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MULTIPLICATIVELY PERIODIC MEROMORPHIC FUNCTIONS IN THE UPPER HALFPLANE

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Multiplicatively periodic meromorphic functions in the upper halfplane are considered. Two theorems which describe some properties and the value distribution of such functions are proved.

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Рассматриваются мультипликативно периодические мероморфные функции в верхней полуплоскости. Описываются некоторые свойства таких функций и распределение их значений.

The theory of multiplicatively periodic meromorphic functions in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ was elaborated by O. Rausenberger ([1]). G. Valiron ([2]) called these functions *loxodromic* because the points at which such a function in the case of non-real q acquires the same value lay on logarithmic spirals. The images of these last on the Riemann sphere intersect each meridian under the same angle, and are called loxodromic curves ($\lambda o\xi o\zeta$ — oblique, $\delta\rho o\mu o\zeta$ — way). In *log*-polar coordinates they are straight lines. The theory of loxodromic functions is tightly connected with the theory of elliptic functions ([3]). We study multiplicatively periodic meromorphic functions in the upper halfplane.

Let $\mathcal{H} = \{z\colon \operatorname{Im} z > 0\}$ and let $\mathcal{H}^* = \overline{\mathcal{H}}\backslash\{0\}$. The set $\mathcal{H}^*$ is invariant with respect to the multiplicative group $\mathbb{R}^+$.

A function f is meromorphic in the closure $\overline{G}$ of a domain $G \subset \mathbb{C}$ if there is an open set G_1, $G_1 \supset \overline{G}$, such that f is meromorphic in G_1.

Let $\mathcal{A}_t = \{z \in \overline{\mathcal{H}}\colon qt < |z| \le t\}$, $t > 0$, $0 < q < 1$. Note that $\bigcup_{t>0} \overline{\mathcal{A}_t} = \mathcal{H}^*$.

Definition 1. A function f is said to be *meromorphic in* $\mathcal{H}^*$ if it is meromorphic in the closure of any horseshoe $\overline{\mathcal{A}_t}$.

Definition 2. A meromorphic in $\mathcal{H}^*$ function f is said to be *multiplicatively periodic of multiplicator* q, $0 < q < 1$, if it satisfies the following condition

$$\forall z \in \mathcal{H}^* \quad f(qz) = f(z). \tag{1}$$

The set of all such functions is denoted by $\mathcal{M}_q$.

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Theorem 1. *Let $f \in \mathcal{M}_q$. Then*

1) *the number of a-points, $a \in \overline{\mathbb{C}}$, of function $f \not\equiv \text{const}$ in the horseshoe $\mathcal{A}_t$ is finite and does not depend on t;*
2) *if the function f is holomorphic and $f(-x) = f(x)$, $x \in \mathbb{R}^+$, then $f \equiv \text{const}$;*
3) *if $f(-x) = f(x)$, $x \in \mathbb{R}^+$, then the function $g(\zeta) = f(\sqrt{\zeta})$, $\sqrt{-1} = i$, is multiplicatively periodic in $\mathbb{C}^*$ of multiplicator q^2. The function $g\left(e^{\frac{2\pi i}{\omega_2}s}\right)$ is double periodic of the period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\frac{\omega_1}{\omega_2} = -\frac{i}{\pi}\log q$.*

Proof. Firstly we prove statement 1). Let m be the number of a-points of the function f in the horseshoe $\mathcal{A}_1$ and let z_j be one of them. According to the uniqueness theorem for analytic functions, $m < +\infty$. Since $f \in \mathcal{M}_q$, we have $f(q^k z_j) = a,\ k \in \mathbb{Z}$.

We must prove that $\exists! k\colon z_j q^k \in \mathcal{A}_t$.

To do this, we use the following inequality

$$\log(qt) < \log|z_j q^k| \leq \log t. \tag{2}$$

Then (2) implies

$$\frac{\log t - \log|z_j|}{\log q} \leq k < 1 + \frac{\log t - \log|z_j|}{\log q}.$$

The preceding inequality shows that such a k is unique.

Let now consider the log-scale and let $\tau = \log t$. Calculating the log-distance between the modules of the neighbouring a-points, we obtain $\log|q^k z_j| - \log|q^{k+1} z_j| = -\log q$.

The width of the horseshoe $\mathcal{A}_t$ in the log-scale also equals $\log t - \log(qt) = -\log q$. Thus, the interval $(\tau + \log q, \tau]$ contains exactly one term of the sequence $\{q^k z_j\}$, $k \in \mathbb{Z}$, which consists of m terms.

