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ON THE DIVISION OF A CHARACTERISTIC FUNCTION BY THE BLASCHKE PRODUCT

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We give an example of a characteristic function f of the probability distribution on the half-line $[0,\infty)$ such that if B is the Blaschke product of f for upper half-plane then f/B is not a characteristic function.

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Приведен пример характеристической функции вероятностного распределения, сосредоточенного на полуоси $[0,\infty)$, результат деления которой на произведение Бляшке, построенное по ее корням, лежащим в верхней полуплоскости, не является характеристической функцией.

Let $\psi(t)$ be an analytic function bounded in the upper half-plane $\mathbb{C}^+ = \{t \in \mathbb{C}\colon \operatorname{Im} t > 0\}$, $B(t)$ be the Blaschke product of $\psi(t)$ for the upper half-plane $\mathbb{C}^+$. Then the function $\psi(t)/B(t)$ is also analytic and bounded in $\mathbb{C}^+$. The characteristic function of a probability distribution concentrated on the half-line $[0,\infty)$ is a function analytic and bounded (by 1) in the half-plane $\mathbb{C}^+$ function. Therefore, if

$$\psi(t) = \int_{[0,\infty)} \exp(itx)\, P(dx)$$

is such a function and $B(t)$ is the Blaschke product for it in the upper half-plane then $\psi(t)/B(t)$ is analytic and bounded in the upper half-plane $\mathbb{C}^+$. Is it true or not that the ratio $\psi(t)/B(t)$ must be a *characteristic function* of the probability distribution concentrated on the half-line $[0,\infty)$? Professor I. V. Ostrovskii has payed an attention of the author to this question. The aim of the present paper is the construct an example which gives a negative answer to the question.

We put following [1], p. 47: $g(z) = 1 + 2z - z^2 + 3z^3 + 3z^4$ and for $\alpha > 0$

$$f_\alpha(z) = \exp\big(\alpha(g(z) - 8)\big).$$

Then $f_\alpha(z) = \sum_{k=0}^{\infty} c_k(\alpha) z^k$ and coefficients $c_k(\alpha)$ satisfy the following conditions

$$\sum_{k=0}^{\infty} c_k(\alpha) = 1, \quad c_k(\alpha) > 0 \ (k \neq 2), \quad c_2(\alpha) < 0 \ (0 < \alpha < 1/2).$$

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A simple calculation shows that

$$c_1(\alpha) \sim 2\alpha, \quad c_2(\alpha) \sim -\alpha, \quad c_3(\alpha) \sim 3\alpha \quad \text{for} \quad \alpha \to 0.$$

Let $\varphi(t)$ be the characteristic function of the uniform distribution on the interval $(0,1)$. We put for $h \in \mathbb{R}$

$$\varphi_h(t) = \varphi(t - ih)/\varphi(-ih).$$

The function $\varphi_h(t)$ is an entire characteristic function with simple zeros $ih + 2\pi n$ ($n = \pm1, \pm2, \ldots$) and with the density

$$p_h(x) = \begin{cases} D_h \exp(hx), & x \in [0,1], \\ 0, & x \notin [0,1], \end{cases}$$

where $D_h = (\int_0^1 \exp(hv)dv)^{-1}$. Now we put

$$\Phi_{\alpha,h}(t) = f_\alpha(\exp(it/2))\varphi_h(t), \quad B(t) = \prod_{k\neq 0}\left(1 - \frac{t}{2\pi k + ih}\right)\left(1 - \frac{t}{2\pi k - ih}\right)^{-1}.$$

Let us show that for $h \in (2\log 2, 2\log 3)$ and for small enough $\alpha > 0$ the function $\Phi_{\alpha,h}(t)$ is a characteristic function of the probability distribution concentrated on the half-line $[0,\infty)$ but the function $\Phi_{\alpha,h}(t)/B(t)$ is not a characteristic function. We have

$$\Phi_{\alpha,h}(t) = \int_{-\infty}^{\infty} \exp(itx) q_{\alpha,h}(x)dx, \quad \text{where} \quad q_{\alpha,h}(x) = \sum_{k=0}^{\infty} c_k(\alpha) p_h\left(x - \frac{k}{2}\right).$$

It is evident that

$$q_{\alpha,h}(x) = 0 \text{ for } x \leqslant 0, \quad q_{\alpha,h}(x) > 0 \text{ for } x \in [0,1] \cup [2,\infty).$$

Let us show that for the above conditions on the parameters h and α the inequality $q_{\alpha,h}(x) > 0$ is also valid for $x \in [1,2]$.

