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ON THE EXISTENCE OF MEROMORPHICALLY STARLIKE AND MEROMORPHICALLY CONVEX SOLUTIONS OF SHAH'S DIFFERENTIAL EQUATION

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We establish conditions under which the differential equation of S. Shah $z^2w'' + (\beta_0z^2 + \beta_1z)w' + (\gamma_0z^2 + \gamma_1z + \gamma_2)w = 0$ has meromorphically starlike and meromorphically convex solutions of order $\alpha \in [0, 1)$.

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Для дифференциального уравнения $z^2w'' + (\beta_0z^2 + \beta_1z)w' + (\gamma_0z^2 + \gamma_1z + \gamma_2)w = 0$ С. Шаха установлены условия, при выполнении которых это уравнение имеет мероморфно звездные и мероморфно выпуклые решения порядка $\alpha \in [0, 1)$.

1. Introduction. An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_nz^n \tag{1}$$

is said to be *convex* if $f(\mathbb{D})$ is a convex domain. It is well known ([1, p. 203]) that the condition $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. Following W. Kaplan ([2]), a function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}(f'(z)/\Phi'(z)) > 0$ $(z \in \mathbb{D})$. A close-to-convex function f has the characteristic property that the complement G to the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that a function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_nz^n, \tag{2}$$

is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$. We remark also, that a function defined by (2) is said to be starlike in $\mathbb{D}$, if $f(\mathbb{D})$ is a starlike domain with respect to the origin. It is clear that every starlike function is close-to-convex.

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S. M. Shah ([3]) indicated conditions on real parameters $\beta_0, \beta_1, \gamma_0, \gamma_1, \gamma_2$ of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

under which there exists an entire transcendental solution given by (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular he obtained the following result: if $\beta_1 + \gamma_2 = 0, -1 \le \beta_0 < 0, \beta_1 > 0$ and $-\beta_1/2 < \gamma_1 \le 0$ then equation (3) has an entire solution (2) such that all derivatives $g^{(n)}(n \ge 0)$ are close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$, where $M_g(r) = \max\{|g(z)| : |z| = r\}$.

The investigations are continued in papers [4]–[9]. In particular in the case of complex parameters $\beta_0, \beta_1, \gamma_0, \gamma_1, \gamma_2$ in [8] it is proved that if $\gamma_0 = 0, \beta_1 + \gamma_2 = 0, \beta_0 \ne 0, |\beta_1| < 2$ and $\frac{2(|\beta_0|+|\gamma_1|)}{2-|\beta_1|} < \ln 2$ then equation (3) has an entire solution (2) such that all the derivatives $g^{(n)}(n \ge 0)$ are starlike and, thus, close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$. An analog of this assertion for convex functions is obtained in [9], where it is proved that if $\gamma_0 = 0, \beta_1 + \gamma_2 = 0, \beta_0 \ne 0, |\beta_1| < 2$ and $\frac{2(|\beta_0|+|\gamma_1|)}{2-|\beta_1|} < \frac{\ln 2}{2}$ then equation (3) has an entire solution (2) such that all the derivatives $g^{(n)}(n \ge 0)$ are convex in $\mathbb{D}$.

Let Σ be the class of functions defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n, \tag{4}$$

analytic in $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma$ is said to be meromorphically starlike of order $\alpha \in [0, 1)$ if

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, \ z \in \mathbb{D}_0, \tag{5}$$

and is said to be meromorphically convex of order $\alpha \in [0, 1)$ if

$$\mathrm{Re}\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, \ z \in \mathbb{D}_0. \tag{6}$$

In the paper we investigate conditions on complex parameters $\beta_0, \beta_1, \gamma_0, \gamma_1, \gamma_2$, under which equation (3) has meromorphically starlike and meromorphically convex solutions of order $\alpha \in [0, 1)$.

2. Auxiliary results. O. Juneja and T. Reddy ([10]) and M. Mogra ([11]) proved that if $f_n \ge 0$ for all $n \ge 1$ then a function defined by (4) is meromorphically starlike of order $\alpha \in [0, 1)$ if and only if $\sum_{n=1}^{\infty}(n + \alpha)f_n \le 1 - \alpha$ and the same function is meromorphically convex of order $\alpha \in [0, 1)$ if and only if $\sum_{n=1}^{\infty} n(n + \alpha)f_n \le 1 - \alpha$.

