $F(z)$, $z\in\mathbb{C}^n$, is called (see [1]–[3]) a function of *bounded L-index in the direction* $\mathbf{b}$ if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$ the following inequality is valid

$$\frac{1}{m!L^m(z)}\Big|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\Big|\le\max\Big\{\frac{1}{k!L^k(z)}\Big|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\Big|\colon 0\le k\le m_0\Big\},\tag{1}$$

where $\frac{\partial^0 F(z)}{\partial\mathbf{b}^0}=F(z),\ \frac{\partial F(z)}{\partial\mathbf{b}}=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j=\langle\mathbf{grad}\ F,\ \overline{\mathbf{b}}\rangle,\ \frac{\partial^k F(z)}{\partial\mathbf{b}^k}=\frac{\partial}{\partial\mathbf{b}}\big(\frac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\big),\ k\ge 2.$

The least such integer $m_0=m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}\in\mathbb{C}^n$ *of the function* $F(z)$ *and is denoted by* $N_{\mathbf{b}}(F,L)=m_0$. If $n=1$ and $L(z)=l(z),\ z\in\mathbb{C}$, we obtain the definition of a function of bounded l-index ([5]), and in the case $L(z)\equiv 1$ we get the definition of a function of bounded index ([7]).

For $\eta>0$, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{0\}$ and a positive continuous function $L\colon\ \mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(\eta)=\inf\Big\{\inf\Big\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}\colon\ t\in\mathbb{C},\ |t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\Big\}\colon\ t_0\in\mathbb{C},\ z\in\mathbb{C}^n\Big\},$$

and also

$$\lambda_2^{\mathbf{b}}(\eta)=\sup\Big\{\sup\Big\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}\colon\ t\in\mathbb{C},\ |t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\Big\}\colon\ t_0\in\mathbb{C},\ z\in\mathbb{C}^n\Big\}.$$

By $Q^n_{\mathbf{b}}$ we denote the class of functions L which for all $\eta\ge 0$ satisfy the condition

$$0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty.$$

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In [1], [2] we considered the following partial differential equation:

$$g_0(z)\frac{\partial^p w}{\partial \mathbf{b}^p} + g_1(z)\frac{\partial^{p-1} w}{\partial \mathbf{b}^{p-1}} + \ldots + g_p(z)w = h(z), \tag{2}$$

where $g_j(z)$, $h(z)$ are entire functions, $z \in \mathbb{C}^n$.

We investigated an L-index boundedness in direction of entire solutions of some partial differential equations. There were obtained sufficient conditions of L-index boundedness of a solution in the following two cases:

1. provided that the coefficients of equation (2) are functions of bounded L-index in direction $\mathbf{b}$ ([1]);
2. did not provide that the coefficients of equation (2) are functions of bounded L-index in direction $\mathbf{b}$ ([2]);

Nevertheless, equation (2) contains a derivative in one direction. It is obvious that equations with one directional derivative constitute a small subclass of partial differential equations. But every partial derivative is a linear combination of directional derivatives. Thus, any partial differential equation can be written as an equation with derivatives in various directions. For example, we consider a partial differential equation with two directional derivatives

$$f_1(z)\frac{\partial F}{\partial \mathbf{b}_1} + f_2(z)\frac{\partial F}{\partial \mathbf{b}_2} = h(z). \tag{3}$$

Problem 1. *Let $f_1(z)$, $f_2(z)$ be entire functions of bounded L-index in corresponding directions $\mathbf{b}_1$, $\mathbf{b}_2$. What are direction $\mathbf{b}$ and additional conditions that an entire solution $F(z)$ of equation (3) has a bounded L-index in the direction $\mathbf{b}$?*

The following equation is a partial case of (3)

$$P_1(z_1,z_2)\frac{\partial F}{\partial z_1} + P_2(z_1,z_2)\frac{\partial F}{\partial z_2} = h(z_1,z_2). \tag{4}$$

