УДК 517.55

P. Wójcicki

WEIGHTED BERGMAN KERNEL FUNCTION, ADMISSIBLE WEIGHTS AND THE RAMADANOV THEOREM

P. Wójcicki. *Weighted Bergman kernel function, admissible weights and the Ramadanov theorem*, Mat. Stud. **42** (2014), 160–164.

The Bergman kernel is an important tool in geometric function theory, both in one and several complex variables. It turned out that not only "regular" Bergman kernel, but also weighted one can be useful, particularly from quantum theory point of view ([3], [7]). But in general, it is difficult to say anything about the kernel of a given domain. One of the classic results is Ramadanov's theorem.

Theorem. *Let $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \dots$ be an increasing sequence of domains and set $\Omega \equiv \bigcup_j \Omega_j$. Then, for all j, $K_{\Omega_j} \to K_\Omega$ uniformly on compact subsets of $\Omega \times \Omega$.*

Some versions of a weighted analog of this theorem are known ([4, Pr. 3.17; Th. 3.18] for instance), but considered weights are of a special form, as a modulus of holomorphic functions or C^2 functions, or as a product of one of those by a given weight ψ. In the present paper we provide a weighted analog of this theorem, for very general weights.

П. Вуйцицки. *Весовые функции ядра Бергмана, допустимые веса и теорема Рамаданова* // Мат. Студiї. – 2014. – Т.42, №2. – C.160–164.

Ядро Бергмана является важным инструментом в геометрической теории функций как одной, так и многих комплексных переменных. Оказалось, что не только "стадартное" ядро Бергмана, но и весовое ядро могут быть полезными с точки зрения квантовой теории ([3], [7]). Однако в общем трудно что-либо сказать о ядре для произвольной области. Одним из классических результатов является теорема Рамаданова.

Теорема. *Пусть $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \dots$ — возрастающая последовательность областей и $\Omega \equiv \bigcup_j \Omega_j$. Тогда $K_{\Omega_j} \to K_\Omega$ равномерно на компактных подмножествах $\Omega \times \Omega$.*

Известны некоторые версии весового аналоги этой теоремы (например, [4, Пр. 3.17; Теор. 3.18]), однако сконструированные веса имеют специальную форму модуля голоморфной функции или C^2 функции, либо произведения одного из таковых на данный вес ψ. В данной статье мы приводим весовой аналог этой теоремы для весьма общих весов.

1. Introduction. We start with explaining what a weighted Bergman space is, and how does the weighted Bergman kernel appear. For a given domain $D \subset \mathbb{C}^N$ consider the space

$$L^2_H(D) = \left\{ f \in \mathrm{Hol}(D); \|f\|^2_D = \int_D |f|^2 dV < \infty \right\}$$

with the inner product $(f, g) = \int_D \bar{f} g dV$. This is a Hilbert space, called the *Bergman space*.

2010 *Mathematics Subject Classification:* 32A25, 32A36.
Keywords: weighted Bergman kernel; Bergman space; Ramadanov theorem.
doi:10.30970/ms.42.2.160-164

Fix a point $t \in D$ and minimize the norm $\|f\|_D$ in the class $E_t = \{f \in L^2_H(D); f(t) = 1\}$. It has been proved that there exists a unique extremal function of the problem posed above ([10, Th. 1, p. 296]). Let us denote it by $\phi(z,t)$.

The *Bergman kernel function* K_D is defined as follows $K_D(z,t) = \frac{\phi(z,t)}{\|\phi\|^2_D}$. We should note that K_D is a function of one variable z, since $t \in D$ is already fixed.

We can define a weighted Bergman space in a similar way. Let $D \subset \mathbb{C}^N$ be a domain, and let $W(D)$ be the set of weights on D, i.e., $W(D)$ is the set of all Lebesgue measurable real–valued positive functions on D (we consider two weights to be equivalent if they are equal almost everywhere with respect to the Lebesgue measure on D). If $\mu \in W(D)$, we denote by $L^2(D,\mu)$ the space of all (equivalence classes of) Lebesgue measurable complex-valued μ-square integrable functions on D, equipped with the norm $\|\cdot\|_\mu$ given by the inner product

$$\langle f|g\rangle_\mu := \int_D \overline{f(z)}g(z)\mu(z)dV, \quad f,g \in L^2(D,\mu).$$

In the present article we deal with the so called admissible weights. It is of importance, since we need such general weights in quantization, for instance. We may define the so called Berezin transform by means of weighted Bergman kernel for a weight which is a positive continuous function ([3]). Next is the formal definition of admissible weights.

