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ESTIMATES FOR THE MAXIMUM MODULUS OF ANALYTIC CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS ON SOME SEQUENCES

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Let φ be the characteristic function of a probability law F analytic in $\mathbb{D}_R = \{z: |z| < R\}$, $0 < R \leq +\infty$, $M(r, \varphi) = \max\{|\varphi(z)|: |z| = r\}$ and $W_F(x) = 1 - F(x) + F(-x)$, $x \geq 0$. We obtain upper estimates for $\lim_{r \uparrow R} (\ln M(r, \varphi)) / \Phi(r)$ for some positive convex on $(0, R)$ function Φ under certain conditions on W_F .

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Пусть φ — характеристическая функция вероятностного закона F , аналитическая в $\mathbb{D}_R = \{z: |z| < R\}$, $0 < R \leq +\infty$, $M(r, \varphi) = \max\{|\varphi(z)|: |z| = r\}$ и $W_F(x) = 1 - F(x) + F(-x)$, $x \geq 0$. Для некоторой положительной выпуклой на $(0, R)$ функции Φ при определенных условиях на W_F получены оценки сверху для $\lim_{r \uparrow R} (\ln M(r, \varphi)) / \Phi(r)$.

1. Introduction and preliminary results. A non-decreasing left continuous on $(-\infty, +\infty)$ function F is said to be a *probability law* ([1, p. 10]) if $\lim_{x \rightarrow +\infty} F(x) = 1$ and $\lim_{x \rightarrow -\infty} F(x) = 0$, and the function $\varphi(z) = \int_{-\infty}^{+\infty} e^{izx} dF(x)$ defined for real z is called the *characteristic function of this law* ([1, p. 12]). If φ has an analytic extension to the disk $\mathbb{D}_R = \{z: |z| < R\}$, $0 < R \leq +\infty$, then we call φ to be the analytic in \mathbb{D}_R characteristic function of the law F . In the sequel we always assume that \mathbb{D}_R is the maximal disk of the analyticity of φ . It is known ([1, p. 37–38]) that φ is the analytic in \mathbb{D}_R characteristic function of a law F if and only if $W_F(x) =: 1 - F(x) + F(-x) = O(e^{-rx})$ as $0 \leq x \rightarrow +\infty$ for every $r \in [0, R)$. Hence it follows that $\lim_{x \rightarrow +\infty} \frac{1}{x} \ln \frac{1}{W_F(x)} = R$. If we put $M(r, \varphi) = \max\{|\varphi(z)|: |z| = r\}$ and $\mu(r, \varphi) = \sup\{W_F(x)e^{rx}: x \geq 0\}$ for $0 \leq r < R$ then ([1, p. 55], see also [2]) $\mu(r, \varphi) \leq 2M(r, \varphi)$. Therefore, the estimates from below for $\ln M(r, \varphi)$ follow from the same estimates for $\ln \mu(r, \varphi)$. Further we assume that $\ln \mu(r, \varphi) \uparrow +\infty$ as $r \uparrow R$, that is

$$\overline{\lim}_{x \rightarrow +\infty} W_F(x)e^{Rx} = +\infty. \quad (1)$$

On the other hand ([1, p. 52]), $M(r, \varphi) \leq I(r, \varphi) + 1 + W_F(0)$ for all $r \in [0, R)$, where $I(r, \varphi) = \int_0^\infty W_F(x)e^{rx} dx$. Since it is possible to estimate $I(r, \varphi)$ via $\mu(r, \varphi)$ we will obtain

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the corresponding estimates for $M(r, \varphi)$ via $\mu(r, \varphi)$. Therefore, the investigation of the relationship between the growth of $M(r, \varphi)$ and the decrease of $W_F(x)$ reduces to the study of the behavior of $\mu(r, \varphi)$.

For entire characteristic functions the relationship between the growth of $M(r, \varphi)$ and the decrease of $W_F(x)$ in terms of the order and the type is investigated by B. Ramachandran ([3], see also [1, p. 54]). N. I. Jakovleva ([4–5]) obtained such a relationship in terms of generalized orders. Some additions to the results of N. I. Jakovleva are obtained by B. V. Vynnyts'kyi ([6]) and M. Dewess ([7]). V. M. Sorokivs'kyi ([8]) investigated the relationship between the growth of $M(r, \varphi)$ and the decrease of $W_F(x)$ for analytic functions in the disk \mathbb{D}_1 . The most general results are obtained ([9]) for entire as well as analytic in \mathbb{D}_R , $R < +\infty$, characteristic functions.

For the lower order

$$\lambda[\varphi] = \varliminf_{r \rightarrow +\infty} \frac{\ln \ln M(r, \varphi)}{\ln r}$$

of an entire characteristic function φ N. I. Jakovleva ([4]) obtained a lower estimate. She proved that if

$$\varliminf_{x \rightarrow +\infty} \frac{\ln x}{\ln \left(\frac{1}{x} \ln \frac{1}{W_F(x)} \right)} \geq \lambda \quad \text{then} \quad \varliminf_{r \rightarrow +\infty} \frac{1}{\ln r} \ln \frac{\ln M(r, \varphi)}{r} \geq \lambda.$$

This result is generalized in [10]; namely, it is proved that if there exists an increasing to $+\infty$ sequence (x_k) such that $x_{k+1} = O(x_k)$ as $k \rightarrow +\infty$ and

$$\ln x_k \geq \lambda \ln \left(\frac{1}{x_k} \ln \frac{1}{W_F(x_k)} \right) \quad \text{then} \quad \varliminf_{r \rightarrow +\infty} \frac{1}{\ln r} \ln \frac{\ln M(r, \varphi)}{r} \geq \lambda.$$

Various generalizations of this result are obtained in [10] for entire as well as for analytic in \mathbb{D}_R , $R < +\infty$, characteristic functions.