We proceed to the proof of statement 2) of Theorem 1.

Consider the function $\sqrt{z}$ in the complex plane with the slit along the non-negative axis denoting this domain by $\widetilde{\mathbb{C}}$ and choosing the branch of $\sqrt{z}$ such that $\sqrt{-1} = i$. That is, $\sqrt{z} = \sqrt{r}e^{i\frac{\varphi}{2}}$, $r > 0$, $0 < \varphi < 2\pi$. Then $\sqrt{z} \in \mathcal{H}$.

Put $g(z) = f(\sqrt{z})$. The function g is defined and holomorphic in $\widetilde{\mathbb{C}}$. Show that it has a holomorphic extension to the positive axis. Indeed, we have for $x > 0$

$$\lim_{\substack{r \to x \\ \varphi \to +0}} f(\sqrt{r}e^{i\frac{\varphi}{2}}) = f(\sqrt{x}), \quad \lim_{\substack{r \to x \\ \varphi \to 2\pi - 0}} f(\sqrt{r}e^{i\frac{\varphi}{2}}) = f(-\sqrt{x}) = f(\sqrt{x})$$

as well as $\lim\limits_{0 < x_n \to x} f(\sqrt{x_n}) = f(\sqrt{x})$.

Then g is continuous in a neighborhood of x. Applying the segment elimination theorem ([4]) we have that g is holomorphic in $\mathbb{C}^*$.

Since the function f is multiplicatively periodic of multiplicator q, the function g is also multiplicatively periodic in $\mathbb{C}^*$ of multiplicator q^2.

Indeed, $g(q^2 z) = f(\sqrt{q^2 z}) = f(q\sqrt{z}) = f(\sqrt{z}) = g(z)$.

Since the function g is holomorphic and multiplicatively periodic in $\mathbb{C}^*$, then ([3]) implies $g(z) \equiv \text{const}$ as well as $f(z) \equiv \text{const}$, which concludes the proof of property 2).

Taking into account the preceding proof, we have already proved that the function $g(z) = f(\sqrt{z})$, $\sqrt{-1} = i$, is multiplicatively periodic in $\mathbb{C}^*$ of multiplicator q^2. It remains to prove that the function $g(e^{2\pi i\frac{s}{\omega_2}})$ is double periodic of the period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\frac{\omega_1}{\omega_2} = -\frac{i}{\pi}\log q$. It is enough to prove that $g(e^{2\pi i\frac{(s+\omega_1+\omega_2)}{\omega_2}}) = g(e^{2\pi i\frac{s}{\omega_2}})$.

Indeed,

$$g\left(e^{2\pi i\frac{(s+\omega_1+\omega_2)}{\omega_2}}\right) = g\left(e^{2\pi i\frac{\omega_1}{\omega_2}}e^{2\pi i}e^{2\pi i\frac{s}{\omega_2}}\right) = g\left(q^2 e^{2\pi i\frac{s}{\omega_2}}\right) = g\left(e^{2\pi i\frac{s}{\omega_2}}\right),$$

where $q^2 = e^{2\pi i\frac{\omega_1}{\omega_2}}$, $\operatorname{Im}\frac{\omega_1}{\omega_2} > 0$. □

Let f be a function meromorphic in $\mathcal{H}^*$. Suppose that f has neither zeros nor poles on $\mathbb{R}\setminus\{0\}$. Let $z_0 \in \mathcal{H}^*$, $f(z_0) \neq 0, \infty$. Let $\log f(z_0)$ be well defined. Put

$$\log f(z) = \log f(z_0) + \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}d\zeta, \tag{3}$$

where the integral is taken along a path which connects the points z_0 and z in $\mathcal{H}^*$ with the radial slits, and

$$\arg f(z) = \operatorname{Im}\log f(z) = \arg f(z_0) + \operatorname{Im}\int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}d\zeta.$$

The value distribution of functions from $\mathcal{M}_q$ describes the following theorem.