Definition 1. A weight $\mu \in W(D)$ is called an *admissible weight*, an *a-weight* for short, if $L^2_H(D,\mu)$ is a closed subspace of $L^2(D,\mu)$ and for any $z \in D$ the evaluation functional E_z is continuous on $L^2_H(D,\mu)$.

The set of all a-weights on D will be denoted by $AW(D)$. The definition of an admissible weight provides us basically with the existence of related Bergman kernel and completeness of the space $L^2_H(D,\mu)$.

Theorem 1 ([9, Corollary 3.1])**.** *Let $\mu \in W(D)$. If the function μ^{-a} is locally integrable on D for some $a > 0$ then $\mu \in AW(D)$.*

The space $L^2_H(D,\mu) = \mathrm{Hol}(D) \cap L^2(D,\mu)$ is called the *weighted Bergman space*. Fix a point $t \in D$ and minimize the norm $\|f\|_\mu$ in the class $E_t = \{f \in L^2_H(D,\mu); f(t) = 1\}$. It can be proved in a similar way as in the regular case, that there exists exactly one function solving the problem posed above. Let us denote it by $\phi_\mu(z,t)$. The *weighted Bergman kernel function* $K_{D,\mu}$ is defined as follows

$$K_{D,\mu}(z,t) = \frac{\phi_\mu(z,t)}{\|\phi_\mu\|^2_\mu}.$$

2. Weighted Ramadanov theorem.

Theorem 2. *Let $D_1 \Subset D_2 \Subset D_3 \Subset \dots$ be an increasing sequence of domains in $\mathbb{C}^N$ and set $D := \bigcup_j D_j$. Let $\mu \in AW(D)$, $\mu_k \in AW(D_k)$. Extend μ_k by μ on $D \setminus D_k$. Assume moreover that*

a) $\mu_i(z) \le \mu_j(z) \le \mu(z)$ *for* $i \le j$, $z \in D_i$.

b) $\mu_k \xrightarrow[k\to\infty]{} \mu$ *pointwise on* D.

Then

$$\lim_{k\to\infty} K_{D_k,\mu_k} = K_{D,\mu}$$

locally uniformly on $D \times D$.

The first step in the proof is to show the monotonicity property for the weighted kernels. Then we should check that the limit of the sequence of weighted kernels of increasing domains, if exists, is equal to $K_{D,\mu}$.

Lemma 1 (Monotonicity property). *For any $n \in \mathbb{N}$, $t \in D$ the inequality $K_{D_n,\mu_n}(t,t) \geq K_{D,\mu}(t,t)$ holds.*

Proof. In the proof we use the simple remark that

$$\frac{1}{K_{D_n,\mu_n}(t,t)} = \int_{D_n} \left| \frac{K_{D_n,\mu_n}(s,t)}{K_{D_n,\mu_n}(t,t)} \right|^2 \mu_n(s)dV$$

since $K_{D_n,\mu_n}(t,t)$ is just a number (positive one) and

$$K_{D_n,\mu_n}(t,t) = \int_{D_n} \overline{K_{D_n,\mu_n}(z,t)} K_{D_n,\mu_n}(z,t)\mu_n(z)dV$$

(since if $\{\varphi_v^m\}_{v=1}^{\infty}$ is an orthonormal complete system on $L^2_H(D_m,\mu_m)$, then

$$K_{D_m,\mu_m}(z,t) = \sum_{v=1}^{\infty} \varphi_v^m(z)\overline{\varphi_v^m(t)},$$

what can be found in [10, Th. 2, p. 297]). Moreover the term $\int_{D_n} \left| \frac{K_{D_n,\mu_n}(s,t)}{K_{D_n,\mu_n}(t,t)} \right|^2 \mu_n(s)dV$ is the only minimal element, by the definition of $K_{D_n,\mu_n}(s,t)$. □

Remark 1. One can show similarly that $K_{D_n,\mu_n}(t,t) \geq K_{D_m,\mu_m}(t,t)$ for $n \leq m$.