Problem 2. *Let $g(z_1,z_2)$ be an entire function of bounded L-index in the directions $\mathbf{b}_1$ and $\mathbf{b}_2$. What are a function L^* and a direction $\mathbf{b}^*$ that an entire solution of equation $\frac{\partial^2 F}{\partial \mathbf{b}_1 \partial \mathbf{b}_2} = g(z_1,z_2)$ has a bounded L^*-index in the direction $\mathbf{b}^*$?*

Problem 3. *Let $P_1(z_1,z_2)$, $P_2(z_1,z_2)$ be entire functions of bounded L-index in the directions $(1,0)$ and $(0,1)$, respectively. What are a direction $\mathbf{b}$ and additional assumptions such that an entire solution $F(z)$ of equation (4) has a bounded L-index in the direction $\mathbf{b}$?*

Consider the ordinary differential equation

$$w' = f(z,w). \tag{5}$$

Shah S. M., Fricke G., Sheremeta M. M., Kuzyk A. D. ([4]–[6]) and others did not investigate an index boundedness of entire solution of (5) because the right hand side of it is a function of two variables. But now in view of entire function theory of bounded L-index in direction it is naturally to pose the following question.

Problem 4. *Let $f(z,w)$ be a function of bounded L-index in the directions $(1,0)$ and $(0,1)$. What is a function l such that an entire solution $w = w(z)$ of equation (5) has a bounded l-index?*

B. Lepson ([7]) studied differential equations of infinite order with constant coefficients and its solutions as hyper-Dirichlet series $\sum P_n(z)e^{-\lambda_n z}$, where $P_n(z)$ are polynomials of degrees μ_n, respectively, and λ_n are positive numbers increasing monotonically to infinity. He introduced a class of entire functions of bounded index to replace $P_n(z)$. Thus we consider the following linear differential equation of infinite order with constant coefficients

$$\sum_{k=0}^{\infty} a_k w^{(k)}(z) = f(z). \tag{6}$$

Problem 5. *Let $f(z)$ be of bounded l-index. What are assumptions on a_k and $f(z)$ such that an entire solution of* (6) *has a bounded l-index?*

We remark that equation (6) can be rewritten for directional derivatives in $\mathbb{C}^n$ and Problem 4 can be reformulated too.

There were obtained some criteria of L-index boundedness in direction ([1]). Later we proved that Theorem 2 and 6 ([1]) have modified versions Theorem 5 ([8]) and Theorem 7 ([1]) that are distinguished the universal quantifiers and the existential quantifiers.

The following theorems were obtained in [1].

Theorem 1 ([1])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F(z)$ is of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for every $R > 0$ there exist $P_2(R) \geq 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and every $t_0 \in \mathbb{C}$ and some $r = r(z^0, t_0) \in [\eta(R), R]$ the following inequality holds*

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t - t_0| = \frac{r}{L(z^0 + t_0\mathbf{b})}\right\} \leq P_2 \min\left\{|F(z^0 + t\mathbf{b})|\colon |t - t_0| = \frac{r}{L(z^0 + t_0\mathbf{b})}\right\}. \tag{7}$$

Denote $g_{z^0}(t) := F(z^0 + t\mathbf{b})$. If for a given $z^0 \in \mathbb{C}^n$ one has $g_{z^0}(t) \neq 0$ for all $t \in \mathbb{C}$, then $G^{\mathbf{b}}_r(F, z^0) := \varnothing$; if for a given $z^0 \in \mathbb{C}^n$ we get $g_{z^0}(t) \equiv 0$, then $G^{\mathbf{b}}_r(F, z^0) := \{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$. And if for a given $z^0 \in \mathbb{C}^n$ we have $g_{z^0}(t) \not\equiv 0$ and a^0_k are zeros of $g_{z^0}(t)$, i. e. $F(z^0 + a^0_k\mathbf{b}) = 0$, then

$$G^{\mathbf{b}}_r(F, z^0) := \bigcup_k \left\{z^0 + t\mathbf{b}\colon |t - a^0_k| \leq \frac{r}{L(z^0 + a^0_k\mathbf{b})}\right\}, \quad r > 0.$$

Let

$$G^{\mathbf{b}}_r(F) = \bigcup_{z^0 \in \mathbb{C}^n} G^{\mathbf{b}}_r(F, z^0). \tag{8}$$

By $n\big(r, z^0, t_0, 1/F\big) = \sum_{|a^0_k - t_0| \leq r} 1$ we denote the counting function of the zero sequence (a^0_k).