In the case of complex coefficients the following lemma is true.

Lemma 1. *If*

$$\sum_{n=1}^{\infty}(n + \alpha)|f_n| \le 1 - \alpha, \ \alpha \in [0, 1), \tag{7}$$

then the function defined by (4) *is meromorphically starlike of order* α*, and if*

$$\sum_{n=1}^{\infty} n(n + \alpha)|f_n| \le 1 - \alpha, \ \alpha \in [0, 1), \tag{8}$$

then the function defined by (4) *is meromorfically convex of order* α*.*

Proof. We use a method from the paper of Sh. Owa and N. Pascu ([12]), where similar results are obtained. Since in view of [7]

$$|zf'(z)+f(z)|-|zf'(z)+(2\alpha-1)f(z)|=$$
$$=\left|\sum_{n=1}^{\infty}(n+1)f_nz^n\right|-\left|\frac{2(\alpha-1)}{z}+\sum_{n=1}^{\infty}(n+2\alpha-1)f_nz^n\right|\le$$
$$\le\frac{1}{|z|}\left(\sum_{n=1}^{\infty}(n+1)|f_n||z|^{n+1}-\left(|2(\alpha-1)|-\sum_{n=1}^{\infty}(|n+2\alpha-1|)|f_n||z|^{n+1}\right)\right)=$$
$$=\frac{1}{|z|}\left(\sum_{n=1}^{\infty}(n+1)|f_n||z|^{n+1}-\left(2(1-\alpha)-\sum_{n=1}^{\infty}(n+2\alpha-1)|f_n||z|^{n+1}\right)\right)=$$
$$=\frac{1}{|z|}\left(\sum_{n=1}^{\infty}(2n+2\alpha)|f_n||z|^{n+1}-2(1-\alpha)\right)<$$
$$<\frac{2}{|z|}\left(\sum_{n=1}^{\infty}(n+\alpha)|f_n|-(1-\alpha)\right)\le 0,$$

we have $\frac{|zf'(z)+f(z)|}{|zf'(z)+(2\alpha-1)f(z)|}<1$ for all $z\in\mathbb{D}_0$, i.e.

$$\left|\frac{zf'(z)}{f(z)}+1\right|<\left|\frac{zf'(z)}{f(z)}+2\alpha-1\right|. \tag{9}$$

We remark that the inequality $|w+1|<|w+2\alpha-1|$ implies the inequality $\operatorname{Re} w<-\alpha$. Therefore (9) implies that the function defined by (4) is meromorphically starlike of order α.

We put $\varphi(z)=-zf'(z)$. Then $\frac{z\varphi'(z)}{\varphi(z)}=1+\frac{zf''(z)}{f'(z)}$ i.e. f is meromorphically convex if and only if φ is meromorphically starlike. Since $\varphi(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\varphi_nz^n$, where $\varphi_n=-nf_n$, from (8) it follows that the function defined by (4) is meromorphically convex. □

Now we find recurrent formulas for the coefficients of a function defined by (4) which satisfies equation (3). We have

$$z^2\left(\frac{2}{z^3}+\sum_{n=2}^{\infty}n(n-1)f_nz^{n-2}\right)+(\beta_0z^2+\beta_1z)\left(-\frac{1}{z^2}+\sum_{n=1}^{\infty}nf_nz^{n-1}\right)+$$
$$+(\gamma_0z^2+\gamma_1z+\gamma_2)\left(\frac{1}{z}+\sum_{n=1}^{\infty}f_nz^n\right)\equiv 0,$$