In [12, p. 36] some characterization lemma for unweighted Bergman kernels is given. It turns out that we can conclude a similar one for weighted Bergman kernels, as the following Lemma 2 shows.

Lemma 2. *Denote by $S_{\mu,t} \subset L^2_H(D,\mu)$ the set of all functions f such that $f(t) \geq 0$ and $\|f\|_\mu \leq \sqrt{f(t)}$, where $t \in D$ is fixed. Then the weighted Bergman function $\varphi_{\mu,t}(z) := K_{D,\mu}(z,t)$ is uniquely characterized by the properties:*

(i) $\varphi_{\mu,t} \in S_{\mu,t}$;

(ii) *if* $f \in S_{\mu,t}$ *and* $f(t) \geq \varphi_{\mu,t}(t)$, *then* $f \equiv \varphi_{\mu,t}$.

Proof. One can easily see that there exists, at most one element $\varphi_{\mu,t} \in L^2_H(D,\mu)$ which satisfies (i) and (ii) (if φ_1, φ_2 satisfies (i) and (ii) then both $\varphi_1(t)$ and $\varphi_2(t)$ are positive, and either $\varphi_1(t) \geq \varphi_2(t)$ and then $\varphi_1 \equiv \varphi_2$ or $\varphi_2(t) \geq \varphi_1(t)$ and then $\varphi_2 \equiv \varphi_1$). We shall show that $\varphi_{\mu,t}(z) = K_{D,\mu}(z,t)$ has both properties.

We have

$$\varphi_{\mu,t}(t) = K_{D,\mu}(t,t) = \sum_{v=1}^{\infty} |\varphi_v^\mu|^2 \geq 0, \ \|K_{D,\mu}(z,t)\|^2_\mu = K_{D,\mu}(t,t).$$

Now let $f \in S_{\mu,t}$. If $f(t) = 0$ then $\varphi_{\mu,t}(t) = 0$. Hence $\|f\|_\mu = \|\varphi_{\mu,t}\|_\mu = 0$, so $f \equiv 0 \equiv \varphi_{\mu,t}$. Assume now $f(t) > 0$. By the definition of the weighted Bergman kernel, the function $\frac{\varphi_{\mu,t}(z)}{\varphi_{\mu,t}(t)}$ is uniquely characterized as an element of the set $\{h \in L^2_H(D,\mu), h(t) = 1\}$ with minimal norm. But $\frac{f(z)}{f(t)}$ belongs to this set as well, moreover

$$\left\|\frac{f(z)}{f(t)}\right\|_\mu = \frac{\|f\|_\mu}{\sqrt{f(t)}\sqrt{f(t)}} \leq \frac{1}{\sqrt{f(t)}} \leq \frac{1}{\sqrt{\varphi_{\mu,t}(t)}} = \left\|\frac{\varphi_{\mu,t}(z)}{\varphi_{\mu,t}(t)}\right\|_\mu.$$

Thus (by minimality)

$$\frac{1}{\sqrt{f(t)}} = \frac{1}{\sqrt{\varphi_{\mu,t}(t)}},\ \frac{f(z)}{f(t)} = \frac{\varphi_{\mu,t}(z)}{\varphi_{\mu,t}(t)}.$$

So $f \equiv \varphi_{\mu,t}$. □

Proof of the main theorem. Let $F \subset D$ be a compact set, and let G be any domain such that $F \subset G \subset D$. If m is large enough then $G \subset D_m$ and we have

$$|K_{D_m,\mu_m}(z,t)| \leq \sqrt{K_{D_m,\mu_m}(z,z)}\sqrt{K_{D_m,\mu_m}(t,t)} \leq \sqrt{K_{G,\mu_m}(z,z)}\sqrt{K_{G,\mu_m}(t,t)} \leq M,$$

