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GENERALIZATION OF CASSINI FORMULAS FOR BALANCING AND LUCAS-BALANCING NUMBERS

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The mathematical identity that connects three adjacent balancing numbers is well known under the name Cassini formula, and is used to establish many important identities involving balancing numbers and their related sequences. This article is an attempt to draw attention to some of the unusual properties of generalized balancing numbers, in particular, to the generalized Cassini formula.

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Математическое тождество, соответствующее трем смежным сбалансированным числам, хорошо известно как формула Кассини и применяется к установлению многих важных тождеств, использующих сбалансированные числа и родственные последовательности. В статье сделана попытка обратить внимание на некоторые необычные свойства обобщенных сбалансированных чисел и, в частности, на обобщенную формулу Кассини.

1. Introduction. Recently, A. Behera et al. ([1]) introduced the sequence of balancing numbers as follows. A positive integer n is called a balancing number with balancer r, if it is a solution of the Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r).$$

The balancing numbers though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is that, n is a balancing number if and only if n^2 is a triangular number i.e. $8n^2+1$ is a perfect square. The square root of $8n^2+1$ also generates a sequence of numbers which are called the Lucas-balancing numbers. The balancing and Lucas-balancing numbers can be generated by the recurrence formulas $B_n = 6B_{n-1} - B_{n-2},\ n \geq 2,\ C_n = 6C_{n-1} - C_{n-2},\ n \geq 2$, with their respective initial terms $B_1 = 1,\ B_2 = 6$ and $C_1 = 3,\ C_2 = 17$ ([1, 8]). The closed forms which are also called as Binet's formulas for balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_1^n}{\lambda_1 - \lambda_2}, \quad C_n = \frac{\lambda_1^n + \lambda_1^n}{2}, \tag{1}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are the roots of the auxiliary equation $\lambda^2 - 6\lambda + 1 = 0$.

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G. K. Panda ([8]) established many important identities concerning balancing numbers and their related sequences. In [4] K. Liptai et al. added another interesting result to the theory of balancing numbers by generalizing these numbers. A. Berczes et al. ([2]) and P. Olajos ([5]) surveyed many interesting properties of generalized balancing numbers. Recently, G. K. Panda et al. ([9]) introduced gap balancing numbers and established many properties of these numbers. Some curious congruence properties of balancing numbers are also studied in ([12]). In ([10, 11]), Ray established new product formulae to generate both balancing and Lucas-balancing numbers. Recently, R. Keskin and O. Karaath ([3]) obtained some new properties of balancing numbers and square triangular numbers. Many interesting properties of balancing matrices and identities involving congruences are also studied in [13]–[16].

Among all the important identities for Fibonacci numbers, one of the most famous identity is the Cassini formula. The Cassini formulas for Fibonacci numbers and their related sequences are available in the literature. They play an important role for finding new identities for these numbers. Cassini formulas for Fibonacci numbers, Lucas numbers, Pell numbers, balancing numbers and Lucas-balancing numbers are respectively given by

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,\ L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1},$$
$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n,\ B_{n+1}B_{n-1} - B_n^2 = (-1),\ C_{n+1}C_{n-1} - C_n^2 = 8.$$

This article is an attempt to draw attention to some of the unusual properties of generalized balancing numbers, in particular, to the generalized Cassini formula.

2. The balancing λ-numbers. For any real number $\lambda > 0$, consider the recurrence relation

$$B_\lambda(n+2) = 6\lambda B_\lambda(n+1) - B_\lambda(n), \tag{2}$$

with initial terms $B_\lambda(0) = 0,\ B_{\lambda(1)} = 1$. Recurrence relation (2) generates an infinite number of numerical sequences for any λ. For $\lambda = 1$, (2) reduces to the recurrence relation for balancing numbers. For $\lambda = 2$, (2) reduces to the following recurrence relation

$$B_2(n+2) = 12B_2(n+1) - B_2(n),\ B_2(0) = 0,\ B_2(1) = 1,$$

which generates the sequence $12, 143, 1704, 20305, 241956$ and so on. Table 1 shows the four expanded sequences of the balancing λ-numbers corresponding to the values $\lambda \in \{1, 2, 3, 4\}$.

n	0	1	2	3	4	5	6
$B_1(n)$	0	1	6	35	204	1189	6930
$B_1(n)$	0	1	6	35	204	1189	6930
$B_2(n)$	0	1	12	143	1704	20305	241956
$B_2(n)$	0	1	12	143	1704	20305	241956
$B_3(n)$	0	1	18	323	5796	104005	1866294
$B_3(n)$	0	1	18	323	5796	104005	1866294
$B_4(n)$	0	1	24	575	13776	3300489	7907400
$B_4(n)$	0	1	24	575	13776	3300489	7907400

2.1. Generating function for balancing λ-numbers. In this section, we will first find the generating function for the balancing λ-numbers and then using this function, we will establish Binet's formulas for these numbers. Let

$$G(x)=\sum_{n=0}^{\infty} B_\lambda(n)x^n = B_\lambda(0)+B_\lambda(1)x+B_\lambda(2)x^2+\ldots+B_\lambda(n)x^n+\ldots$$

be the generating function for balancing λ-numbers. By simple algebraic manipulation, we obtain $G(x)-6\lambda xG(x)+x^2G(x)=x$, which implies that $G(x)=\frac{x}{1-6\lambda x+x^2}$.