that is

$$\frac{2}{z}+\sum_{n=2}^{\infty}n(n-1)f_nz^n-\beta_0+\sum_{n=1}^{\infty}\beta_0nf_nz^{n+1}-\frac{\beta_1}{z}+\sum_{n=1}^{\infty}\beta_1nf_nz^n+$$
$$+\gamma_0z+\gamma_1+\frac{\gamma_2}{z}+\sum_{n=1}^{\infty}\gamma_0f_nz^{n+2}+\sum_{n=1}^{\infty}\gamma_1f_nz^{n+1}+\sum_{n=1}^{\infty}\gamma_2f_nz^n\equiv 0$$

and

$$\frac{2-\beta_1+\gamma_2}{z}-\beta_0+\gamma_1+\gamma_0z+\sum_{n=2}^{\infty}n(n-1)f_nz^n+\sum_{n=2}^{\infty}\beta_0(n-1)f_{n-1}z^n+\sum_{n=1}^{\infty}\beta_1nf_nz^n+$$

$$+\sum_{n=3}^{\infty}\gamma_0 f_{n-2}z^n+\sum_{n=2}^{\infty}\gamma_1 f_{n-1}z^n+\sum_{n=1}^{\infty}\gamma_2 f_n z^n\equiv 0.$$

Hence the following lemma follows.

Lemma 2. *A function defined by* (4) *is a solution of equation* (3) *if and only if*

$$\begin{gathered}2-\beta_1+\gamma_2=0,\ -\beta_0+\gamma_1=0,\\ \gamma_0+2(1+\gamma_2)f_1=0,\ 3(2+\gamma_2)f_2+2\gamma_1 f_1=0\end{gathered}\tag{10}$$

and for $n\ge 3$

$$(n+1)(n+\gamma_2)f_n+n\gamma_1 f_{n-1}+\gamma_0 f_{n-2}=0.\tag{11}$$

3. Main results. We assume that

$$n+\gamma_2\ne 0,\ n\ge 1.\tag{12}$$

Then equalities (10) and (11) yield that if $\gamma_0=0$ then all $f_n=0$, that is, conditions (7) and (8) are equivalent to the condition $0\le 1-\alpha$. Therefore, the following statement is true.

Proposition 1. *If* $\beta_1=2+\gamma_2,\beta_0=\gamma_1,\gamma_0=0$ *and condition* (12) *holds then differential equation* (3) *has the solution* $f(z)=1/z$, *which is meromorphically starlike and meromorphically convex of order* α *for each* $\alpha\in[0,1)$.

Now we assume that $\gamma_0\ne 0$. Then

$$f_1=-\frac{\gamma_0}{2(1+\gamma_2)},\ f_2=-\frac{2\gamma_1}{3(2+\gamma_2)}f_1,\ f_n=-\frac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1}-\frac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}.$$

Using these equalities and Lemma 1 we prove the following theorem.

Theorem 1. *Let* $\alpha\in[0,1)$. *If* $\beta_1=2+\gamma_2,|\gamma_2|<1,\beta_0=\gamma_1$ *and* $\gamma_0\ne 0$ *then differential equation* (3) *has a solution given by* (4), *which by the condition*

$$\frac{(1+\alpha)|\gamma_0|}{2(1-|\gamma_2|)}\le(1-\alpha)\left(1-\frac{4|\gamma_1|}{3(2-|\gamma_2|)}-\frac{3|\gamma_0|}{4(3-|\gamma_2|)}\right)\tag{13}$$

is meromorphically starlike of order α *and by the condition*

$$\frac{(1+\alpha)|\gamma_0|}{2(1-|\gamma_2|)}\le(1-\alpha)\left(1-\frac{8|\gamma_1|}{3(2-|\gamma_2|)}-\frac{9|\gamma_0|}{4(3-|\gamma_2|)}\right)\tag{14}$$

is meromorphically convex of order α.