For an upper estimate of $\lambda[\varphi]$ it is proved in [4] that if

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln \ln M(r, \varphi)}{\ln r} = \varrho > 1 \quad \text{and} \quad \overline{\lim}_{x \rightarrow +\infty, x \in U} \frac{\ln x}{\ln \left(\frac{1}{x} \ln \frac{1}{W_F(x)} \right)} \leq \delta < \varrho - 1,$$

where $U = \bigcup_j (a_{2j}, a_{2j+1})$ and $\overline{\lim}_{j \rightarrow \infty} (a_{2j}/a_{2j+1}) \leq \delta/(\varrho - 1)$ then $\lambda[\varphi] \leq 1 + \delta$.

Below we will show that, in this result the set U of intervals can be replaced with a sequence that increases to $+\infty$ not very quickly.

As in [10], by $\Omega(0, R)$, $0 < R \leq +\infty$, we denote the class of positive unbounded functions Φ on $[r_0, R)$ for some $r_0 \in [0, R)$ such that the derivative Φ' is positive, continuously differentiable and increasing to $+\infty$ on (r_0, R) . For $\Phi \in \Omega(0, R)$ let $\Psi(r) = r - \frac{\Phi(r)}{\Phi'(r)}$ be the function associated with Φ in the sense of Newton and ϕ be the inverse function to Φ' . It is known ([11]) that the function Ψ is continuously differentiable on $[r_0, R)$, $\Psi(r) \uparrow R$ as $r \uparrow R$, the function ϕ is continuously differentiable and increasing to R on $(x_0, +\infty)$ and the following lemma is true.

Lemma 1 ([11], Theorem 2.1). *Let $\Phi \in \Omega(0, R)$, $0 < R < +\infty$, and φ be an analytic in \mathbb{D}_R characteristic function of a probability law F , which satisfies (1). Then in order that $\ln \mu(r, \varphi) \leq \Phi(r)$ for all $r \in [r_0, R)$ it is necessary and sufficient that $\ln W_F(x) \leq -x\Psi(\phi(x))$ for all $x \geq x_0$.*

The following assertion is also true.

Proposition 1. *Let $\Phi \in \Omega(0, R)$, $0 < R \leq +\infty$, and φ be an analytic in \mathbb{D}_R characteristic function of a probability law F , which satisfies (1). Then if $\ln \mu(r_k, \varphi) \leq \Phi(r_k)$ for some sequence (r_k) increasing to R then*

$$\ln W_F(x_k) \leq -x_k \Psi(\phi(x_k)) \quad (2)$$

for all k , where $x_k = \Phi'(r_k)$.

Indeed, the condition $\ln \mu(r_k, \varphi) \leq \Phi(r_k)$ implies that $\ln W_F(x) \leq \Phi(r_k) - xr_k$ for all $x \geq 0$ and $k \geq 1$. Therefore,

$$\ln W_F(x_k) = \ln W_F(\Phi'(r_k)) \leq \Phi(r_k) - r_k \Phi'(r_k) = -\Phi'(r_k) \Psi(r_k) = -x_k \Psi(\phi(x_k)).$$

In view of Proposition 1 the question arises whether inequality (2) for some increasing to $+\infty$ sequence (x_k) implies the estimate $\ln \mu(r_k, \varphi) \leq \Phi(r_k)$ for some sequence (r_k) increasing to R . The answer is negative in general because the following statement is true.

Proposition 2. *For every function $\Phi \in \Omega(0, R)$, $0 < R \leq +\infty$, and increasing to $+\infty$ sequence (x_k) of positive numbers there exists a probability law F such that (2) holds for all $k \geq 1$ and $\ln \mu(r, \varphi) > \Phi(r)$ for all $r < R$.*

Proof. Indeed, let $F(x) = 0$ for $x \leq x_1$ and $F(x) = 1 - \exp\{-x_k \Psi(\phi(x_k))\}$ for $x \in [x_k, x_{k+1})$, $k \geq 1$. Then $\ln W_F(x) = \ln W_F(x_k) = -x_k \Psi(\phi(x_k))$ for all $x \in [x_k, x_{k+1})$ and $k \geq 1$. Therefore, if $r \in [\phi(x_k), \phi(x_{k+1})]$ then

$$\begin{aligned} \ln \mu(r, \varphi) &\geq \sup\{\ln W_F(x) + rx : x_k \leq x < x_{k+1}\} = \\ &= \sup\{\ln W_F(x_k) + rx : x_k \leq x < x_{k+1}\} = -x_k \Psi(\phi(x_k)) + rx_{k+1}. \end{aligned} \quad (3)$$

On $[\phi(x_k), \phi(x_{k+1})]$ we consider the function $A(r) = (-x_k \Psi(\phi(x_k)) + rx_{k+1})/\Phi(r)$. Then $A'(r) = a(r)/\Phi(r)^2$, where $a(r) = \Phi(r)x_{k+1} - \Phi'(r)(rx_{k+1} - x_k \Psi(\phi(x_k)))$. Since

$$\begin{aligned} a(\phi(x_k)) &= \Phi(\phi(x_k))x_{k+1} - x_k(\phi(x_k) - x_k \Psi(\phi(x_k))) = \\ &= x_{k+1}(\Phi(\phi(x_k)) - x_k(\phi(x_k))) + x_k^2 \Psi(\phi(x_k)) = \\ &= -x_{k+1}x_k \Psi(\phi(x_k)) + x_k^2 \Psi(\phi(x_k)) = -(x_{k+1} - x_k)x_k \Psi(\phi(x_k)) < 0, \\ a(\phi(x_{k+1})) &= \Phi(\phi(x_{k+1}))x_{k+1} - x_{k+1}(x_{k+1}\phi(x_{k+1}) - x_k \Psi(\phi(x_k))) = x_{k+1}(\Phi(\phi(x_{k+1})) - \\ &\quad - x_{k+1}\phi(x_{k+1})) + x_k x_{k+1} \Psi(\phi(x_k)) = x_{k+1}(x_k \Psi(\phi(x_k)) - x_{k+1} \Psi(\phi(x_{k+1}))) < 0, \\ a'(r) &= x_{k+1}\Phi'(r) - \Phi''(r)(rx_{k+1} - x_k \Psi(\phi(x_k))) - \Phi'(r)x_{k+1} \leq -\Phi''(r)(\phi(x_k)x_{k+1} - \\ &\quad - x_k \phi(x_k) + \Phi(\phi(x_k))) = -\Phi''(r)((x_{k+1} - x_k)\phi(x_k) + \Phi(\phi(x_k))) < 0 \end{aligned}$$