Theorem 2. *Let $f \in \mathcal{M}_q$, $f \not\equiv \text{const}$ and $f(z) \not\equiv 0, \infty$ on $\mathbb{R}\setminus\{0\}$. Then*

1) *the sum of the $\arg f$ increments along the segments $[qt, t]$ and $[-t, -qt]$ does not depend on t and equals $2\pi\big(n_0(f) - n_\infty(f)\big)$, where $n_0(f)$, $n_\infty(f)$ are the numbers of zeros and poles of the function f in the horseshoe $\mathcal{A}_1 = \{z \in \overline{\mathcal{H}}\colon q < |z| \leq 1\}$ respectively;*
2) *let $z_n = r_n e^{i\alpha_n}$ be a-points of f, $a \in \mathbb{C}$, and $w_n = \rho_n e^{i\beta_n}$ be the poles of f in $\mathcal{A}_t$. Then*

$$\int_{qr}^{r} \sum_{qt<r_n\leq t}\left(\frac{q}{r_n} - \frac{r_n}{t^2}\right)\sin\alpha_n dt = \int_{qr}^{r} \sum_{qt<\rho_n\leq t}\left(\frac{q}{\rho_n} - \frac{\rho_n}{t^2}\right)\sin\beta_n dt$$

for any positive r.

Proof. Let $\partial\mathcal{A}_t = \Gamma_{qt} \cup \Gamma_t \cup [qt, t] \cup [-t, -qt]$, where Γ_{qt}, Γ_t are the semicircles of radii qt, t, respectively, centered at the origin. We have by the argument principle

$$\int_{\partial\mathcal{A}_t} \frac{f'(z)}{f(z)}dz = 2\pi i\big(n_0(f) - n_\infty(f)\big). \tag{4}$$

Thus, we can rewrite (4) as follows

$$it\int_0^\pi \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}e^{i\varphi}d\varphi - iqt\int_0^\pi \frac{f'(qte^{i\varphi})}{f(qte^{i\varphi})}e^{i\varphi}d\varphi + \int_{qt}^{t}\frac{f'(\tau)}{f(\tau)}d\tau + \int_{-t}^{-qt}\frac{f'(z)}{f(z)}dz = \\ = 2\pi i\big(n_0(f) - n_\infty(f)\big). \tag{5}$$

Using equalities $f(qz) = f(z)$, $f'(qz) = \frac{1}{q}f'(z)$ and (3), we conclude that (5) implies

$$[\log f(t) - \log f(qt)] + [\log f(-qt) - \log f(-t)] = 2\pi i\big(n_0(f) - n_\infty(f)\big). \tag{6}$$

Taking the imaginary parts of both sides of (6), we have

$$[\arg f(t) - \arg f(qt)] + [\arg f(-qt) - \arg f(-t)] = 2\pi\big(n_0(f) - n_\infty(f)\big).$$

So, the first statement of Theorem 2 is proved.

Let firstly z_n be the zeros of f in $\mathcal{A}_t$. Applying the residue theorem to the integrals $\int_\Gamma z\frac{f'(z)}{f(z)}dz$ and $\int_\Gamma \frac{1}{z}\frac{f'(z)}{f(z)}dz$, we get

$$\int_\Gamma z\frac{f'(z)}{f(z)}dz = 2\pi i\Bigg(\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n\Bigg),$$
$$\int_\Gamma \frac{1}{z}\frac{f'(z)}{f(z)}dz = 2\pi i\Bigg(\sum_{qt<|z_n|\le t} \frac{1}{z_n} - \sum_{qt<|w_n|\le t} \frac{1}{w_n}\Bigg).$$

The left hand side of each of these equalities equals

$$\begin{aligned} it^2\int_0^\pi \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}e^{2i\varphi}d\varphi - iqt^2\int_0^\pi \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}e^{2i\varphi}d\varphi + \int_{qt}^t \tau\frac{f'(\tau)}{f(\tau)}d\tau + \\ + \int_{-t}^{-qt} z\frac{f'(z)}{f(z)}dz = 2\pi i\Bigg(\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n\Bigg), \end{aligned} \tag{7}$$

$$\begin{aligned} i\int_0^\pi \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}d\varphi - \frac{i}{q}\int_0^\pi \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}d\varphi + \int_{qt}^t \frac{f'(\tau)}{f(\tau)}\frac{d\tau}{\tau} + \int_{-t}^{-qt} \frac{f'(z)}{f(z)}\frac{dz}{z} = \\ = 2\pi i\Bigg(\sum_{qt<|z_n|\le t} \frac{1}{z_n} - \sum_{qt<|w_n|\le t} \frac{1}{w_n}\Bigg), \end{aligned}$$

respectively.