Proof. Since $|\gamma_2|<1$, (12) holds and from the indicated above equalities for f_j we obtain

$$\sum_{n=1}^{\infty}(n+\alpha)|f_n|\le(1+\alpha)|f_1|+(2+\alpha)|f_2|+$$

$$+\sum_{n=3}^{\infty}(n+\alpha)\left|\frac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1}+\frac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}\right|\le(1+\alpha)|f_1|+(2+\alpha)|f_2|+$$

$$+\sum_{n=3}^{\infty}\frac{(n+\alpha)n|\gamma_1|}{(n+1)(n-|\gamma_2|)}|f_{n-1}|+\sum_{n=3}^{\infty}\frac{(n+\alpha)|\gamma_0|}{(n+1)(n-|\gamma_2|)}|f_{n-2}|=$$

$$=(1+\alpha)|f_1|+(2+\alpha)|f_2|+\sum_{n=2}^{\infty}\frac{(n+1+\alpha)(n+1)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n|+$$

$$+\sum_{n=1}^{\infty}\frac{(n+2+\alpha)|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n|=(1+\alpha)|f_1|+(2+\alpha)|f_2|-\frac{(2+\alpha)2|\gamma_1|}{3(2-|\gamma_2|)}|f_1|+$$

$$+\sum_{n=1}^{\infty}\frac{(n+1+\alpha)(n+1)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n|+\sum_{n=1}^{\infty}\frac{(n+2+\alpha)|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n|,$$

whence

$$\sum_{n=1}^{\infty}\left(1-\frac{(n+1+\alpha)(n+1)|\gamma_1|}{(n+\alpha)(n+2)(n+1-|\gamma_2|)}-\frac{(n+2+\alpha)|\gamma_0|}{(n+\alpha)(n+3)(n+2-|\gamma_2|)}\right)(n+\alpha)|f_n|\le$$

$$\le(1+\alpha)|f_1|+\frac{(2+\alpha)2|\gamma_1|}{3(2-|\gamma_2|)}|f_1|-\frac{(2+\alpha)2|\gamma_1|}{3(2-|\gamma_2|)}|f_1|=(1+\alpha)|f_1|\le\frac{(1+\alpha)|\gamma_0|}{2(1-|\gamma_2|)}. \quad (15)$$

Since $\frac{n+1+\alpha}{n+\alpha}\le\frac{n+1}{n}$ and in view of the inequality $|\gamma_2|<1$ the sequence $\frac{(n+1)^2}{n(n+2)(n+1-|\gamma_2|)}$ decreases, we have

$$\frac{(n+1+\alpha)(n+1)|\gamma_1|}{(n+\alpha)(n+2)(n+1-|\gamma_2|)}\le\frac{4|\gamma_1|}{3(2-|\gamma_2|)}. \quad (16)$$

Analogously,

$$\frac{(n+2+\alpha)|\gamma_0|}{(n+\alpha)(n+3)(n+2-|\gamma_2|)}\le\frac{3|\gamma_0|}{4(3-|\gamma_2|)}. \quad (17)$$

Condition (13) implies the inequality

$$\frac{4|\gamma_1|}{3(2-|\gamma_2|)}+\frac{3|\gamma_0|}{4(3-|\gamma_2|)}<1.$$

Therefore, from (15) in view of (16) and (17) we have

$$\left(1-\frac{4|\gamma_1|}{3(2-|\gamma_2|)}-\frac{3|\gamma_0|}{4(3-|\gamma_2|)}\right)\sum_{n=1}^{\infty}(n+\alpha)|f_n|\le\frac{(1+\alpha)|\gamma_0|}{2(1-|\gamma_2|)},$$

whence in view of (13) we obtain the inequality $\sum_{n=1}^{\infty}(n+\alpha)|f_n|\le 1-\alpha$, and by Lemma 1 function (4) is meromorphically starlike of order α. The first part of Theorem 1 is proved.

The proof of the second part is analogous. As above, we have

$$\sum_{n=1}^{\infty}n(n+\alpha)|f_n|\le(1+\alpha)|f_1|+2(2+\alpha)|f_2|+$$

$$+\sum_{n=3}^{\infty}n(n+\alpha)\left|\frac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1}+\frac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}\right|\le(1+\alpha)|f_1|+2(2+\alpha)|f_2|+$$

$$+\sum_{n=3}^{\infty}\frac{n(n+\alpha)n|\gamma_1|}{(n+1)(n-|\gamma_2|)}|f_{n-1}|+\sum_{n=3}^{\infty}\frac{n(n+\alpha)|\gamma_0|}{(n+1)(n-|\gamma_2|)}|f_{n-2}|=$$