we obtain that $a(r) < 0$ on $[\phi(x_k), \phi(x_{k+1})]$, the function $A(r)$ decreases on $[\phi(x_k), \phi(x_{k+1})]$ and, thus,

$$A(r) \geq \frac{-x_k \Psi(\phi(x_k)) + \phi(x_{k+1})x_{k+1}}{\Phi(\phi(x_{k+1}))} > \frac{-x_{k+1} \Psi(\phi(x_{k+1})) + x_{k+1}\phi(x_{k+1})}{\Phi(\phi(x_{k+1}))} = 1.$$

Therefore, in view of (3) and of the definition of $A(r)$ for $r \in [\phi(x_k), \phi(x_{k+1})]$ we have $(\ln \mu(r, \varphi))/\Phi(r) = A(r) > 1$. \square

2. Main results. Under additional assumptions on the decrease of W_F (i. e. on the growth of $\ln \mu(r, \varphi)$) it is possible to get from (2) estimates on $\ln \mu(r_k, \varphi)$ from above for some sequence $(r_k) \uparrow R$. Here we will suggest two related solutions of this problem. One of them is based on results from [12].

For $\Phi \in \Omega(0, R)$ and $\Phi'(x_0) \leq a < b < +\infty$ we put

$$G_1(a, b, \Phi) = \frac{ab}{b-a} \int_a^b \frac{\Phi(\varphi(t))}{t^2} dt, \quad G_2(a, b, \Phi) = \Phi \left(\frac{1}{b-a} \int_a^b \varphi(t) dt \right).$$

It is known ([13]) that $G_1(a, b, \Phi) < G_2(a, b, \Phi)$, and in [12] the following lemma is proved.

Lemma 2. *Let (x_k) be an increasing to $+\infty$ sequence of positive numbers, $\Phi \in \Omega(0, R)$, $0 < R < +\infty$, and $\mu_D(r)$ be the maximal term of formal Dirichlet series*

$$D(s) = \sum_{k=1}^{\infty} \exp\{-x_k \Psi(\phi(x_k)) + s x_k\}, \quad s = r + it.$$

Then

$$\overline{\lim}_{r \uparrow R} \frac{\ln \mu_D(r)}{\Phi(r)} = 1, \quad \overline{\lim}_{r \uparrow R} \frac{\ln \ln \mu_D(r)}{\ln \Phi(r)} = 1, \quad (4)$$

$$\underline{\lim}_{r \uparrow R} \frac{\ln \mu_D(r)}{\Phi(r)} = \underline{\lim}_{k \rightarrow \infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \quad (5)$$

and if

$$\ln \mu_D(r) + \left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1 \right) \ln \Phi(r) \geq 0, \quad r \in [r_0, R), \quad (6)$$

then

$$\underline{\lim}_{r \uparrow R} \frac{\ln \ln \mu_D(r)}{\ln \Phi(r)} = \underline{\lim}_{k \rightarrow \infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \quad (7)$$

Using Lemma 2 we prove the following theorem.

Theorem 1. *Let $\Phi \in \Omega(0, R)$, $0 < R \leq +\infty$, and φ be an analytic in \mathbb{D}_R characteristic function of a probability law F , which satisfies (1). We suppose that $\ln \mu(r, \varphi) \leq \Phi(r)$ for all $r \in [r_0, R)$ and $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \rightarrow \infty$, for some increasing to $+\infty$ sequence $X = (x_k)$ of positive numbers. Then*

$$\underline{\lim}_{r \uparrow R} \frac{\ln \mu(r, \varphi)}{\Phi(r)} \leq \underline{\lim}_{k \rightarrow \infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \quad (8)$$

and if

$$Q(r) + \left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1 \right) \ln \Phi(r) \geq q > -\infty, \quad r \in [r_0, R), \quad (9)$$

where $Q(r) \equiv 0$ if $R < +\infty$ and $Q(r) \equiv \ln r$ if $R = +\infty$, then

$$\underline{\lim}_{r \uparrow R} \frac{\ln \ln \mu(r, \varphi)}{\ln \Phi(r)} \leq \underline{\lim}_{k \rightarrow \infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \quad (10)$$

Proof. We put $x_0 = 0$ and $\mu(r, \varphi; X) = \max \{W_F(x_k)e^{rx_k} : k \geq 1\}$. Then the condition $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \rightarrow \infty$ yields that

$$\begin{aligned} \ln \mu(r, \varphi) &= \sup_{x \geq 0} (\ln W_F(x) + rx) = \max_{k \geq 0} \sup_{x_k \leq x < x_{k+1}} (\ln W_F(x) + rx) \leq \\ &\leq \max_{k \geq 0} (\ln W_F(x_k) + rx_{k+1}) = \max_{k \geq 0} (\ln W_F(x_{k+1}) + rx_{k+1} + \ln W_F(x_k) - \ln W_F(x_{k+1})) \leq \\ &\leq \max_{k \geq 0} (\ln W_F(x_{k+1}) + rx_{k+1}) + \text{const} \leq \ln \mu(r, \varphi; X) + \text{const}. \end{aligned} \quad (11)$$

