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QUESTIONS RELATED TO THE K-THEORETICAL ASPECT OF BEZOUT RINGS WITH VARIOUS STABLE RANGE CONDITIONS

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We provide a list of open problems that are connected to the commutative and noncommutative ring theory, K-theory and homological algebra. Some problems are solved now completely, some partially, and most of them, remaining still open, are supplemented with ideas and hints.

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В статье сформулированы ряд задач, связанных с теорией коммутативных и некоммутативных колец, K-теорией и гомологической алгеброй. Некоторые из задач решены частично, некоторые — полностью, но большинство сформулированных проблем остаются открытыми.

Introduction. The present paper contains a list of questions that were posed by the author to the participants of the scientific seminar "The Problems of Elementary Divisor Rings" over the years. The formulated questions vary from the simple checkup questions to the shorten versions of the further researches. Some questions are just exercises, and some other are deep problems such that their solutions will be a nontrivial contribution to the general theory.

Eventually, the author would like to receive from the reader any information about the possible errors and inaccuracies in these questions. Moreover, it will be much better to receive any ideas or even complete solutions to the posed questions, but it is useful to remark that an answer to a question marked as "Open Problem" sometimes cannot be a sufficient reason of its publication.

One of the main sources of almost all questions in the present paper is a rather old problem of a full description of the elementary divisor rings. The notion of an elementary divisor ring was introduced by Kaplansky in [15]. There are a lot of researches that deal with the matrix diagonalization in different cases (the most comprehensive history of these researches can be found in [32]. There are still a lot of publications concerning the topic. On the other hand, most of them are outdate in the ideological aspect. This is connected with the appearance of a K-theoretical invariant such as the stable range that was established in 1960 by Bass. One of the most fruitful aspects of Bass' studies was the following fact: a lot of answers to the problems of the linear algebra over the rings becomes simpler if we increase the dimension of the considered object (the rank of the projective module, the size

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of the matrix etc.), and, furthermore, the answer is independent on the choice of the base ring for rather big dimensions' values, as well as is independent on the dimension of the current object — it functions only in terms of the geometry of a given module.

Moreover, it have been discovered that in the commutative rings' case there exist structural theorems on these objects starting from some small values of a stable range (for example, 1 or 2) which depends only on the considered problem, but does not depend on the dimension or structure of the base ring.

Thus, for example, in the case of commutative Bezout domains the elementary divisor rings are precisely the rings of neat range 1 (see [34]).

1. Preliminaries. All the rings considered in the article are supposed to be associative with nonzero identity element. Let $U(R)$ be the set of all invertible elements of a ring R. By the *Jacobson radical* $J(R)$ of a ring R we mean the set $J(R) = \{x \in R | \forall a, b \in R: \ 1 - axb \in U(R)\}$, and the *nilradical* rad(R) is defined to be the set of all nilpotent elements of a ring R. A ring R is called a *reduced* ring if rad(R) $= \{0\}$. Also by spec(R) (resp. mspec(R), mspec(a)) we denote the space of all prime ideals (resp. maximal ideals, maximal ideals that contain an element a) in the case of a commutative ring.

Suppose that A is a subset of a ring R. The sets $r(A) = \{x | Ax = 0\}$ and $l(A) = \{x | xA = 0\}$ are called the *right and left annihilators* of the set A, and they are a right and a left ideal respectively. We say that an element $a \in R$ is a *regular element*, if it is not a left and a right zero divisor, that is $r(a) = l(a) = 0$.

We start with recalling some definitions and facts that we will need below in our proofs.

Definition 1. 1) We say that a ring R is a *right (left) quasi-duo* ring if every its maximal right (left) ideal is 2-sided. If all right (left) ideals in R are 2-sided then we call it a *right (left) duo-ring*. If a ring R is both a left and a right (quasi-)duo-ring we just say that it is a *(quasi-)duo-ring* ([17]).
2) A nonzero element $a \in R$ is called a *right (left) duo-element*([4]), if $Ra \subseteq aR$ ($aR \subseteq Ra$) and it is a *duo-element*, if $Ra = aR$.
3) A ring R is said to be *right (left) distributive* if the lattice of its right (left) ideals is distributive. If a ring R has both left and right distributivity then we say that it is *distributive* ([26, 27]).
4) If any right (left) finitely generated ideal of a ring R is principal then R is said to be a *right (left) Bezout ring.*
5) We say that a rectangular matrix A over a ring R *admits a canonical diagonal reduction* if there are two invertible matrices P, Q of appropriate sizes such that the matrix $PAQ = D = (d_i)$ is a diagonal matrix with the additional condition: for all indices we have the inclusion $Rd_i \cap d_iR \subseteq Rd_{i+1}R$. If every row matrix (a, b) (column matrix $(a, b)^T$) admits a canonical diagonal reduction then we say that R is a *right (left) Hermite ring.* If every matrix over a ring R admits a canonical diagonal reduction then R is said to be an *elementary divisor ring.*