$$= (1+\alpha)|f_1| + 2(2+\alpha)|f_2| + \sum_{n=2}^{\infty} \frac{(n+1)(n+1+\alpha)(n+1)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n| +$$

$$+ \sum_{n=1}^{\infty} \frac{(n+2)(n+2+\alpha)|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n| = (1+\alpha)|f_1| + 2(2+\alpha)|f_2| - \frac{2(2+\alpha)2|\gamma_1|}{3(2-|\gamma_2|)}|f_1| +$$

$$+ \sum_{n=1}^{\infty} \frac{(n+1)(n+1+\alpha)(n+1)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n| + \sum_{n=1}^{\infty} \frac{(n+2)(n+2+\alpha)|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n|,$$

whence

$$\sum_{n=1}^{\infty} \left(1 - \frac{(n+1+\alpha)(n+1)^2|\gamma_1|}{n(n+\alpha)(n+2)(n+1-|\gamma_2|)} - \frac{(n+2)(n+2+\alpha)|\gamma_0|}{n(n+\alpha)(n+3)(n+2-|\gamma_2|)}\right) n(n+\alpha)|f_n| \leq$$

$$\leq \frac{(1+\alpha)|\gamma_0|}{2(1-|\gamma_2|)}. \tag{18}$$

Since, as above,

$$\frac{(n+1+\alpha)(n+1)^2|\gamma_1|}{n(n+\alpha)(n+2)(n+1-|\gamma_2|)} \leq \frac{(n+1)^3|\gamma_1|}{n^2(n+2)(n+1-|\gamma_2|)} \leq \frac{8|\gamma_1|}{3(2-|\gamma_2|)}$$

and

$$\frac{(n+2)(n+2+\alpha)|\gamma_0|}{n(n+\alpha)(n+3)(n+2-|\gamma_2|)} \leq \frac{9|\gamma_0|}{4(3-|\gamma_2|)},$$

from condition (14) it follows that

$$\frac{8|\gamma_1|}{3(2-|\gamma_2|)} + \frac{9|\gamma_0|}{4(3-|\gamma_2|)} < 1.$$

Therefore, (18) implies

$$\left(1 - \frac{8|\gamma_1|}{3(2-|\gamma_2|)} - \frac{9|\gamma_0|}{4(3-|\gamma_2|)}\right) \sum_{n=1}^{\infty} n(n+\alpha)|f_n| \leq \frac{(1+\alpha)|\gamma_0|}{2(1-|\gamma_2|)},$$

whence in view of condition (14) we obtain the inequality $\sum_{n=1}^{\infty} n(n+\alpha)|f_n| \leq 1-\alpha$, and by Lemma 1 the function defined by (4) is meromorphically convex of order α. □

Remark 1. By analogue with the definitions of starlike and convex functions we say that a function given by (4) is meromorphically starlike if $\mathrm{Re}\{-zf'(z)/f(z)\} > 0$ and meromorphically convex if $\mathrm{Re}\{-(1+zf''(z))/f'(z)\} > 0$ for all $z \in \mathbb{D}_0$.

Then from Theorem 1 for $\alpha = 0$ we obtain the following corollary.

Corollary 1. *If $\beta_1 = 2+\gamma_2, \gamma_2 = 0, \beta_0 = \gamma_1$ and $\gamma_0 \neq 0$ then the differential equation $zw'' + (\gamma_1 z + 2)w' + (\gamma_0 z + \gamma_1)w = 0$ has the solution*

$$f(z) = \frac{1}{z} - \frac{\gamma_0}{2}z + \frac{\gamma_0\gamma_1}{6}z^2 + \sum_{n=3}^{\infty} f_n z^n,$$

where $f_n = -\frac{\gamma_1}{(n+1)}f_{n-1} - \frac{\gamma_0}{(n+1)n}f_{n-2}$ for $n \geq 3$, which by the condition $9|\gamma_0| + 8|\gamma_1| \leq 12$ is meromorphically starlike and by the condition $15|\gamma_0| + 16|\gamma_1| \leq 12$ is meromorphically convex in $\mathbb{D}_0$.