On the other hand,

$$\ln \mu(r, \varphi) = \max_{k \geq 0} \sup_{x_k < x \leq x_{k+1}} (\ln W_F(x) + rx) \geq \max_{k \geq 0} (\ln W_F(x_{k+1}) + rx_{k+1}) \geq \ln \mu(r, \varphi; X)$$

and since $\ln \mu(r, \varphi) \leq \Phi(r)$ we have $\ln \mu(r, \varphi; X) \leq \Phi(r)$ for $r \in [r_0, R)$. Therefore, by Lemma 1 $\ln W_F(x_k) \leq -x_k \Psi(\phi(x_k))$ for all $k \geq k_0$. Hence it follows that $\ln \mu(r, \varphi; X) \leq \ln \mu_D(r)$ for $r \in [r_0, R)$. Therefore, by Lemma 2 from (5) we obtain

$$\lim_{r \uparrow R} \frac{\ln \mu(r, \varphi; X)}{\Phi(r)} \leq \lim_{k \rightarrow \infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \quad (12)$$

and if condition (6) holds then (7) implies

$$\lim_{r \uparrow R} \frac{\ln \ln \mu(r, \varphi; X)}{\ln \Phi(r)} \leq \lim_{k \rightarrow \infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \quad (13)$$

We remark that (9) implies (6), because if $R < +\infty$ then (4) implies $\ln \mu_D(r) \uparrow +\infty$ as $r \uparrow R$, and if $R = +\infty$ then $(\ln \mu_D(r))/r \rightarrow \infty$ as $r \rightarrow +\infty$, that is $\ln \ln \mu_D(r) - \ln r \rightarrow +\infty$ as $r \rightarrow +\infty$.

Inequalities (8) and (10) follow from (11)–(13). \square

If $R = +\infty$ then the condition $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \rightarrow \infty$, can be replaced with some weaker condition provided that the function $\Phi \in \Omega(0, +\infty)$ grows not very quickly.

Let L be the class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$, $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \rightarrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \rightarrow +\infty$, and $\alpha \in L_{si}$ if $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \rightarrow +\infty$ for each $c \in (0, +\infty)$.

Theorem 2. Let $\Phi \in \Omega(0, +\infty)$ and φ be an entire characteristic function of a probability law F and $\ln \mu(r, \varphi) \leq \Phi(r)$ for all $r \geq r_0$. Then:

- 1) if $\Phi \in L^0$ and $\ln W_F(x_k) = (1 + o(1)) \ln W_F(x_{k+1})$, $k \rightarrow \infty$, for some increasing to $+\infty$ sequence $X = (x_k)$ of positive numbers then inequality (8) holds;
- 2) if $\ln \Phi \in L_{si}$,

$$\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1 \right) \ln \Phi(r) + \ln r \geq q > -\infty, \quad r \geq r_0, \quad (14)$$

and if there exists an increasing to $+\infty$ sequence $X = (x_k)$ of positive numbers such that $\ln W_F(x_k) \leq a \ln W_F(x_{k+1})$ for some $a \in (0, 1)$ and all $k \geq k_0$ then inequality (10) holds.

Proof. We begin with the first part. Since $\ln W_F(x_k) \leq (1 - \varepsilon) \ln W_F(x_{k+1})$ for each $\varepsilon \in (0, 1)$ and all $k \geq k_0 = k_0(\varepsilon)$, instead of (11) now we have

$$\begin{aligned} \ln \mu(r, \varphi) &\leq \max_{k \geq 0} (\ln W_F(x_k) + rx_{k+1}) = \\ &= \max \left\{ \max_{0 \leq k \leq k_0} (\ln W_F(x_k) + rx_{k+1}), \max_{k \geq k_0} \left(\frac{\ln W_F(x_k)}{\ln W_F(x_{k+1})} \ln W_F(x_{k+1}) + rx_{k+1} \right) \right\} \leq \\ &\leq \max \{ rx_{k_0+1}, \max_{k \geq k_0} ((1 - \varepsilon) \ln W_F(x_{k+1}) + rx_{k+1}) \} \leq (1 - \varepsilon) \max_{k \geq 0} (\ln W_F(x_k) + rx_{k+1}) + \\ &\quad + rx_{k_0+1} \leq (1 - \varepsilon) \ln \mu(r/(1 - \varepsilon), \varphi; X) + rx_{k_0+1}. \end{aligned} \quad (15)$$

Therefore, from (12) we obtain

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{\ln \mu(r, \varphi)}{\Phi(r)} &\leq (1 - \varepsilon) \lim_{r \rightarrow +\infty} \frac{\ln \mu(r/(1 - \varepsilon), \varphi; X)}{\Phi(r)} \leq \\ &\leq (1 - \varepsilon) \lim_{r \rightarrow +\infty} \frac{\ln \mu(r, \varphi; X)}{\Phi(r)} \lim_{r \rightarrow +\infty} \frac{\Phi(r/(1 - \varepsilon))}{\Phi(r)} \leq (1 - \varepsilon) A(\varepsilon) \lim_{k \rightarrow \infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}, \end{aligned} \quad (16)$$

where $A(\varepsilon) = \overline{\lim}_{r \rightarrow +\infty} \frac{\Phi(r/(1 - \varepsilon))}{\Phi(r)}$. For $\Phi \in L^0$ in [14] it is proved that $A(\varepsilon) \searrow 1$ as $\varepsilon \downarrow 0$. Therefore, (16) implies (8).

For the proof of the second part we remark that now instead of (15) we have $\ln \mu(r, \varphi) \leq a \ln \mu(r/a, \varphi; X)$, and (14) implies (9). Therefore, from (13) we obtain

$$\lim_{r \rightarrow +\infty} \frac{\ln \ln \mu(r, \varphi)}{\ln \Phi(r)} \leq \lim_{r \rightarrow +\infty} \frac{\ln \ln \mu(r/a, \varphi; X)}{\ln \Phi(r/a)} \lim_{r \rightarrow +\infty} \frac{\ln \Phi(r/a)}{\ln \Phi(r)} \leq \lim_{k \rightarrow \infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \quad \square$$

If the function $\ln W_F(x)$ is smooth enough, then it is possible to get an estimate of $\ln \mu(r, \varphi)$.