Definition 2. 1) We say that an element $a \neq 0$ of a commutative Bezout ring R is *adequate to an element* $b \in R \backslash \{0\}$ if there exist some elements $r, s \in R$ such that $a = rs$, $rR + bR = R$, $s'R + bR \neq R$, for any $sR \subseteq s'R \neq R$. The latter fact will be denoted by ${}_aA_b$.
2) If for any $b \in R \setminus \{0\}$ an element $a \neq 0$ of a commutative Bezout ring R is adequate to b, then a is called an *adequate element.*
3) A commutative Bezout ring R is called an *adequate ring* if all its nonzero elements are adequate. If the element 0 is adequate in the ring R then R is called a *zero-adequate* ring. In

the case where all the elements (including zero) are adequate a ring R is called *everywhere adequate*.

Definition 3. We say that a ring R has *the stable range* n ($st.r.(R) = n$ in notations) if for any elements $a_1, \dots, a_{n+1} \in R$ the equality $a_1R + \dots + a_{n+1}R = R$ implies that there are some elements $b_1, \dots, b_n \in R$ such that $(a_1 + a_{n+1}b_1)R + \dots + (a_n + a_{n+1}b_n)R = R$ ([2]).

Remark that the latter definition is left-right symmetric due to [30]. Below we will use the stable range condition for specific values of n, in fact for $n = 1$ and $n = 2$.

We say that a ring R has *stable range 1* if for any elements $a, b \in R$ the equality $aR+bR = R$ implies that there is some $x \in R$ such that $(a+bx)R = R$. If such an element $x \in R$ always can be taken to be an idempotent (resp. a unit) then it is said that R has *the idempotent* (resp. *unit*) *stable range 1*.

If for any elements a, b, c of a ring R the equality $aR + bR + cR = R$ implies that there are some elements $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$ then we say that *the stable range* of R is equal to *2*.

Definition 4. An element $a \neq 0$ of a ring R is called an *almost stable range 1 element* if the stable range of R/aR is equal to 1. If all nonzero elements of a ring R are elements of almost stable range 1 then we say that R is an *almost stable range 1 ring*.

Definition 5. 1) A ring R is said to be a *right (left) almost Baer ring* if the right (left) annihilator of each element of R is a principal ideal.
2) A ring R is said to be a *right (left) P-injective ring* if any homomorphism from any principal ideal aR (or Ra in left case) into R can be extended to an endomorphism of the ring R, namely, using multiplication by some fixed element of the ring R (see [22]).
3) A ring R is called a *left (right) morphic ring* if for any $a \in R$ there is an isomorphism $R/Ra \cong l(a)$ (or $R/aR \cong r(a)$ respectively) of left (right) modules, where $l(a)$ and $r(a)$ are the left and right annihilators of the element a respectively ([21]).
4) A ring R is called a *right (left) coherent* if the right (left) annihilator of any element of R is a finitely generated right (left) ideal, and the intersection of any two finitely generated right (left) ideals is again a finitely generated right (left) ideal. In the case of a right (left) Bezout ring it is equivalent to being a right (left) almost Baer ring and the intersection of two principal right (left) ideals is again a right (left) principal one.
5) A ring R is a *right (left) IF-ring* if every injective right (left) module over R is flat ([3]).
6) A ring R is called a right (left) Kasch ring if all maximal right (left) ideals of R are right annihilators of ring elements, that is, each one is of the form $r(x)$ (respectively $l(x)$), where $x \in R$.

Below we gather some results concerning our topic.

Theorem 1. *1) Over a right Bezout ring R of stable range n any row $a_1, \dots, a_m \in R$ such that $a_1R + \dots + a_mR = R$, where $m \geq n + 1$, can be completed to an invertible matrix generated by some elementary matrices ([32]).*
2) A right Bezout ring of stable range 1 is a right Hermite ring ([32]).
3) For any elements a, b in a right Bezout ring R of stable range 1 there are $x, d \in R$ such that $a + bx = d$ and $aR + bR = dR$ (see [32]).
4) Every matrix A over a right Hermite ring R can be reduced to the lower triangular matrix AU via the right multiplication by some invertible matrix U (see [15]).
5) If all 2×2, 2×1 and 1×2 matrices over a commutative ring R admit a canonical diagonal reduction then R is an elementary divisor ring ([15]).