Theorem 2 ([1])**.** *Let $F(z)$ be an entire function on $\mathbb{C}^n$, $L \in Q^n_{\mathbf{b}}$ and $\mathbb{C}^n \setminus G^{\mathbf{b}}_r(F) \neq \varnothing$. Then $F(z)$ is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if:*

1) *for every $r > 0$ there exists $P = P(r) > 0$ such that for each $z \in \mathbb{C}^n \backslash G^{\mathbf{b}}_r(F)$*

$$\left|\frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}}\right| \leq P L(z); \tag{9}$$

2) *for every $r > 0$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for every $z^0 \in \mathbb{C}^n$, for which $F(z^0 + t\mathbf{b}) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$*

$$n\left(\frac{r}{|\mathbf{b}| L(z^0 + t^0\mathbf{b})}, z^0, t_0, \frac{1}{F}\right) \leq \widetilde{n}(r). \tag{10}$$

Therefore the next problem arises.

Problem 6. *Is Conjecture 1 true?*

Conjecture 1. *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist $R > 0$, $P_2(R) \geq 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and every $t_0 \in \mathbb{C}$ and some $r = r(z^0, t_0) \in [\eta(R), R]$ inequality (7) holds.*

Problem 7. *Is Conjecture 2 true?*

Conjecture 2. *Let $F(z)$ be an entire in $\mathbb{C}^n$ function, $L \in Q^n_{\mathbf{b}}$ and $\mathbb{C}^n \setminus G^{\mathbf{b}}_r(F) \neq \varnothing$. $F(z)$ is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if:*

1) there exist $r > 0$, $P = P(r) > 0$ such that for each $z \in \mathbb{C}^n \backslash G^{\mathbf{b}}_r(F)$ inequality (9) holds;

2) there exist $r > 0$, $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for every $z^0 \in \mathbb{C}^n$, for which $F(z^0 + t\mathbf{b}) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$ inequality (10) holds.

Problem 8. *Are there an entire function $F(z)$, a positive continuous function L and unbounded domains G_1, G_2, $\overline{G_1} \cup \overline{G_2} = \mathbb{C}^n$, $G_1 \cap G_2 = \varnothing$ with the following properties: inequality (1) holds for all $z \in G_1$, $\mathbf{b} = \mathbf{b}_1$, inequality (1) holds for all $z \in G_2$, $\mathbf{b} = \mathbf{b}_2$, but inequality (1) does not hold for all $z \in G_1$, $\mathbf{b} = \mathbf{b}_2$, inequality (1) does not hold for all $z \in G_2$, $\mathbf{b} = \mathbf{b}_1$, i. e. F is of bounded L-index in the direction $\mathbf{b}_1$ in the domain G_1 and F is of bounded L-index in the direction $\mathbf{b}_2$ in the domain G_2, but F is of unbounded L-index in the direction $\mathbf{b}_2$ in the domain G_1 and F is of unbounded L-index in the direction $\mathbf{b}_1$ in the domain G_2?*

If the answer to this question is the positive then we can consider entire functions of bounded L-index in the direction $\mathbf{b}$ in some domain.

The following assertion can be easily obtained using the definition of bounded L-index in direction.

Proposition 1. *Let $L(z)$ be a positive continuous function. An entire function $F(z)$, $z \in \mathbb{C}^n$, is of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n$ if and only if the function $G(z) = F(\mathbf{a}z + \mathbf{c})$ is of bounded L_*-index in the direction $\frac{\mathbf{b}}{\mathbf{a}}$ for any $\mathbf{c} \in \mathbb{C}^n$ and $\mathbf{a} \in \mathbb{C}^n$, such that $a_j \neq 0$ $(\forall j)$, where $\mathbf{a}z + \mathbf{c} = (a_1 z_1 + c_1, \ldots, a_n z_n + c_n)$, $\frac{\mathbf{b}}{\mathbf{a}} = (\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})$, $L_*(z) = L(\mathbf{a}z + \mathbf{c})$.*