Using the generating function for balancing λ-numbers, we can find Binet's formula as follows: rewrite the factor $1-6\lambda x+x^2$ as $1-6\lambda x+x^2=(1-\alpha x)(1-\beta x)$, where $\alpha = 3\lambda+\sqrt{9\lambda^2-1}$, $\beta=3\lambda-\sqrt{9\lambda^2-1}$. Therefore, $G(x)$ can be written as

$$G(x)=\frac{x}{1-6\lambda x+x^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x},$$

which implies that $x=A(1-\beta x)+B(1-\alpha x)=(A+B)-x(A\beta+B\alpha)$. Therefore, $A+B=0$ and $A\beta+B\alpha=-1$.

Solving these two equations, we get $A=\frac{1}{2\sqrt{9\lambda^2-1}}$, $B=\frac{-1}{2\sqrt{9\lambda^2-1}}$. Thus, we have

$$G(x)=\frac{x}{1-6\lambda x+x^2}=\frac{1}{\alpha-\beta}\left[\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right]=\frac{1}{\alpha-\beta}\sum(\alpha^n-\beta^n)x^n,$$

hence $B_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, which is Binet's formula for balancing λ-numbers. It can be observed that $\alpha+\beta=6\lambda, \alpha-\beta=2\sqrt{9\lambda^2-1}$ and $\alpha\beta=1$.

2.2. The Cassini formula for the balancing λ-numbers. The Cassini formula can be generalized for the case of the balancing λ-numbers.

Theorem 1. *If $B_\lambda(n)$ is the n^{th} balancing λ-numbers then the generalized Cassinni formula is*

$$B_\lambda^2(n)-B_\lambda(n+1)B_\lambda(n-1)=1. \quad (3)$$

Proof. Identity (3) can be proved by mathematical induction. Recurrence relation (1) takes the following values $B_\lambda(0)=0, B_\lambda(1)=1, B_\lambda(0)=6\lambda$, which implies that $B_\lambda^2(1)-B_\lambda(2)\times$ $\times B_\lambda(0)=1$, and therefore the base of the induction is proved. We assume that identity (3) is valid for any positive integer n and prove the validity for the case $(n+1)$; i.e. the identity

$$B_\lambda^2(n+1)-B_\lambda(n+2)B_\lambda(n)=1, \quad (4)$$

is valid too. In order to prove identity (4), we represent the left hand part of (4) as follows

$$B_\lambda^2(n+1)-B_\lambda(n+2)B_\lambda(n)=B_\lambda^2(n+1)-B_\lambda(n)(6B_\lambda(n+1)-B_\lambda(n))=$$
$$=B_\lambda(n+1)-(B_\lambda(n+1)-6B_\lambda(n))+B_\lambda^2(n)=B_\lambda^2(n)-B_\lambda(n+1)B_\lambda(n-1)=1,$$

which completes the proof. □

Example 1. Consider the examples of the validity of identity (3) for various sequences shown in Table 1. Consider the balancing 2-number $B_2(n)$ for the case of $n=4$ to get $B_2(4)=1704, B_2(5)=20305, B_2(6)=241956$.

By performing calculations over them according to (3), we obtain the following result $B_2^2(5)-B_2(4)B_2(6)=20305^2-(1704\times 241956)=1$. Further, consider the $B_3(n)$ sequence from Table 1 for the case $n = 3$. For this case, we should choose the following balancing 3-numbers $B_3(n)$, $B_3(3) = 323, B_3(4) = 5796, B_3(5) = 104005$. By performing calculations over them according to (3), we obtain the following result: $B_3^2(4) - B_3(3)B_3(5) = 1$. Finally, consider the $B_4(-n)$ sequence from Table 1 for the case $n = 4$: for this case, we should choose the following balancing 4-numbers $B_4(-n)$. $B_4(-4) = -13776, B_4(-3) = -575, B_4(-2) = -24$. By performing calculations over them according to equation (3), we obtain the following result $B_4^2(-3) - B_4(-2)_4(-4) = -575^2 - (-24 \times -13776) = 1$.