Theorem 3. Let $\Phi \in \Omega(0, R)$, $0 < R \leq +\infty$, and φ be an analytic in \mathbb{D}_R characteristic function of a probability law F such that $\ln W_F(x) = -V(x)$ for all $x \geq a$, where the function V is positive, continuously differentiable and $V'(x) \uparrow R$ as $0 < x \uparrow +\infty$. If conditions (1) and (2) hold for some sequence (x_k) of positive numbers then $\ln \mu(r_k, \varphi) \leq \Phi(r_k) + ar_k$, where $r_k = V'(x_k)$.

Proof. Clearly

$$\ln \mu(r, \varphi) \leq \max \left\{ \sup_{0 \leq x \leq a} (\ln W_F(x) + rx), \sup_{x \geq a} (\ln W_F(x) + rx) \right\} \leq ar + \max_{x \geq a} (-V(x) + rx)$$

and $\max_{x \geq a} (-V(x) + rx) = (-V(x) + rx)|_{x=v(r)}$, where $v(r)$ is the inverse function to V' . Therefore,

$$\begin{aligned} \ln \mu(r_k, \varphi) &\leq -V(v(r_k)) + r_k v(r_k) + ar_k = -V(v(V'(x_k))) + V'(x_k) v(V'(x_k)) + ar_k = \\ &= V(x_k) + r_k x_k + ar_k \leq \max_{j \geq 1} (V(x_j) + x_j r_k) + ar_k = \max_{j \geq 1} (\ln W_F(x_j) + x_j r_k) + ar_k \leq \\ &\leq \max_{j \geq 1} (-x_j \Psi(\phi(x_j)) + x_j r_k) + ar_k \leq \max_{x \geq a} (-x \Psi(\phi(x)) + xr_k) + ar_k \leq \Phi(r_k) + ar_k, \end{aligned}$$

because $(x \Psi(\phi(x)))' = \phi(x)$ and $(-x \Psi(\phi(x)) + xr)|_{x=\Phi'(r)} = \Phi(r)$. \square

3. Corollaries. Examining the scale of growth in Theorems 1–2 it is possible to get a number of results for analytic in \mathbb{D}_R characteristic functions. Here we will restrict ourselves only by three cases which arise often in mathematical literature. The most often used characteristics of growth for analytic in \mathbb{D}_R , $0 < R < +\infty$ functions φ are the order $\varrho_*[\varphi]$, the lower order $\lambda_*[\varphi]$ and (if $0 < \varrho_*[\varphi] < +\infty$) the type $T_*[\varphi]$ and the lower type $t_*[\varphi]$, which are defined by the formulas

$$\varrho_*[\varphi] = \overline{\lim}_{r \uparrow R} \frac{\ln \ln M(r, \varphi)}{\ln(1/(R-r))}, \quad \lambda_*[\varphi] = \lim_{r \uparrow R} \frac{\ln \ln M(r, \varphi)}{\ln(1/(R-r))}, \quad (17)$$

$$T_*[\varphi] = \overline{\lim}_{r \uparrow R} (R-r)^{\varrho_*[\varphi]} \ln M(r, \varphi), \quad t_*[\varphi] = \lim_{r \uparrow R} (R-r)^{\varrho_*[\varphi]} \ln M(r, \varphi). \quad (18)$$

We will show that in these formulas $\ln M(r, \varphi)$ can be replaced with $\ln \mu(r, \varphi)$. Indeed ([1, p. 55])

$$\ln \mu(r, \varphi) \leq \ln M(r, \varphi) + \ln 2. \quad (19)$$

On the other hand ([1, p. 52]), if $0 < \eta(r) < R-r$, then

$$\begin{aligned} M(r, \varphi) &\leq \int_0^\infty W_F(x) e^{rx} dx + 1 + W_F(0) = \int_0^\infty W_F(x) e^{(r+\eta(r))x} e^{-\eta(r)x} dx + 1 + W_F(0) \leq \\ &\leq \frac{1}{\eta(r)} \mu(r + \eta(r), \varphi) + 1 + W_F(0), \end{aligned}$$

that is

$$\ln M(r, \varphi) \leq \ln \mu(r + \eta(r), \varphi) - \ln \eta(r) + o(1), \quad r \uparrow R. \quad (20)$$

We choose $\eta(r) = (R-r)^2$. Then for $r > R-1$ from (20) we obtain

$$\ln M(r, \varphi) \leq \ln \mu(r + (R-r)^2, \varphi) + 2 \ln(1/(R-r)) + o(1), \quad r \uparrow R. \quad (21)$$

Since $\frac{R-r+(R-r)^2}{R-r} \rightarrow 1$, $\frac{\ln(R-r+(R-r)^2)}{\ln(R-r)} \rightarrow 1$, $\frac{\ln \ln(1/(R-r))}{\ln(1/(R-r))} \rightarrow 0$ and $(R-r)^{\varrho_*[\varphi]} \ln(1/(R-r)) \rightarrow 0$ as $r \uparrow R$, from (19) and (21) it follows that in formulas (17) and (18) $\ln M(r, \varphi)$ can be replaced with $\ln \mu(r, \varphi)$.