Proof of Proposition 1. Let an entire function $F(z)$ be of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$. Observe that

$$\frac{\partial G(z)}{\partial (\frac{\mathbf{b}}{\mathbf{a}})} = \sum_{j=1}^{n} \frac{\partial G(z)}{\partial z_j} \frac{b_j}{a_j} = \sum_{j=1}^{n} \frac{\partial F(\mathbf{a}z + \mathbf{c})}{\partial z_j} a_j \frac{b_j}{a_j} = \frac{\partial F(\mathbf{a}z + \mathbf{c})}{\partial \mathbf{b}}.$$

We can prove by induction that $\frac{\partial^k G(z)}{\partial (\frac{\mathbf{b}}{\mathbf{a}})^k} = \frac{F(\mathbf{a}z + \mathbf{c})}{\partial \mathbf{b}^k}$ for all $k \in \mathbb{N}$. From inequality (1) at $\mathbf{a}z + \mathbf{c}$ instead of z we have

$$\frac{1}{m! L_*^m(z)} \left| \frac{\partial^m G(z)}{\partial (\frac{\mathbf{b}}{\mathbf{a}})^m} \right| \leq \max \left\{ \frac{1}{k! L^k(\mathbf{a}z + \mathbf{c})} \left| \frac{\partial^k F(\mathbf{a}z + \mathbf{c})}{\partial \mathbf{b}^k} \right| : 0 \leq k \leq m_0 \right\} =$$
$$= \max \left\{ \frac{1}{k! L_*^k(z)} \left| \frac{\partial^k G(z)}{\partial (\frac{\mathbf{b}}{\mathbf{a}})^k} \right| : 0 \leq k \leq m_0 \right\}.$$

The last inequality means that the function $G(z)$ is of bounded L_*-index in the direction $\frac{\mathbf{b}}{\mathbf{a}}$ and vice versa. □

Proposition 1 induces the following problem.

Problem 9. *Are there numbers $a_1, a_2, c_1, c_2 \in \mathbb{C}$ and a function $F(z_1, z_2)$ such that $F(z_1, z_2)$ is of bounded L-index in a direction $\mathbf{b} = (b_1, b_2)$ but $F(a_1z_1 + c_1, a_2z_2 + c_2)$ is of unbounded L-index in the same direction $\mathbf{b} = (b_1, b_2)$?*

Problem 10 ([1])**.** *What is the least set A with following property: if for every $\mathbf{b} \in A$ an entire in $\mathbb{C}^n$ function F is of bounded L-index in the direction $\mathbf{b}$ then F is of bounded L-index in any direction $\mathbf{b} \in \mathbb{C}^n$?*

A partial answer to this question is contained in the following theorem.

Theorem 3 ([1])**.** *An entire function $F(z), z \in \mathbb{C}^n$, is a function of bounded L-index in all directions in $\mathbb{C}^n$ if and only if this function is a function of bounded L-index in every direction $\mathbf{b} \in \mathbb{C}^n, |\mathbf{b}| = 1$, such that that the sum of the values of the main arguments of all components of the vector $\mathbf{b}$ is a multiple of 2π, i. e. $\sum_{j=1}^{n} \arg(b_j) = 2\pi m$, where $m \in \mathbb{Z}$.*

Problem 11 ([1])**.** *Is Conjecture 3 true?*

Conjecture 3. *Let $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis in $\mathbb{C}^n$ and let $F(z)$, $z \in \mathbb{C}^n$, be an entire function of bounded L-index in every direction $\mathbf{b}_i \in \mathbb{C}^n$, $L \in Q^n_{\mathbf{b}_i}$, $i \in \{1, 2, \dots, n\}$. Then the function $F(z)$ is of bounded L-index in any direction $\mathbf{b} = \lambda_1\mathbf{b}_1 + \ldots + \lambda_n\mathbf{b}_n$, where $\lambda_i \in \mathbb{C}$ (at least one $\lambda_i \neq 0$).*

Our proof of Conjecture 3 in [1, Theorem 11] contains a mistake and a correct proof is unknown.