where $M = M(F)$ is a constant such that $\max_{z\in F} |K_{G,\mu_m}(z,z)| \leq M$. By Montel's property, any subsequence of $\{K_{D_m,\mu_m}\}$ has a subsequence convergent locally uniformly on $D \times D$. With no loss of generality let us consider $\{K_{D_m,\mu_m}\}$ itself and assume that is does converge to some k. Using Lemma 2 we will show that $k = K_{D,\mu}$. Given any $G \Subset D$, we have that $G \Subset D_k$ for large enough k. Hence, by Fatou's lemma,

$$\int_G |k(z,t)|^2 \mu(z) dV \leq \liminf_{k\to\infty} \int_G |K_{D_k,\mu_k}(z,t)|^2 \mu_k(z) dV \leq$$
$$\leq \liminf_{k\to\infty} \int_{D_k} |K_{D_k,\mu_k}(z,t)|^2 \mu_k(z) dV = \liminf_{k\to\infty} K_{D_k,\mu_k}(t,t) = k(t,t).$$

Since G is arbitrary, we have that $\|k(\cdot,t)\|^2_\mu \leq k(t,t) < \infty$. Obviously, $k(t,t) \geq 0$, and thus $k(\cdot,t) \in S_{\mu,t}$. Moreover, $K_{D,\mu}(t,t) \leq K_{D_k,\mu_k}(t,t)$ implies (taking $k \to \infty$) that $K_{D,\mu}(t,t) \leq k(t,t)$. By Lemma 2, $k(z,t) = K_{D,\mu}(z,t)$. □

Remark 2. From Theorem 3 it follows that the counterexample to the Lu Qi-Keng conjecture given in [2] works for the weighted Bergman kernel as well. The general theorem given in [5] does not hold for general admissible weights, so we need further assumptions, for instance, that $\mu_{\Omega_j} \circ \Theta_j = \mu_\Omega$ for any diffeomorphism $\Theta_j : \Omega \to \Omega_j$.

3. Acknowledgements. I express my thanks to the Referee for valuable comments and careful reading of this paper. I am very grateful to my advisor prof. Z. Pasternak-Winiarski for very useful remarks on applications of the theory of weighted Bergman kernels in quantum mechanics and suggesting me the articles. I am very grateful to prof. M. Skwarczyński (†) for his passion and a great book for students ([11]).

REFERENCES

1. S. Bergman, The kernel function and conformal mapping, A.M.S. Survey Number V, 2^{nd} Edition, 1970.
2. H.P. Boas, *Counterexample to the Lu Qi-Keng conjecture*, Proc. Amer. Math. Soc., **97** (1986), №2, 374–375.
3. M. Englis, *Weighted Bergman kernels and quantization*, Comm. Math. Phys., **227** (2002), №2, 211–241.
4. R.L. Jacobson, Weighted Bergman Kernel Functions and the Lu Qi-Keng problem, Thesis (Ph.D.) – Texas A&M University, 2012.
5. S.G. Krantz, *A new proof and a generalization of Ramadanov's theorem*, Complex variables and elliptic equations, **51** (2006), №12, 1125–1128.
6. S.G. Krantz, Geometric analysis of the Bergman kernel and metric, New York, Springer, 2013.
7. A. Odzijewicz, *On reproducing kernels and quantization of states*, Commun. Math. Phys., **114** (1988), 577–597.
8. Z. Pasternak-Winiarski, *On the dependence of the reproducing kernel on the weight of integration*, Journal of Functional Analysis, **94** (1990), №1, 110–134.
9. Z. Pasternak-Winiarski, *On weights which admit the reproducing kernel of Bergman type*, Internat. J. Math & Math. Sci., **15** (1992), №1, 1–14.
10. B.V. Shabat, Introduction to Complex Analysis, Part II : Functions of several variables, Translations of Mathematical Monographs, V.110, A.M.S.
11. M. Skwarczyński, T. Mazur, Wstępne twierdzenia teorii funkcji wielu zmiennych zespolonych, Wyd. Krzysztof Biesaga, Warszawa, 2001.
12. Skwarczyński, M., Biholomorphic invariants related to the Bergman function, Dissertationes Mathematicae (Rozprawy Matematyczne), V.173, Warsaw, Polish Scientific Publishing Company, 1980.

Faculty of Mathematics and Information Science
Warsaw University of Technology
p.wojcicki@mini.pw.edu.pl

Received 10.07.2013
Revised 7.09.2014
Revised 1.12.2014