Therefore, if $\varrho_*[\varphi] < +\infty$ ($T_*[\varphi] < +\infty$) then $\ln \mu(r, \varphi) \leq \frac{T}{(R-r)^\varrho}$ for all $r \in [r_0(\varepsilon), R)$, where either $\varrho = \varrho_*[\varphi] + \varepsilon$ and $T = 1$ or $\varrho = \varrho_*[\varphi]$ and $T = T_*[\varphi] + \varepsilon$. For a function $\Phi \in \Omega(0, R)$ such that $\Phi(r) = \frac{T}{(R-r)^\varrho}$ for all $r \in [r_0(\varepsilon), R)$ we have

$$\Phi'(r) = \frac{T\varrho}{(R-r)^{\varrho+1}}, \quad \phi(x) = R - (T\varrho/x)^{1/(\varrho+1)}, \quad \frac{\Phi(\phi(x))}{x^2} = T(T\varrho)^{-\varrho/(\varrho+1)} x^{\varrho/(\varrho+1)-2}$$

for $x \geq x_0(\varepsilon)$. Hence it follows that for $k \geq k_0(\varepsilon)$

$$G_1(x_k, x_{k+1}, \Phi) = \frac{T(\varrho+1)}{(T\varrho)^{\varrho/(\varrho+1)}} \frac{x_k x_{k+1}}{x_{k+1} - x_k} \left(\frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}} \right), \quad (22)$$

$$G_2(x_k, x_{k+1}, \Phi) = T \left(\frac{(\varrho+1)(T\varrho)^{1/(\varrho+1)}}{\varrho} \frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1} - x_k} \right)^{-\varrho}. \quad (23)$$

Further we remark that

$$\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1 \right) \ln \Phi(r) = \frac{1}{\varrho} \ln \frac{T}{(R-r)^\varrho} \uparrow +\infty, \quad r \uparrow R,$$

that is (9) holds. Therefore, if $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \rightarrow \infty$, then by Theorem 1 in view of (22)–(23) and of arbitrariness of ε ,

$$\lambda_*[\varphi] \leq \varrho_*[\varphi] \lim_{k \rightarrow \infty} \frac{\ln \left(\frac{x_k x_{k+1}}{x_{k+1} - x_k} \left(\frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}} \right) \right)}{\ln \left(\frac{x_{k+1} - x_k}{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}} \right)^\varrho}, \quad (24)$$

$$t_*[\varphi] \leq T_*[\varphi] \frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} \lim_{k \rightarrow \infty} \frac{x_k x_{k+1}}{x_{k+1} - x_k} \left(\frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}} \right) \left(\frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1} - x_k} \right)^\varrho. \quad (25)$$

We suppose that

$$\beta =: \lim_{k \rightarrow \infty} \frac{\ln x_k}{\ln x_{k+1}} < 1.$$

Then there exists a number $\beta^* \in (\beta, 1)$ and an increasing sequence (k_j) of positive integers such that $\ln x_{k_j} \leq \beta^* \ln x_{k_j+1}$, that is $x_{k_j} = o(x_{k_j+1})$ as $j \rightarrow \infty$. Therefore, from (24) we obtain

$$\begin{aligned} \lambda_*[\varphi] &\leq \varrho_*[\varphi] \lim_{j \rightarrow \infty} \frac{\ln \left(\frac{x_{k_j} x_{k_j+1}}{x_{k_j+1} - x_{k_j}} \left(\frac{1}{x_{k_j}^{1/(\varrho+1)}} - \frac{1}{x_{k_j+1}^{1/(\varrho+1)}} \right) \right)}{\ln \left(\frac{x_{k_j+1} - x_{k_j}}{x_{k_j+1}^{\varrho/(\varrho+1)} - x_{k_j}^{\varrho/(\varrho+1)}} \right)^\varrho} = \\ &= \varrho_*[\varphi] \lim_{j \rightarrow \infty} \frac{\ln x_{k_j}^{\varrho/(\varrho+1)}}{\varrho \ln x_{k_j+1}^{1/(\varrho+1)}} = \varrho_*[\varphi] \lim_{j \rightarrow \infty} \frac{\ln x_{k_j}}{\ln x_{k_j+1}} \leq \varrho_*[\varphi] \beta^*, \end{aligned}$$

i. e. in view of arbitrariness of β^* we have the inequality $\lambda_*[\varphi] \leq \beta \varrho_*[\varphi]$. For $\beta = 1$ this inequality is trivial.

Now we suppose that

$$\gamma =: \lim_{k \rightarrow \infty} \frac{x_k}{x_{k+1}} \in (0, 1).$$

Then there exist an increasing sequence (k_j) of positive integers such that $x_{k_j} = (1 + o(1)) \times \gamma x_{k_j+1}$ as $j \rightarrow \infty$. Therefore, from (25) we obtain

$$\begin{aligned} t_*[\varphi] &\leq T_*[\varphi] \frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} \lim_{j \rightarrow \infty} \frac{x_{k_j} x_{k_j+1}}{x_{k_j+1} - x_{k_j}} \left(\frac{1}{x_{k_j}^{1/(\varrho+1)}} - \frac{1}{x_{k_j+1}^{1/(\varrho+1)}} \right) \left(\frac{x_{k_j+1}^{\varrho/(\varrho+1)} - x_{k_j}^{\varrho/(\varrho+1)}}{x_{k_j+1} - x_{k_j}} \right)^\varrho \leq \\ &\leq T_*[\varphi] \frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} \frac{\gamma}{\gamma - 1} \left(\frac{1}{\gamma^{1/(\varrho+1)}} - 1 \right) \frac{(1 - \gamma^{\varrho/(\varrho+1)})^\varrho}{(1 - \gamma)^\varrho} = T_*[\varphi] \frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} A(\gamma), \quad (26) \end{aligned}$$

where

$$A(\gamma) =: \frac{\gamma^{\varrho/(\varrho+1)} (1 - \gamma^{1/(\varrho+1)}) (1 - \gamma^{\varrho/(\varrho+1)})^\varrho}{(1 - \gamma)^{\varrho+1}}.$$

It is easy to show that $A(\gamma) \rightarrow \frac{\varrho^\varrho}{(\varrho+1)^{\varrho+1}}$ as $\gamma \rightarrow 1$, that is, (26) is transformed in the obvious inequality $t_*[\varphi] \leq T_*[\varphi]$ as $\gamma \rightarrow 1$. If $\gamma = 0$ then $x_{k_j} = O(x_{k_j+1})$ as $j \rightarrow \infty$ and from (25) we obtain easily that $t_*[\varphi] = 0$. This equality follows from (26), because $A(0) = 0$. Thus, the following corollary is proved.