Problem 12 ([2])**.** *What are minimal requirements on a set A such that*

$$N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) \colon \ z^0 \in A\},$$

where $l_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$, $g_{z^0}(t) = F(z^0 + t\mathbf{b})$, $N(f, l)$ is the l-index of function f?

Our best result concerning this problem is the following

Proposition 2 ([2])**.** *Let $\mathbf{b} \in \mathbb{C}^n$ be a given direction, A_0 be a dense subset of some hyperplane, i. e. its closure satisfies $\overline{A}_0 = \{z \in \mathbb{C}^n \colon \langle z, c\rangle = 1\}$, where $\langle c, \mathbf{b}\rangle \neq 0$. An entire function $F(z)$, $z \in \mathbb{C}^n$ is a function of bounded L-index in direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exists a number $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ is of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of one variable $t \in \mathbb{C}$. Thus $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) \colon z^0 \in A_0\}$.*

But we do not know whether the density of the set A in a hyperplane can be replaced with a weaker assumption.

Let π be an entire function in $\mathbb{C}^n$ *of genus p with "planar" zeros*

$$\pi(z) = \prod_{k=1}^{\infty} g(\langle z, a^k|a^k|^{-2}\rangle, p), \tag{11}$$

$$g(u, p) = (1-u)\exp\left\{u + \frac{u^2}{2} + \dots + \frac{u^p}{p}\right\}, \quad p \neq 0, \quad g(u, 0) = (1-u),$$

where $a^k \in \mathbb{C}^n$ is a sequence of genus p, i.e.

$$\sum_{k=1}^{\infty} |a^k|^{-p-1} < +\infty, \ \sum_{k=1}^{\infty} |a^k|^{-p} = +\infty. \tag{12}$$

We assume that the sequence (a^k) is ordered such that $|a^k| \le |a^{k+1}|$ $(k \ge 1)$. Besides we suppose that the elements of the sequence (a^k) are located on some ray

$$a_j^k = m_j |a^k| \text{ for all } k \ge 1, \tag{13}$$

$m = (m_1, m_2, \dots, m_n)$.

We obtained some sufficient conditions of L-index boundedness in direction for entire functions with "planar" zeros ([1], [9], [10]) with condition (13). It is obvious that (13) does not provide the L-index boundedness in direction. In practice, it is related with the method of proof. Thus, the following problem is interesting.

Problem 13. *Are there sufficient conditions of L-index boundedness in direction for infinite products* (11) *without condition* (13)*?*

Problem 14. *For given $\mathbf{b}_1 \not\parallel \mathbf{b}_2$ construct an entire function with 'planar' zeros of bounded L-index in the direction $\mathbf{b}_1$ and of unbounded L-index in the direction $\mathbf{b}_2$.*

Problem 15. *Let $F\colon \mathbb{C}^{n+m} \to \mathbb{C}$ be an entire function, $L_1\colon \mathbb{C}^n \to \mathbb{R}_+$, $L_2\colon \mathbb{C}^m \to \mathbb{R}_+$, for all $(z_{n+1}, z_{n+2}, \dots, z_{n+m}) \in \mathbb{C}^m$, F be of uniformly bounded L_1-index in the direction $\mathbf{b}_1 = (b_1, b_2, \dots, b_n, \underbrace{0, \dots, 0}_{m-\text{times}}) \in \mathbb{C}^{n+m}$, for all $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, F be of uniformly bounded L_2-index in the direction $\mathbf{b}_2 = (\underbrace{0, \dots, 0}_{n-\text{times}}, b_{n+1}, b_{n+2}, \dots, b_{n+m},) \in \mathbb{C}^{n+m}$. What is a function $L\colon \mathbb{C}^{n+m} \to \mathbb{R}_+$ such that F is of bounded L-index in the direction $\mathbf{b} = (b_1, b_2, \dots, b_{n+m})$?*

Denote $\mathbf{e}_j = (0, \dots, \underbrace{1}_{j\text{-th place}}, \dots, 0)$, $l_j = l(z_j)$.