Here are some Nicholson's criteria for the right P-injective and the left morphic rings.

Theorem 2. *1) The following statements are equivalent for a ring R:*

a) R is a right P-injective ring;

b) for any $a \in R$: $lr(a) = Ra$;

c) if some elements $a, b \in R$ are such that $r(a) \subseteq r(b)$ then $Rb \subseteq Ra$;

d) for any $a, b \in R$ we have $l(bR \cap r(a)) = l(b) + Ra$ (see [22]).

2) The following statements are equivalent for a ring R:

a) R is a left morphic ring;

b) for any $a \in R$ there is $b \in R$ such that $l(a) = Rb$, $l(b) = Ra$;

c) for any $a \in R$ there is $b \in R$ such that $l(a) = Rb$, $l(b) \cong Ra$ (see [21]).

Also in [21] it is proved that the left morphic rings are the right P-injective ones.

Definition 6. An element a of a ring R is said to be a *von Neumann regular element* if there is some $b \in R$ such that $aba = a$. If this element b commutes with a, then a is called a *strongly regular element*. An element a of a ring R is said to be a *unit-regular element* if the mentioned above element b is invertible. If all elements of a ring R are von Neumann regular (resp. unit-regular, strongly regular) then R is called a *von Neumann regular* (resp. *unit-regular, strongly regular*) *ring*.

Note that all strongly regular rings are unit-regular, and the unit regular rings are von Neumann regular.

Moreover, we will need some other classes of rings.

Definition 7. An element $a \in R \setminus J(R)$ is called a *semipotent element* if there is some element $b \in R$ such that $bab = b$. If all elements of a ring R that are not in $J(R)$ are semipotent then R is said to be a *semipotent ring*. If additionally idempotents can be lifted modulo $J(R)$ then R is called a *potent ring* ([19]).

Definition 8. A commutative ring R is called *semilocal* if it has finite number of maximal ideals. A ring R is said to be *local* if it has exactly one maximal right/left ideal. A ring R is called a *semiregular ring* if $R/J(R)$ is a von Neumann regular ring and idempotents can be lifted modulo $J(R)$.

2. Rings ruled by annihilators and idempotents. The main attention here will be paid to the exchange, clean, regular, semiregular, unit-regular, coherent, P-injective, morphic, almost Baer rings. Although there are different ways to define these classes of ring, but they all have one common property: there are zero divisors inside them, and these divisors determine their structure. Each of the mentioned classes can be described using a condition on annihilators or/and idempotent. The latter features determine the title of the current section.

The adequate rings can be precisely described as rings whose finite homomorphic images are the semiregular rings ([34]). Also it is known that finite homomorphic images of commutative Bezout domain R are Kasch rings if and only if all maximal ideals of R are principal ones ([34]). Thus all finite homomorphic images of the adequate Bezout domains whose maximal ideals are principal are semiregular Kasch rings. Any commutative principal ideal domain can serve as an evident example.

Question 1. How to describe the rings whose finite homomorphic images are semiregular Kasch rings, but they are not commutative principal ideal domains? Can anyone find an example of such a ring?

Question 2. Describe necessary and sufficient conditions for a P-injective ring to be a semi-potent one?

As the NJ-rings are a slight generalization of the von Neumann regular rings, the following group of questions is posted below.

Definition 9. A ring R is called an *NJ-ring* if every element $a \in R \setminus J(R)$ is a von Neumann regular element ([20]).

Question 3. Suppose that R is an NJ-ring, not necessarily commutative. If R is additionally a right Bezout ring of stable range 2 then is it necessarily a right Hermite one?

Question 4. If a NJ-ring R is a Hermite ring then is it true that R is an elementary divisor ring?

Definition 10. We say that R is a *separative ring* if the following condition holds for all finitely generated projective R-modules A and B

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B.$$

Question 5. Is it true that the stable range of any separative NJ-ring equals to $1, 2$ or infinity?

Question 5 is inspired by the result of K. R. Goodlearl and P. Ara ([1]), who proved that stable range of a separative von Neumann regular ring equals to $1, 2$ or ∞. We just want to know how the regularity condition can be weakened.

Definition 11. An element a of a ring R is said to be an *exchange element* if there is some idempotent $e = e^2 \in R$ such that $eR \subseteq aR,\ (1-e)R \subseteq (1-a)R$. A ring R is called an exchange ring if all its elements are exchange elements ([19]).

The next several problems are related to the complete description of the Jacobson radical problem for the exchange ring. All these problems can be formulated for both commutative and noncommutative cases, but even the commutative one is not so easy.