Remark 2. In Theorem 1 the parameter γ_2 satisfies the condition $|\gamma_2| < 1$. We can obtain an analog of this theorem for $n < |\gamma_2| < n+1$ with some $n \geq 1$. We demonstrate this assertion in the case where $1 < |\gamma_2| < 2$. By this condition, in the proof of Theorem 1 we replace the estimates $|n+\gamma_2| \geq n - |\gamma_2|$ for all $n \geq 1$ with the estimates $|1+\gamma_2| \geq |\gamma_2| - 1$ and $|n+\gamma_2| \geq n - |\gamma_2|$ for all $n \geq 2$. Then conditions (13) and (14) take the form

$$\frac{(1+\alpha)|\gamma_0|}{2(|\gamma_2|-1)} \leq (1-\alpha)\left(1 - \frac{4|\gamma_1|}{3(2-|\gamma_2|)} - \frac{3|\gamma_0|}{4(3-|\gamma_2|)}\right),$$
$$\frac{(1+\alpha)|\gamma_0|}{2(|\gamma_2|-1)} \leq (1-\alpha)\left(1 - \frac{8|\gamma_1|}{3(2-|\gamma_2|)} - \frac{9|\gamma_0|}{4(3-|\gamma_2|)}\right).$$

Now we consider the cases where condition (12) does not hold. At first, we assume that $1+\gamma_2 = 0$. Then in view of (10) $\gamma_0 = 0$ and we can choose $f_1 \neq 0$, because if $f_1 = 0$ then in view of (10) $f_2 = 0$, and in view of (11) all $f_n = 0$ for $n \geq 3$ and we come to the case where $f(z) = 1/z$, which we considered above. We assume that $f_1 = a^2$ and $\gamma_1 = 0$. Since $2+\gamma_2 \neq 0$, we have $f_2 = 0$ and in view of the equality $\gamma_0 = 0$, all $f_n = 0$ for $n \geq 3$. Thus, the solution has the form $f(z) = 1/z + a^2 z = a(1/(az) + az) = 2aJ(az)$, where J is the function of Joukowsky. Therefore, using Lemma 1, we get the following statement.

Proposition 2. *If $\beta_1 = 1$, $\gamma_2 = -1$ and $\beta_0 = \gamma_1 = \gamma_0 = 0$ then differential equation* (3) *has the solution $f(z) = J(az)$, which by the condition $(1+\alpha)|a|^2 \leq 1-\alpha$ is meromorphically starlike and meromorphically convex of order α.*

Remark 3. From Proposition 2 it follows that the function of Zhukowski is meromorphically starlike and meromorphically convex. Using the definition it is easy to show that the function $f(z) = J(az)$ with $|a| > 1$ is not meromorphically starlike and meromorphically convex. We remark also that differential equation (3) by the assumptions of Proposition 2 has the form $z^2 w'' + zw' - w = 0$.

If $1+\gamma_2 = 0$ and $\gamma_1 \neq 0$ then $\gamma_0 = 0$, $f_2 = -\frac{2\gamma_1}{3} f_1$ and $f_n = -\frac{n\gamma_1}{n^2-1} f_{n-1}$ for $n \geq 3$. Therefore, as in the proof of Theorem 1 we have

$$\sum_{n=1}^{\infty}(n+\alpha)|f_n| = (1+\alpha)|f_1| + \sum_{n=1}^{\infty}\frac{|\gamma_1|(n+1)(n+1+\alpha)}{((n+1)^2-1)(n+\alpha)}(n+\alpha)|f_n| \leq$$
$$\leq (1+\alpha)|f_1| + \sum_{n=1}^{\infty}\frac{2|\gamma_1|(2+\alpha)}{3(1+\alpha)}(n+\alpha)|f_n|,$$

that is

$$\sum_{n=1}^{\infty}\left(1 - \frac{2|\gamma_1|(2+\alpha)}{3(1+\alpha)}\right)(n+\alpha)|f_n| \leq (1+\alpha)|f_1|.$$