We obtain for $x \in [1, 3/2]$

$$q_{\alpha,h}(x) = c_1(\alpha)p_h\left(x - \frac{1}{2}\right) + c_2(\alpha)p_h(x-1) =$$
$$= D_h\left(c_1(\alpha)e^{-h/2} + c_2(\alpha)e^{-h}\right)\exp(hx) \approx D_h\alpha\left(2e^{h/2} - 1\right)e^{-h}\exp(hx),$$

where the sign $A \approx B$ denotes that $A/B \to 1$ for $\alpha \to 0$ uniformly with respect to h and x on every finite segment of the half-line $(0,\infty)$. Therefore $q_{\alpha,h}(x) > 0$ if $x \in [1,3/2]$ for every $h > 0$ and for sufficiently small $\alpha > 0$.

For $x \in [3/2, 2]$ we have

$$q_{\alpha,h}(x) = c_3(\alpha)p_h\left(x - \frac{3}{2}\right) + c_2(\alpha)p_h(x-1) =$$
$$= D_h\left(c_3(\alpha)e^{-3h/2} + c_2(\alpha)e^{-h}\right)\exp(hx) \approx D_h\alpha\left(3e^{-h/2} - 1\right)e^{-h}\exp(hx).$$

Since $h < 2\log 3$, we see that $q_{\alpha,h}(x) > 0$ for $x \in [3/2, 2]$, if the parameter α is small enough.

Let us show that if $h > 2\log 2$ then the function $\Phi_{\alpha,h}(t)/B(t)$ is not a characteristic function. Using the formula

$$\frac{\sin z}{z} = \prod_{k\neq 0}\left(1 - \frac{z}{\pi k}\right),$$

it is easy to verify that $\varphi_h(t)/B(t) = \varphi_{-h}(t)$. Therefore

$$\Phi_{\alpha,h}(t)/B(t) = f_\alpha(\exp(it/2))\varphi_{-h}(t) = \int_{-\infty}^{\infty}\exp(itx)q_{\alpha,-h}(x)dx.$$

We have for $x \in [1, 3/2]$

$$q_{\alpha,-h}(x) = c_1(\alpha)p_{-h}\left(x - \frac{1}{2}\right) + c_2(\alpha)p_{-h}(x-1) =$$
$$= D_h\left(c_1(\alpha)e^{h/2} + c_2(\alpha)e^h\right)\exp(-hx) \approx D_h\alpha\left(2 - e^{h/2}\right)\exp(-hx).$$

We obtain $2 - e^{h/2} < 0$ because $h > 2\log 2$. Therefore the value $q_{\alpha,-h}(x)$ is negative for $x \in [1, 3/2]$ if α is small enough. It follows that the function $\Phi_{\alpha,h}(t)/B(t)$ is not a characteristic function.

Remark. As was mentioned by A. M. Ulanovskii, one can construct by the above reasoning an analogous example with the characteristic function of a probability distribution concentrated on a finite segment.

Question. For $h < 2\log 2$ we obtain an example such that the ratio $\Phi_{\alpha,h}(t)/B(t)$ is a characteristic function. So, we have two nonempty classes of probability distributions concentrated on the half-line $[0, \infty)$. The first (second) class includes the distributions with the characteristic function $\psi(t)$ such that $\psi(t)/B(t)$ is (is not) a characteristic function. What is their "capacity"? Is it true that two these classes are dense in the topology of weak convergence in the set of all distributions concentrated on the half-line $[0, \infty)$?

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