Question 6. Suppose that R is a Bezout exchange ring. Is it a semiregular ring? Under what conditions it becomes a zero-adequate ring? If R is additionally a coherent / P-injective / morphic / IF ring, is it true that it is a semiregular one? Are the inclusions

$$\begin{gathered}(\text{Bezout morphic exchange}) \subset (\text{Bezout P}-\text{injective exchange}) \subset \\ \subset (\text{Bezout coherent exchange}) \subset (\text{Bezout exchange})\end{gathered}$$

proper?

We finish this section with few problems concerning morphic rings.

Definition 12. A right (left) ideal I is called *right (left) pure* if the ring modulo this ideal is a flat left module.

Question 7. If R is a commutative morphic ring, then is it true that any finitely generated pure ideal is projective?

Question 8. What can one say about a noncommutative morphic unit stable range 1 ring? Under what conditions a coherent / P-injective / morphic noncommutative ring R becomes unit stable range 1 ring?

The latter question is rather important, as any element of unit stable range 1 ring can be represented as a sum of two units (these rings are called 2-*good* due to Vamos [28]). If we additionally suppose that any unit stable range 1 ring is a unit-central ring (a ring R is said to be *unit-central* if its units are central elements) then it becomes commutative.

3. Commutative Bezout Rings. The commutative Bezout rings are a natural generalization of the principal ideal rings and they are not so "finite" as noetherian rings. This class of rings has not been so effectively studied as the principal ideal rings, noetherian, local and semilocal rings, but it has close relations with these classes as well as Bezout rings interpret GCD notion from integers to the abstract case.

It is known that all elementary divisor rings are Hermite ones, and any Hermite ring is a Bezout ring. Thus it is in particular interesting to obtain these inclusions, and examine the internal structure of a Bezout ring. Thus some related open problems are presented below.

Definition 13. A commutative ring R is called a PM-*ring* if any its prime ideal is contained in a unique maximal one ([5]). If the latter condition holds only for nonzero prime ideals then R is called a PM*-ring ([13]).

Question 9. Suppose that R is a commutative Bezout PM-ring. Is it true that R is an Hermite ring if and only if R is an elementary divisor ring?

Since any Bezout domain is an Hermite domain, then we have some modification of the previous problem.

Question 10. Is any commutative Bezout PM*-domain an elementary divisor ring?

Classical definitions of an Hermite and elementary divisor rings are given using matrix language. Thus we cannot omit matrix problems connected with Bezout rings.

Question 11. Under what assumptions the matrix ring $M_2(R)$ over a commutative Bezout domain R has no infinite set of orthogonal idempotents?

Question 12. Suppose that R is an integral domain, and I is its ideal. Let us consider some ring-theoretical property S. Can we justify that the matrix ring $M_n(R)$ satisfies the property S whether for any ideal I of R the matrix ring $M_n(R/I)$ satisfies the property S? What if I is a principal ideal?

For commutative Bezout domains we obtain the following questions.

Question 13. Suppose that R is a commutative Bezout domain, and for any $a \in R \setminus \{0\}$ the matrix ring $M_n(R/aR)$ is semipotent. Is $M_n(R)$ necessarily a semipotent ring?

Question 14. Suppose that we have a commutative Bezout domain R and we consider the set S of all elements a such that any matrix $A \in M_2(R)$ with a at some position (i, j) admits a canonical diagonal reduction. Is S a multiplicatively closed set? Under what assumptions the set S becomes a saturated one? Is the localization $S^{-1}R$ a field?

Question 15. Suppose that R is a commutative Bezout PM*-domain, with $J(R) = 0$, and the set mspec(R) is at most countable. Is it true that R satisfies the Henriksen condition, i.e. for any elements $a, b \in R$ there is some element $r \in R$ such that

$$\text{mspec(r)} = \text{mspec(a)} \setminus \text{mspec(b)}?$$

Question 16. Describe necessary and sufficient conditions for an element a of a commutative Bezout domain R such that $J(aR) = J(R/aR)$ is a noetherian module.

Question 17. It is known ([12]) that if any maximal ideal M of a ring R is pure then R is a von Neumann regular ring. Therefore, what property of a maximal ideal of a commutative ring R is sufficient to become an exchange ring?

4. Nontrivial finite homomorphic images of commutative Bezout rings. There is the well know theorem of T. Shores ([24]) which asserts that a commutative Bezout domain R is an elementary divisor ring if and only if so is any R/aR, where a is a nonzero element of R. Thus, a lot of properties of commutative Bezout rings will be studied using their finite homomorphic images R/aR.