Corollary 1. *Let the characteristic function φ of a probability law F be analytic in \mathbb{D}_R , $0 < R < +\infty$, have order $\varrho_*[\varphi]$ and lower order $\lambda_*[\varphi]$. Assume that $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \rightarrow \infty$, for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\lim_{k \rightarrow \infty} \frac{\ln x_k}{\ln x_{k+1}} = \beta$, where β is some nonnegative constant. Then $\lambda_*[\varphi] \leq \beta \varrho_*[\varphi]$. If, moreover, φ has type $T_*[\varphi]$ and lower type $t_*[\varphi]$ and $\lim_{k \rightarrow \infty} \frac{x_k}{x_{k+1}} = \gamma$ then $\tau_*[\varphi] \leq T_*[\varphi] \frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}} A(\gamma)$.*

For an entire characteristic function φ of order $\varrho[\varphi] \in (1, +\infty)$ the quantities

$$T[\varphi] = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, \varphi)}{r^{\varrho[\varphi]}}, \quad t[\varphi] = \underline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, \varphi)}{r^{\varrho[\varphi]}} \quad (27)$$

are called the *type* and the *lower type* of φ . From (20) for $\eta(r) = 1$ we obtain

$$\ln M(r, \varphi) \leq \ln \mu(r+1, \varphi) + o(1), \quad r \rightarrow +\infty.$$

Combining this with (19) we conclude that in (27) $\ln M(r, \varphi)$ can be replaced with $\ln \mu(r, \varphi)$. Therefore, we choose $\Phi \in \Omega(0, +\infty)$ such that $\Phi(r) = Tr^{\varrho}$ for $r \geq r_0 = r_0(\varepsilon)$, where either $\varrho = \varrho[\varphi] + \varepsilon$ and $T = 1$ or $\varrho = \varrho[\varphi]$ and $T = T[\varphi] + \varepsilon$. Then $\Phi \in L^0$, $\ln \Phi \in L_{si}$ and $\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1 \right) \ln \Phi(r) = \frac{\varrho-1}{\varrho} \ln \Phi(r) \rightarrow +\infty$ as $r \rightarrow +\infty$. It is known [15] that for this function

$$G_1(x_k, x_{k+1}, \Phi) = (\varrho - 1)T^{-1/(\varrho-1)}\varrho^{-\varrho/(\varrho-1)} \frac{x_k x_{k+1}}{x_{k+1} - x_k} \left(x_{k+1}^{1/(\varrho-1)} - x_k^{1/(\varrho-1)} \right)$$

and

$$G_2(x_k, x_{k+1}, \Phi) = (\varrho - 1)^{\varrho} T^{-1/(\varrho-1)} \varrho^{-\varrho^2/(\varrho-1)} \left(\frac{x_{k+1}^{\varrho/(\varrho-1)} - x_k^{\varrho/(\varrho-1)}}{x_{k+1} - x_k} \right)^{\varrho}.$$

Therefore, if $x_{k_j} = (1 + o(1))\gamma x_{k_j+1}$ as $j \rightarrow \infty$, where $0 < \gamma < 1$, then

$$\lim_{j \rightarrow \infty} \frac{G_1(x_{k_j}, x_{k_j+1}, \Phi)}{G_2(x_{k_j}, x_{k_j+1}, \Phi)} = \frac{\varrho^{\varrho}}{(\varrho - 1)^{\varrho-1}} \frac{\gamma(1 - \gamma)^{\varrho-1}(1 - \gamma^{1/(\varrho-1)})}{(1 - \gamma^{\varrho/(\varrho-1)})^{\varrho}}$$

and if $\ln x_{k_j} \leq \beta^* \ln x_{k_j+1}$, where $0 < \beta^* < 1$, then $x_{k_j} = o(x_{k_j+1})$, $j \rightarrow \infty$, and

$$\lim_{j \rightarrow \infty} \frac{\ln G_1(x_{k_j}, x_{k_j+1}, \Phi)}{\ln G_2(x_{k_j}, x_{k_j+1}, \Phi)} \leq \frac{\beta^*(\varrho - 1) + 1}{\varrho}.$$

So, as in the proof of Corollary 1, using Theorem 2 in view of arbitrariness of β^* we obtain the following corollary.

Corollary 2. *Let the entire characteristic function φ of a probability law F have the order $\varrho[\varphi] > 1$, the lower order $\lambda[\varphi]$, the type $T[\varphi]$ and the lower type $t[\varphi]$. Then:*

1) *if $\ln W_F(x_k) \leq a \ln W_F(x_{k+1})$, $0 < a < 1$, for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\lim_{k \rightarrow \infty} \frac{\ln x_k}{\ln x_{k+1}} = \beta$ then $\lambda[\varphi] - 1 \leq \beta(\varrho[\varphi] - 1)$;*

2) *if $\ln W_F(x_k) = (1 + o(1)) \ln W_F(x_{k+1})$ as $k \rightarrow \infty$ for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\lim_{k \rightarrow \infty} \frac{x_k}{x_{k+1}} = \gamma$ then $t[\varphi] \leq T[\varphi] \frac{\varrho^{\varrho}}{(\varrho-1)^{\varrho-1}} A_1(\gamma)$, where $A_1(\gamma) = \frac{\gamma(1-\gamma)^{\varrho-1}(1-\gamma^{1/(\varrho-1)})}{(1-\gamma^{\varrho/(\varrho-1)})^{\varrho}}$.*

If we define the modified order $\varrho_m[\varphi] = \overline{\lim}_{r \rightarrow +\infty} \frac{1}{\ln r} \ln \frac{M(r, \varphi)}{r}$ and the modified lower order $\lambda_m[\varphi] = \underline{\lim}_{r \rightarrow +\infty} \frac{1}{\ln r} \ln \frac{M(r, \varphi)}{r}$, then $\varrho_m[\varphi] = \varrho[\varphi] - 1$, $\lambda_m[\varphi] = \lambda[\varphi] - 1$ and under the assumptions of item 1) of Corollary 2 we have the inequality $\lambda_m[\varphi] \leq \beta \varrho_m[\varphi]$, which is an analog of the inequality from Corollary 1.