Dividing (7) by t^2 and integrating over t from qr to r, we have

$$\begin{aligned} 2\pi i\int_{qr}^r \Bigg[\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n\Bigg]\frac{dt}{t^2} = \int_{qr}^r \left[\int_{qt}^t z\frac{f'(z)}{f(z)}dz\right]\frac{dt}{t^2} + \\ + \int_{qr}^r \left[\int_{-t}^{-qt} z\frac{f'(z)}{f(z)}dz\right]\frac{dt}{t^2} + i(1-q)\int_{qr}^r \left[\int_0^\pi \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}e^{2i\varphi}d\varphi\right]dt. \end{aligned}$$

By the Fubini theorem, the last equality implies

$$\begin{aligned} 2\pi i\int_{qr}^r \Bigg[\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n\Bigg]\frac{dt}{t^2} = \int_{qr}^r \left[\int_{qt}^t z\frac{f'(z)}{f(z)}dz\right]\frac{dt}{t^2} + \\ + \int_{qr}^r \left[\int_{-t}^{-qt} z\frac{f'(z)}{f(z)}dz\right]\frac{dt}{t^2} + i(1-q)\int_0^\pi e^{2i\varphi}\left[\int_{qr}^r \frac{f'(te^{i\varphi})}{f(te^{i\varphi})}dt\right]d\varphi. \end{aligned} \tag{8}$$

Denote

$$I_1 = \int_{qr}^r \left[\int_{qt}^t z\frac{f'(z)}{f(z)}dz\right]\frac{dt}{t^2}, \quad I_2 = \int_{qr}^r \left[\int_{-t}^{-qt} z\frac{f'(z)}{f(z)}dz\right]\frac{dt}{t^2}. \tag{9}$$

Applying (3) and (9) to (8), we get

$$2\pi i \int_{qr}^{r} \left[\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n \right] \frac{dt}{t^2} =$$
$$= I_1 + I_2 + i(1-q) \int_0^{\pi} \left[\log f(re^{i\varphi}) - \log f(qre^{i\varphi}) \right] e^{i\varphi} d\varphi. \tag{10}$$

Applying similar consideration to the integral $\int_{\partial \mathcal{A}_t} \frac{1}{z} \frac{f'(z)}{f(z)} dz$, we deduce

$$2\pi q i \int_{qr}^{r} \left[\sum_{qt<|z_n|\le t} \frac{1}{z_n} - \sum_{qt<|w_n|\le t} \frac{1}{w_n} \right] dt =$$
$$= qI_3 + qI_4 + i(q-1) \int_0^{\pi} \left[\log f(re^{i\varphi}) - \log f(qre^{i\varphi}) \right] e^{-i\varphi} d\varphi, \tag{11}$$

where $I_3 = \int_{qr}^{r} [\int_{qt}^{t} \frac{1}{z} \frac{f'(z)}{f(z)} dz] dt,\ I_4 = \int_{qr}^{r} [\int_{-t}^{-qt} \frac{1}{z} \frac{f'(z)}{f(z)} dz] dt.$

Summing up equalities (10) and (11), we obtain

$$2\pi i \int_{qr}^{r} \left[\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n \right] \frac{dt}{t^2} + +2\pi q i \int_{qr}^{r} \left[\sum_{qt<|z_n|\le t} \frac{1}{z_n} - \sum_{qt<|w_n|\le t} \frac{1}{w_n} \right] dt =$$
$$= I_1 + I_2 + qI_3 + qI_4 - 2(1-q) \int_0^{\pi} \left[\log f(re^{i\varphi}) - \log f(qre^{i\varphi}) \right] \sin \varphi d\varphi. \tag{12}$$

Since $\log |f(re^{i\varphi})| - \log |f(qre^{i\varphi})| = 0$, taking the real part of (12), we obtain

$$2\pi \operatorname{Re} \left[i \int_{qr}^{r} \left(\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|w_n|\le t} w_n \right) \frac{dt}{t^2} \right] +$$
$$+2\pi \operatorname{Re} \left[iq \int_{qr}^{r} \left(\sum_{qt<|z_n|\le t} \frac{1}{z_n} - \sum_{qt<|w_n|\le t} \frac{1}{w_n} \right) dt \right] = \operatorname{Re} \left(I_1 + I_2 + qI_3 + qI_4 \right). \tag{13}$$

The calculation of the real parts of the integrals I_1, I_2, I_3, I_4 gives the following equalities

$$\operatorname{Re} I_1 = \int_{qr}^{r} \frac{1}{t^2} \left[t \log |f(t)| - qt \log |f(qt)| - \int_{qt}^{t} \log |f(\tau)| d\tau \right] dt =$$
$$= (1-q) \int_{qr}^{r} \log |f(t)| \frac{dt}{t} + \int_{qr}^{r} \left(\int_{qt}^{t} \log |f(\tau)| d\tau \right) d\left(\frac{1}{t} \right) =$$
$$= (1-q) \int_{qr}^{r} \log |f(t)| \frac{dt}{t} + \frac{1}{r} \int_{qr}^{r} \log |f(t)| dt - \frac{1}{qr} \int_{q^2 r}^{qr} \log |f(t)| dt -$$
$$- \int_{qr}^{r} \frac{1}{t} \left[\log |f(t)| - q \log |f(qt)| \right] dt = \frac{1}{r} \int_{qr}^{r} \log |f(t)| dt - \frac{1}{qr} \int_{q^2 r}^{qr} \log |f(t)| dt.$$