Since, for a commutative Bezout domain R and any $a \in R \setminus \{0\}$ it is know that $Q_{cl}(R/aR) = R/aR$, studying an internal structure of R/aR is itself an interesting question.

Question 18. Suppose that R is a commutative Bezout domain, and $a \in R \setminus \{0\}$ is its fixed element.

a) How can one describe all $\bar{b} \in \overline{R} = R/aR$ such that if $\bar{b} \notin J(\overline{R})$ then $\overline{bR}$ contains a nonzero idempotent? (i.e. under what assumptions $\overline{R}$ is a semipotent ring?)
b) What are the elements $b \in R$ such that their images $\bar{b} \in \overline{R}$ are idempotents? a von Neumann regular elements?
c) What can be said about principal ideals $\overline{bR}$ of R/aR if all maximal ideals of R are principal?
d) If the maximal ideal $\overline{M} = \overline{M}^2$ of the ring $\overline{R}$ is an idempotent ideal, then how can one describe its preimage?

Definition 14. An element c of a commutative Bezout ring R is called an *avoidable element* if for any elements $a, b \in R$ such that $aR + bR + cR = R$ there are some elements $r, s \in R$ such that $c = rs$ and $aR + rR = bR + sR = rR + sR = R$.

Question 19. If R/aR is a PM-ring, then is it true the same about R? Is it true that R/aR is an exchange ring if and only if a is an avoidable element of R?

Question 20. Let R be a commutative Bezout domain, and $a \in R \setminus \{0\}$. Is it true that R/aR is semipotent if and only if R is an almost stable range 1 ring?

Question 21. Suppose that R is a commutative Bezout ring, and $a \in R \setminus \{0\}$. Is it true that R/aR is a morphic ring in the case where R is a stable range 2 (or even almost stable range 1) ring?

Question 22. Suppose that R is a commutative Bezout domain, and $a = bc \in R \setminus \{0\}$, where $bR + cR = R,\ b \notin U(R),\ c \notin U(R)$. Is R/aR an elementary divisor ring?

Definition 15. A commutative ring R is said to be an *(f)-ring* if any its pure ideal is generated by idempotents ([14]).

As proved in [18] for a commutative ring R the following conditions are equivalent:

— R is an exchange ring;

— R is a clean ring;

— R is an idempotent stable range 1 ring;

— R is an (f)-ring and PM-ring.

Thus in the following group of problems these conditions are equivalent.

Question 23. Let R be a commutative Bezout domain and $a \in R \setminus \{0\}$. Under what conditions on the element a the ring R/aR is an (f)-ring?

Question 24. Suppose that any finite homomorphism image R/aR of a commutative Bezout domain R is an exchange ring. Is it true that R/aR is a semiregular ring?

The latter problem has a tight connection with adequate domains that will be stated in the next section.

Question 25. Considering the set S of all elements a of a commutative ring R such that R/aR is a PM-ring it is interesting to know: is S a multiplicatively closed set? saturated set? What useful properties of S can be found?

Question 26. It is proved ([11]) that if the Jacobson radical $J(R)$ of a commutative coherent ring is finitely generated or flat then $R/J(R)$ is again a coherent ring. Using this fact for a commutative Bezout PM*-domain R and its element $a \in R \setminus \{0\}$ we need to know: if rad(R/aR) or $J(R/aR)$ is principal, then is a necessarily an adequate element?

5. Adequate rings and their generalizations. We recall that a commutative Bezout ring R is adequate, if any $a \in R \setminus \{0\}$ is an adequate element. If all elements including 0 are adequate, we say that R is everywhere adequate. If we know only about 0 that it is an adequate element then we say that R is zero adequate. If for any pair of nonzero elements $a, b \in R \setminus \{0\}$ one of them is adequate to another one, then R is called a *generalized adequate ring*. Also other modifications of the notions of adequacy will be discussed below.

We start with open problems for the generalized adequate rings.

Question 27. Is an ultraproduct of principal ideal rings a generalized adequate ring?

Question 28. If R is a commutative Bezout ring that is an (f)-ring then is R necessarily a generalized adequate ring?

Question 29. Is any commutative Bezout exchange ring necessarily a generalized adequate ring?

Question 30. Is it true that a generalized adequate PM*-domain is necessarily an adequate ring?

Here are a few open problems concerning the adequate rings.

Question 31. Is it true that a subdirect product of adequate rings is adequate as well?

Question 32. Is a homomorphic image of an adequate element again adequate?

While all the previous problems were about "globally" adequate rings, next we will consider some "locally" adequate ones.

Question 33. Suppose that for every nonzero element a of a commutative Bezout domain R and for every its nonzero ideal I there is some element $b \in I$ such that ${}_aA_b$. Is R an elementary divisor ring?