If for an entire characteristic function φ the function $\ln M(r, \varphi)$ increases faster than the power functions it is possible to use Theorem 1. We will demonstrate this by the example of R -order $\varrho_R[\varphi]$, lower R -order $\lambda_R[\varphi]$, R -type $T_R[\varphi]$ and lower R -type $t_R[\varphi]$, which are defined by the formulas

$$\varrho_R[\varphi] = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln \ln M(r, \varphi)}{r}, \quad \lambda_R[\varphi] = \underline{\lim}_{r \rightarrow +\infty} \frac{\ln \ln M(r, \varphi)}{r},$$

$$T_R[\varphi] = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, \varphi)}{\exp\{r \varrho_R[\varphi]\}}, \quad t_R[\varphi] = \underline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, \varphi)}{\exp\{r \varrho_R[\varphi]\}}.$$

For $\eta(r) = 1/r$ (20) implies the inequality $\ln M(r, \varphi) \leq \ln \mu(r + 1/r, \varphi) + \ln r + o(1)$, $r \rightarrow +\infty$. From here and (19) it follows that in the formulas for $\varrho_R[\varphi]$, $\lambda_R[\varphi]$, $T_R[\varphi]$ and $t_R[\varphi]$, the function $\ln M(r, \varphi)$ can be replaced with the function $\ln \mu(r, \varphi)$. Therefore, we choose $\Phi(r) = Te^{r\varrho}$ for $r \geq r_0 = r_0(\varepsilon)$, where either $\varrho = \varrho_R[\varphi] + \varepsilon$ and $T = 1$ or $\varrho = \varrho_R[\varphi]$ and $T = T_R[\varphi] + \varepsilon$. It is known ([15]) that for this function

$$G_1(x_k, x_{k+1}, \Phi) = \frac{1}{\varrho} \frac{x_k x_{k+1}}{x_{k+1} - x_k} \ln \frac{x_{k+1}}{x_k}, \quad G_2(x_k, x_{k+1}, \Phi) = \frac{1}{e\varrho} \exp \left\{ \frac{x_{k+1} \ln x_{k+1} - x_k \ln x_k}{x_{k+1} - x_k} \right\}.$$

If now $\ln x_{k_j} \leq \beta^* \ln x_{k_j+1}$, where $0 < \beta^* < 1$, then $x_{k_j} = o(x_{k_j+1})$, $j \rightarrow \infty$, and

$$\underline{\lim}_{j \rightarrow \infty} \frac{\ln G_1(x_{k_j}, x_{k_j+1}, \Phi)}{\ln G_2(x_{k_j}, x_{k_j+1}, \Phi)} \leq \beta^*$$

and if $x_{k_j} = (1 + o(1))\gamma x_{k_j+1}$ as $j \rightarrow \infty$, where $0 < \gamma < 1$, then

$$\underline{\lim}_{j \rightarrow \infty} \frac{G_1(x_{k_j}, x_{k_j+1}, \Phi)}{G_2(x_{k_j}, x_{k_j+1}, \Phi)} = \frac{\gamma}{1 - \gamma} \exp \left\{ 1 + \frac{\gamma \ln \gamma}{1 - \gamma} \right\} \ln \frac{1}{\gamma}.$$

Repeating the proof of Corollary 1, we obtain the following corollary.

Corollary 3. *Let an entire characteristic function φ of a probability law F have R -order $\varrho_R[\varphi]$, lower R -order $\lambda_R[\varphi]$, R -type $T_R[\varphi]$ and lower R -type $t_R[\varphi]$. We suppose that $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$ as $k \rightarrow \infty$ for some increasing to $+\infty$ sequence (x_k) of positive numbers. If $\underline{\lim}_{k \rightarrow \infty} \frac{\ln x_k}{\ln x_{k+1}} = \beta$ then $\lambda_R[\varphi] \leq \beta \varrho_R[\varphi]$ and if*

$$\underline{\lim}_{k \rightarrow \infty} \frac{x_k}{x_{k+1}} = \gamma \quad \text{then} \quad t_R[\varphi] \leq T_R[\varphi] \frac{\gamma}{1 - \gamma} \exp \left\{ 1 + \frac{\gamma \ln \gamma}{1 - \gamma} \right\} \ln \frac{1}{\gamma}.$$

We demonstrate the application of Theorem 3 only for an entire characteristic function of finite R -order. It is easy to verify that for the function $\Phi(r) = e^{r\varrho}$ we have $x\Psi(\phi(x)) = \frac{x}{\varrho} \ln \frac{x}{e\varrho}$. Therefore, under the corresponding assumptions on W_F , Theorem 3 implies that if $\ln W_F(x_k) \leq -\frac{x_k}{\varrho} \ln \frac{x_k}{e\varrho}$ then $\ln \mu(r_k, \varphi) \leq (1 + o(1))e^{\varrho r_k}$, $k \rightarrow \infty$, where $r_k = V'(x_k)$. Hence the following corollary follows.

Corollary 4. *Let the characteristic function φ of a probability law F be entire and analytic in \mathbb{D}_R . Assume that $\ln W_F(x) = -V(x)$ for all $x \geq a$, where the function V is positive, continuously differentiable and $V'(x) \uparrow R$ as $0 < x \uparrow +\infty$. Then $\lambda_R[\varphi] \leq \lim_{x \rightarrow +\infty} \frac{x \ln x}{-\ln W_F(x)}$.*

One can obtain analogues of Corollary 3 for other scales of growth, but we are not going to discuss this here.

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