Problem 16. *Prove the following Conjecture 4.*

Conjecture 4. *Let $l\colon \mathbb{C} \to \mathbb{R}_+$ be a continuous function and for every $j \in \{1, \dots, n-1\}$ an entire function F is of bounded l_j-index in the direction $\mathbf{e}_j$, and for every $(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$, F is of bounded l_n-index as a function of the variable z_n. Then F is of bounded l_n-index in the direction $\mathbf{e}_n$.*

We proved the following assertion in [1].

Theorem 4 ([1])**.** *An entire function $F(z)$, $z \in \mathbb{C}^n$ is a function of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exists a number $M > 0$ such that for all $z^0 \in \mathbb{C}^n$ the function $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ is a function of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of variable $t \in \mathbb{C}$ $(l_{z^0}(t) \equiv L(z^0 + t\mathbf{b}))$. Thus $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0})\colon z^0 \in \mathbb{C}^n\}$.*

In view of this theorem the following *question* naturally arises: are there an entire function $F(z)$, $z \in \mathbb{C}^n$ and $\mathbf{b} \in \mathbb{C}^n$ such that $N(g_{z^0}, l_{z^0}) < +\infty$ for all $z^0 \in \mathbb{C}^n$, but $N_{\mathbf{b}}(F, L) = +\infty$?

Later we gave a positive answer ([3]): the function $\cos\sqrt{z_1 z_2}$ has the described properties for $\mathbf{b} = (1, 1)$ and $L(z) = 1$.

But traditionally a solution of some problem leads to new problems. In our case there are interesting questions:

Problem 17. *What are conditions on zero set and growth of entire functions providing the index boundedness of* $F(z_1^0+b_1t, z_2^0+b_2t)$ *for every* $(z_1^0, z_2^0) \in \mathbb{C}^2$ *and the index unboundedness of* $F(z_1, z_2)$ *in the direction* $\mathbf{b} = (b_1, b_2)$*?*

Problem 18. *Construct an entire function* F *of* n *variables such that* $F(z^0 + t\mathbf{b})$ *is of bounded* l_{z^0}*-index for any* $z^0 \in \mathbb{C}^n$*, but* $F(z)$ *is of unbounded* L*-index in the direction* $\mathbf{b} = (b_1, \ldots, b_n)$*, where* $n \geq 3$*,* $l_{z^0}(t) = L(z^0 + t\mathbf{b})$*.*

REFERENCES

1. A.I. Bandura, O. B. Skaskiv, *Entire functions of bounded L-index in direction,* Mat. Stud., **27** (2007), №1, 30–52. (in Ukrainian)
2. A.I. Bandura, O.B. Skaskiv, *Sufficient sets for boundedness L-index in direction for entire functions,* Mat. Stud., **30** (2008), №2, 177–182.
3. A.I. Bandura, O.B. Skaskiv, *Entire functions of bounded and unbounded index in direction,* Mat. Stud., **27** (2007), №2, 211–215. (in Ukrainian)
4. G.H. Fricke, S.M. Shah, *Entire functions satisfying a linear differential equation,* Indag. Math., **37** (1975), 39–41.
5. A.D. Kuzyk, M.N. Sheremeta, *On entire functions, satisfying linear differential equations,* Diff. equations, **26** (1990), №10, 1716–1722. (in Russian)
6. S.M. Shah, *Entire solutions of linear differential equations and bounds for growth and index numbers,* Proc. Sect. A: Mathematics, Royal Soc. Edinburgh, **93A** (1983), 49–60.
7. B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index,* Proc. Sympos. Pure Math., V.2., Amer. Math. Soc.: Providence, Rhode Island, 1968, 298–307.
8. A.I. Bandura, *A modified criterion of boundedness of L-index in direction,* Mat. Stud., **39** (2013), №1, 99–102.
9. A.I. Bandura, *Sufficient conditions of boundedness L-index in direction for entire functions with "planar" zeros of genus p,* Mathem. bulletin SSS, **6** (2009), 44–49. (in Ukrainian)
10. A.I. Bandura, O.B. Skaskiv, *Boundedness of L-index in direction of functions of the form* $f(\langle z, m\rangle)$ *and existence theorems,* Mat. Stud., **41** (2014), №1, 45–52.

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