Question 34. Suppose that R is a commutative Bezout domain and $a \in R \setminus \{0\}$. Is it true that ${}_aA_b$ if and only if $\bar{b}$ is an exchange element of the quotient ring R/aR?

Question 35. Let R be a commutative Bezout domain. Let us consider several axioms defined for elements of R.

a. For any triple of coprime elements $aR + bR + cR = R$ there is some $x \in R$ such that ${}_aA_{bx}$ or ${}_aA_{cx}$.
b. For any triple of coprime elements $aR + bR + cR = R$ we have that ${}_aA_b$ or ${}_aA_c$.
c. For any triple of coprime elements $aR + bR + cR = R$ there are some $x, y \in R$ such that ${}_{(a+cx)}A_{(b+cy)}$.
d. For any triple of coprime elements $aR + bR + cR = R$ there is some $p \in R$ such that $pR + bR + cR = R$ and ${}_cA_{ap}$.
e. For any triple of coprime elements $aR+bR+cR = R$ there are some $p, q \in R$ such that $pR + bR + cR = R$, $qR + aR + cR = R$, ${}_cA_{ap}$, ${}_cA_{bq}$.
f. Various modifications by increasing the number of coprime elements or adding x, y, p, q as factors in different combinations.

Therefore, the question is: is it true that any commutative Bezout domain admits some type of the adequacy (like some of axioms from a to f)? Are there analogues with the stable range theory? Which of the axioms lead to the well-known types of adequate rings? What are the examples of rings such that using them we can differ one of the written above axioms from the other ones?

The following question ties the previous one with the topology.

Question 36. As it is well known, a lot of topological results depend on the type of points separation in the given topological space. The separation type is defined by the separation axioms $T_i,\ i \in \{0, 1, 2, 3, 3\frac{1}{2}, 4, 5, 6\}$, which are called T-axioms. They define how closely are situated the points of the topological space, and can we separate them by some of their neighborhoods. The condition of adequateness of an element a to an element b similarly defines how can the element a be represented as a product of 2 cofactors such that one of them is coprime (separated) with b, and the other one is never coprime (arbitrary close) to b. Therefore, the question is: are there any connections of the adequacy axioms with the famous T-axioms of separation for some topological spaces?

As it was proved by S. I. Bilyavska and B. V. Zabavsky ([35]) every zero-adequate ring is an exchange one. Thus if R is a commutative Bezout domain and $a \in R \setminus \{0\}$ is an adequate element then R/aR is a ring in which 0 is an adequate element and therefore is an exchange one.

We are trying to modify this result by the following question.

Definition 16. A ring R is called a *von Neumann local (VNL) ring* if either a either $1-a$ is a von Neumann regular element, for any $a \in R$ (see [6]).

Question 37. Let R be a commutative Bezout domain and $a, b, c \in R \setminus \{0\}$ are such that $aR + bR + cR = R$. Is it true that R/cR is a VNL-ring iff ${}_cA_a$ or ${}_cA_b$?

Question 38. Suppose that for every pair of coprime elements $a, b \in R \setminus \{0\}$ of a commutative Bezout domain R one of them is adequate. Is R/aR a VNL-ring for any $a \in R \setminus \{0\}$? Does the converse inclusion hold?

Question 39. Let us consider the following condition for a commutative Bezout ring R: for any element $c \in R \setminus \{0\}$ any pair of elements a, b in R such that $aR + bR + cR = R$ there are some elements $r, s \in R$ such that $c = rs$, and $rR + aR = sR + bR = rR + sR = R$. Such rings are PM-rings and any adequate ring satisfies the mentioned property. We ask the converse: are such rings adequate?

Definition 17. An element a of a ring R is called a *clean element* if it can be represented as a sum of an invertible element and an idempotent. If all elements of a ring R are clean then R is called a *clean ring* ([19, 27]).

Definition 18. We say that an element a of a commutative ring R is a *neat element* if the quotient ring R/aR is a clean ring. We say that R is a *neat ring* if all its elements are neat.

Due to the remark at the beginning of the article we say that a commutative ring R is a neat ring if for any $c \in R \setminus \{0\}$ the quotient ring R/cR is clean. Thus, we ask the following question.

Question 40. Suppose that R is a commutative Bezout domain. Is it true that R is a neat ring if and only if R is an adequate one?

By the above mentioned result of S. I. Bilyavska and B. V. Zabavsky the converse inclusion becomes obvious. Thus we only need to know: if R/cR is an exchange ring then is it zero-adequate? It is the same as asking about semiregularity of the morphic exchange ring.