Making the substitution $t = q\tau$ in the second integral, we obtain $\operatorname{Re} I_1 = 0$. Similarly we deduce that $\operatorname{Re} I_2 = 0$.

Further,

$$\operatorname{Re} I_3 = \int_{qr}^{r} \left[\frac{1}{t} \log|f(t)| - \frac{1}{qt} \log|f(qt)| + \int_{qt}^{t} \log|f(\tau)| \frac{d\tau}{\tau^2} \right] dt =$$
$$= \left(1 - \frac{1}{q}\right) \int_{qr}^{r} \log|f(t)| \frac{dt}{t} + r \int_{qr}^{r} \log|f(t)| \frac{dt}{t^2} - qr \int_{q^2 r}^{qr} \log|f(t)| \frac{dt}{t^2} -$$
$$- \int_{qr}^{r} t \left(\frac{\log|f(t)|}{t^2} - q \frac{\log|f(qt)|}{q^2 t^2} \right) dt = (1 - \frac{1}{q}) \int_{qr}^{r} \log|f(t)| \frac{dt}{t} -$$
$$- \left(1 - \frac{1}{q}\right) \int_{qr}^{r} \log|f(t)| \frac{dt}{t} + r \int_{qr}^{r} \log|f(t)| \frac{dt}{t^2} - qr \int_{q^2 r}^{qr} \log|f(t)| \frac{dt}{t^2} = 0.$$

Similarly we obtain that $\operatorname{Re} I_4 = 0$. Equality (13) implies

$$2\pi \operatorname{Re} \left[i \int_{qr}^{r} \left(\sum_{qt<|z_n|\le t} z_n - \sum_{qt<|z_n|\le t} w_n \right) \frac{dt}{t^2} \right] +$$
$$+ 2\pi \operatorname{Re} \left[iq \int_{qr}^{r} \left(\sum_{qt<|z_n|\le t} \frac{1}{z_n} - \sum_{qt<|w_n|\le t} \frac{1}{w_n} \right) dt \right] = 0. \qquad (14)$$

In view of the notation $z_n = r_n e^{i\alpha_n}$ and $w_n = \rho_n e^{i\beta_n}$ the calculation of the real part of (14) gives

$$\int_{qr}^{r} \sum_{qt<r_n\le t} \left(\frac{q}{r_n} - \frac{r_n}{t^2} \right) \sin\alpha_n dt - \int_{qr}^{r} \sum_{qt<\rho_n\le t} \left(\frac{q}{\rho_n} - \frac{\rho_n}{t^2} \right) \sin\beta_n dt = 0.$$

Applying the obtained result to the function $f(z) - a$, we deduce statement 2) of Theorem 2. □

Corollary 1. *Let f be a holomorphic and multiplicatively periodic function of multiplicator q, $0 < q < 1$, in $\mathcal{H}^*$, $f \ne \mathrm{const}$ and $f(z) \ne 0, \infty$ on $\mathbb{R}\backslash\{0\}$. Let $z_n = r_n e^{i\alpha_n}$ be a-points of f, $a \in \mathbb{C}$. Then $\int_{qr}^{r} \sum_{qt<r_n\le t} (\frac{q}{r_n} - \frac{r_n}{t^2}) \sin\alpha_n dt = 0$ for any positive r.*

REFERENCES

1. Rausenberger O., Lehrbuch der Theorie der periodischen Funktionen einer Variabeln, Leipzig, Druck und Ferlag von B.G.Teubner, 1884, 470 p.
2. Valiron G., Cours d'analyse mathematique, theorie des fonctions, 3nd Edition, Masson et.Cie., Paris, 1966, 522 p.
3. Hellegouarch Y., Invitation to the mathematics of Fermat-Wiles, Academic Press, 2002, 381 p.
4. Goldberg A.A., Sheremeta M.M., Zabolotskij M.V., Skaskiv O.B., Complex analysis, Textbook, Lviv, Afisha, 2008, 203 p.

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