3. Lucas-balancing-λ numbers. For any real number $\lambda > 0$; consider the recurrence relation

$$C_\lambda(n+2) = 6\lambda C_\lambda(n+1) - C_\lambda(n),\ C_\lambda(0) = 1,\ C_\lambda(1) = 3. \tag{5}$$

The recurrence relation (5) generates an infinite number of new numerical sequences for any real number λ. For $\lambda = 1$, (5) reduces the following relation $C_1(n+2) = 6C_1(n+1)C_1(n)$, $C_1(0) = 1$, $C_1(1) = 3$, which generates the Lucas balancing numbers. For $\lambda = 2, \lambda = 3$, and $\lambda = 4$, we have the following sequences:

$$1, 3, 35, 417, 4969, 59211, \ldots,$$
$$1, 3, 53, 951, 17065, \ldots,$$
$$1, 3, 71, 1701, 40753, \ldots.$$

Table 2 shows four expanded sequences of the Lucas-balancing λ-numbers, corresponding to the values $\lambda \in \{1, 2, 3, 4\}$.

n	0	1	2	3	4	5	6
$C_1(n)$	1	3	17	99	577	3363	19601
$C_1(n)$	1	3	17	99	577	3363	19601
$C_2(n)$	1	3	35	417	4969	59211	705563
$C_2(n)$	1	3	35	417	4969	59211	705563
$C_3(n)$	1	3	53	951	17065	306219	5494877
$C_3(n)$	1	3	53	951	17065	306219	5494877
$C_4(n)$	1	3	71	1701	40753	976371	23392151
$C_4(n)$	1	3	71	1701	40753	976371	23392151

3.1. Generating function for Lucas-balancing λ-numbers. In this section, we first find the generating function for Lucas-balancing numbers and then with the help of this function, we find Binet's formula for these numbers. Let

$$g(x) = \sum_{n=0}^{\infty} C_\lambda(n) x^n$$

be the generating function for Lucas-balancing λ-numbers. By simple algebraic manipulation, we get $g(x) - 6\lambda x g(x) + x^2 g(x) = 1 + x(3 - 6\lambda)$, which implies that $g(x) = \frac{1+x(3-6\lambda)}{1-6\lambda x+x^2}$. Using this generating function and the previous described method, we can find Binet's formula for Lucas-balancing λ-numbers as $C_\lambda^n = \frac{(3-\beta)\alpha^n-(3-\alpha)\beta^n}{\alpha-\beta}$, where $\alpha = 3\lambda + \sqrt{9\lambda^2 - 1}$ and $\beta = 3\lambda - \sqrt{9\lambda^2 - 1}$.

We notice that $\alpha + \beta = 6\lambda$, $\alpha - \beta = 2\sqrt{9\lambda^2 - 1}$ and $\alpha\beta = 1$.

3.2. Cassini formula for Lucas-balancing λ-numbers. The Cassini formula can also be generalized for the case of the Lucas-balancing λ-numbers.

Theorem 2. *If $C_\lambda(n)$ is the n^{th} Lucas-balancing λ-numbers then the generalized Cassinni formula for Lucas-balancing λ-numbers is*

$$C_\lambda(n+1)C_\lambda(n-1) - C_\lambda^2(n) = 18\lambda - 10. \tag{6}$$

Proof. Once again induction comes into picture. According to recurrence relation (5) we have $C_\lambda(0) = 1, C_\lambda(1) = 3, C_\lambda(2) = 18\lambda - 1$, follows that identity (6) for the case $n = 1$ is equal to

$$C_\lambda(2)C_\lambda(0) - C_\lambda^2(1) = (18\lambda - 1) - 9 = 18\lambda - 10.$$

Thus, the result is true for $n = 1$. For the inductive assumption, suppose that identity (6) is valid for any given positive integer n. We notice that

$$C_\lambda(n)C_\lambda(n+2) - C_\lambda^2(n+1) = C_\lambda(n)(6C_\lambda(n+1) - C_\lambda(n)) - C_\lambda^2(n+1) =$$
$$= C_\lambda(n+1)(6C_\lambda(n) - C_\lambda(n+1)) - C_\lambda^2(n) = C_\lambda(n+1)C_\lambda(n-1) - C_\lambda^2(n) = 18\lambda - 10,$$

from which the proof follows. □

Example 2. Consider the examples of the validity of identity (6) for various sequences shown in Table 1. Let us consider the $C_2(n)$ sequence for the case $n = 3$. For this case, we should choose the following Lucas-balancing 2-numbers $C_2(n)$ as follows $C_2(4) = 4969$, $C_2(5) = 59211$, $C_2(3) = 417$.

By performing calculations over them according to (6), we obtain the following result $C_2(5)C_2(3) - C_2^2(4) = (59211 \times 417) - 4969^2 = 26$. And the right hand side is $18\lambda - 10 = 18(2) - 10 = 26$.

Consider $C_3(n)$ sequence for the case $n = 4$. For this case, we should choose the following Lucas-balancing 3-numbers $C_3(n)$, $C_3(6) = 5494877, C_3(4) = 17065, C_3(5) = 306219$. In a similar way, we obtain the following result $C_3(6)C_3(4) - C_3^2(5) = (306219 \times 17065) - 5494877^2 = 44$, and the right hand side of it is given by $18\lambda - 10 = 18(3) - 10 = 44$.

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