6. Local properties and ranges of commutative rings. As it was mentioned, a ring R is called a VNL-ring if for any $a \in R$ either a or $1-a$ is a von Neumann regular element. Here the von Neumann regularity holds only "locally" and from this fact their name becomes clear. But what will happen if we replace the von Neumann regularity with the exchange, clean, semipotent, avoidable, adequate, stable range 1 or neat element property? What classes of rings would we obtain? Do some of these notions coincide or even are redundant?

Definition 19. We say that a ring R is a *locally P ring* if for any element $a \in R$ either a or $1-a$ satisfies the property P, where P is some ring-theoretical property that can be defined for a single element of a ring.

Definition 20. An element a of a ring R is called a *2-simple element* if there are some elements $u_1, u_2, v_1, v_2 \in R$ such that $u_1 a v_1 + u_2 a v_2 = 1$.

Definition 21. A ring R is called a *locally adequate (unit-regular,2-simple) ring* if for any element $a \in R$ either a or $1-a$ is an adequate (unit-regular, 2-simple) element.

This is the first point that we will discuss. The second one is the range property of a ring. The famous Dirichlet theorem tells us that if a and b are coprime integers then in the arithmetical progression $\{a+bt\}$ there are infinitely many primes. We modify and extend this property in the following way.

Definition 22. A commutative ring R is of P *range* 1 if for any pair of coprime elements $a, b \in R$ there is some $t \in R$ such that element $a+bt$ satisfies the property P, where P is some ring-theoretical property that can be defined for a single element of a ring.

It is well-known that if the total ring of quotients $Q(R)$ of a commutative ring R is semilocal, then R has regular range 1, that is if $a, b \in R$ are regular elements (are not zero divisors) then there is some $t \in R$ such that $a+bt$ is again a regular element.

Definition 23. A commutative ring R is of *neat (clean, irreducible, adequate) range 1* if for any pair of coprime elements $a, b \in R$ there is some $t \in R$ such that the element $a+bt$ is a neat (clean, irreducible, adequate) element ([34]).

Thus we obtain several open problems.

Question 41. By [5] a commutative ring R is a PM-ring if and only if for any elements $a, b \in R$ such that $a+b=1$ there are some elements $x, y \in R$ such that $(1-ax)(1-by)=0$. Are there elements $x, y \in R$ such that $(1-ax)R+(1-by)R=R$? As a consequence we ask: is it true that R is a ring of neat range 1 if and only if for any element $c \in R$ and any coprime elements $a, b \in R$ there are some elements $x, y \in R$ such that $c=xy$ and $(1-ax)R+(1-by)R=R$?

Definition 24. An element a of a commutative ring R is called a *PM-element* if R/aR is a PM-ring.

Question 42. Does the condition $aR+bR=R$ imply that both elements are PM-elements, whenever one of them is a PM-element? (A ring R is supposed to be commutative.)

Question 43. Does any locally adequate ring have an adequate range 1?

Question 44. Let R be a commutative Bezout domain and suppose that every maximal ideal M of R is principal. Under what conditions R is a ring of irreducible range 1?

Now we will formulate several open problems connected to the ones above but for the noncommutative case.

Question 45. Suppose that R is a noncommutative locally unit-regular ring. What is the stable range of R? If it is additionally an abelian ring then is R an exchange ring? A unit-regular ring?

Question 46. Is it true that any locally unit-regular Bezout ring is an elementary divisor ring if and only if it is an abelian ring (a ring R is called abelian if its idempotents are central)?

Question 47. Under what conditions any simple locally 2-simple Bezout ring is an elementary divisor ring?

7. Noncommutative Bezout rings. It is proved in [22, 29, 33, 34] that if R is a commutative Bezout domain and $a \in R \setminus \{0\}$ then the ring R/aR has several good properties: it is almost Baer, coherent, P-injective, morphic, IF-ring and $Q_{cl}(R/aR) = R/aR$. The following question naturally arises: what properties are preserved in the noncommutative case, especially if R is a duo-domain, or a is a duo-element of R?

Question 48. Suppose that R is a Bezout duo-domain and $a \in R \setminus \{0\}$. Has the ring R/aR one or two-sided properties from the next list: almost Baer property, IF-ring property, coherence, P-injectivity, morphic property? Does $Q_{cl}(R/aR) = R/aR$?

Definition 25. An element a of a ring R is called a *square-free element* if whenever $a = xy$ for some $x, y \in R$ then $xR + yR = Rx + Ry = R$.

Question 49. It is known that the ring $\mathbb{Z}/n\mathbb{Z}$ is a von Neumann regular ring if and only if n is a square-free integer. It is true the same for the case of commutative/duo/noncommutative Bezout domain R and its element $a \in R \setminus \{0\}$?

Question 50. Suppose that a is a duo-element of the noncommutative Bezout domain R. Under what conditions R/aR is a PM-ring? P-injective ting? Morphic ring? Clean ring? Exchange ring? What can be said about the ring R or R/aR if R/aR is additionally a right Kasch ring?

B. V. Zabavsky and M. Ya. Komarnytskii in [37] proved that the distributive Bezout domain is an elementary divisor ring if and only if it is a duo-ring. This result is quite good since being an elementary divisor ring requires some slight commutativity conditions in the case of distributive Bezout domain. On the other hand, the absence of zero divisors makes the theorem useful in a more narrow case than it is desired in general. Therefore some our hypothesis will be formulated in the following group of problems.

Question 51. Is it true that every quasi-duo stable range 1 Bezout ring is an elementary divisor ring if and only if it is a duo-ring?

Question 52. Let R be a quasi-duo Bezout ring such that $R/J(R)$ is an exchange ring. Is R necessarily an elementary divisor ring?

Question 53. Is a noncommutative semipotent stable range 1 Bezout ring necessarily an elementary divisor ring?

Question 54. Suppose that R is a noncommutative Bezout domain. Does it follow that R is an elementary divisor ring if so is $R/J(R)$? What about the case where $R/J(R)$ is a unit regular ring?

We will finish this section with a few commutativity problems for some special classes of rings.

Definition 26. A ring R is called an *abelian (unit-central)* ring if its idempotents (units) are central ([16]).

Question 55. It is known ([16]) that any unit-central ring R has stable range 1 if $R/J(R)$ is an exchange ring. Is every unit-central ring R a distributive ring whenever $R/J(R)$ is an exchange ring?

9. Ultimate problems. We have called this section "Ultimate problems" since the solution of one of them will imply the solutions for many other problems in commutative (and noncommutative) ring theory as well as it will be a nontrivial contribution in the general algebraic investigations.

The first ultimate problem is the following one.

Open Problem 1. Is every right Bezout ring of stable range 2 a right Hermite one?

Sometimes a complete solution is too hard to find it at one time, therefore we have some other related questions.

Open Problem 1′. Is every Bezout duo-ring of stable range 2 an Hermite ring?

Definition 27. A right R-module M is called a *perspective module* if for any two its summands A, B there exists a submodule C of M such that $A \oplus C = B \oplus C = M$. A ring R is called *right perspective* if R_R is a perspective module.

Open Problem 1″. Is every right perspective Bezout ring of stable range 2 a right Hermite ring?

The second ultimate problem is the following one.

Open Problem 2. Suppose that R is a commutative morphic ring. Is there any commutative Bezout domain K and some its element $a \in K \setminus \{0\}$ such that $R \cong K/aK$?

This means that the class of morphic rings coincides with the class of all nontrivial finite homomorphic images of the commutative Bezout domains.

Again here we have other helpful questions.

Open Problem 2′. It is well-known that $R[x]$ is a commutative Bezout ring if and only if R is a von Neumann regular ring. Let R be a morphic ring of stable range 1. Is there any von Neumann regular ring K such that $R \cong K[x]/(x^n)$ for some $n \in \mathbb{N}$?

Open Problem 2″. How can be chosen a noncommutative ring R such that $R[x]$ is a right morphic right Bezout ring?

The third ultimate problem will help to import the results proved for the commutative Bezout domains to the Prüffer ones, in the case of an affirmative (or even partial) answer.

Open Problem 3. As any semilocal Prüffer ring is a Bezout one, in the case where all maximal ideals are principal, then is it true the same if we allow the ring R to be not necessarily semilocal? (the domain condition can be also added).

In the forth ultimate problem we ask about the existence of a commutative ring of finite stable range greater than 2.

Open Problem 4. Is it true that the stable range of any commutative Bezout ring equals 1, 2 or ∞?

The fifth ultimate problem connects analysis and algebra.

Open Problem 5. Suppose that for a given topological space X the ring $C(X)$ of all continuous real-valued functions defined on X is a Bezout ring. Is there any commutative Bezout PM*-domain R and some element $a \in R \setminus \{0\}$ such that $C(X) \cong R/aR$?

In the case of a positive answer to the latter problem we obtain that there is some commutative Bezout domain which is not an elementary divisor ring. (M. Henriksen ([10]) found an example of $C(X)$ that is not an elementary divisor ring). Thus it would be a negative answer to the last ultimate problem!

Open Problem 6. Is every commutative Bezout domain an elementary divisor ring?

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