If $\frac{2+\alpha}{1+\alpha} < \frac{3}{2|\gamma_1|}$ then we have

$$\sum_{n=1}^{\infty}(n+\alpha)|f_n| \leq \frac{(1+\alpha)|f_1|}{1 - \frac{2|\gamma_1|(2+\alpha)}{3(1+\alpha)}} \leq 1-\alpha$$

provided $(1+\alpha)|f_1| \leq \left(1 - \frac{2|\gamma_1|(2+\alpha)}{3(1+\alpha)}\right)(1-\alpha)$. Analogously, if $\frac{2+\alpha}{1+\alpha} < \frac{3}{4|\gamma_1|}$ then

$$\sum_{n=1}^{\infty}n(n+\alpha)|f_n| \leq \frac{(1+\alpha)|f_1|}{1 - \frac{4|\gamma_1|(2+\alpha)}{3(1+\alpha)}} \leq 1-\alpha$$

provided $(1+\alpha)|f_1| \le \left(1-\frac{4|\gamma_1|(2+\alpha)}{3(1+\alpha)}\right)(1-\alpha)$. Thus, by the arbitrariness of f_1 we obtain the following theorem.

Theorem 2. *If $\beta_1 = 1, \gamma_2 = -1, \gamma_0 = 0$ and $\beta_0 = \gamma_1 \neq 0$ then for each $\alpha \in [0,1)$ there exists a solution (4) of differential equation (3), which by the condition $|\gamma_1| < \frac{3(1+\alpha)}{2(2+\alpha)}$ is meromorphically starlike and by the condition $|\gamma_1| < \frac{3(1+\alpha)}{4(2+\alpha)}$ is meromorphically convex of order α.*

We remark that differential equation (3) by the assumptions of Theorem 2 has the form $z^2w'' + (\gamma_1 z^2 + z)w' + (\gamma_1 z - 1)w = 0$.

Now let $2+\gamma_2 = 0$. Then $\beta_1 = 0$ and from (10) and (11) we obtain $f_1 = \gamma_0/2, \gamma_1 f_1 = 0$ and $f_n = -\frac{n\gamma_1}{(n+1)(n-2)}f_{n-1} - \frac{\gamma_0}{(n+1)(n-2)}f_{n-2}$ for $n \ge 3$. Hense it follows that either $f_1 = 0$ or $\gamma_1 = 0$ and f_2 may be arbitrary.

At first we assume that $f_1 = 0$. Then $\gamma_0 = 0$ and for $n \ge 3$

$$|f_n| = \frac{n|\gamma_1|}{(n+1)(n-2)}|f_{n-1}| \le \frac{|\gamma_1|}{n-2}|f_{n-1}| \le \frac{|\gamma_1|^2}{(n-2)(n-3)}|f_{n-2}| \le \ldots \le \frac{|\gamma_1|^{n-2}}{(n-2)!}|f_2|.$$

Hence it follows that

$$\sum_{n=1}^{\infty}(n+\alpha)|f_n| \le |f_2|\sum_{n=1}^{\infty}\frac{n+\alpha}{(n-2)!}|\gamma_1|^{n-2} = K_1(|\gamma_1|)|f_2|$$

and similarly

$$\sum_{n=1}^{\infty}n(n+\alpha)|f_n| \le K_1(|\gamma_1|)|f_2|,$$

where $K_j(|\gamma_1|)$ are positive constants. Since f_2 may be arbitrary, the following proposition holds.

Theorem 3. *If $\gamma_2 = -2, \beta_1 = \gamma_0 = 0, \beta_0 = \gamma_1 \neq 0$ then for each $\alpha \in [0,1)$ there exists a solution (4) of differential equation (3) which is meromorphically starlike and meromorphically convex of order α.*

We remark that differential equation (3) by the assumptions of Theorem 3 has the form $z^2w'' + \gamma_1 z^2 w' + (\gamma_1 z - 2)w = 0$.

Finally, we assume that $\gamma_1 = 0$. Then $f_1 = \gamma_0/2, f_2$ may be arbitrary and $|f_n| = \frac{|\gamma_0|}{(n+1)(n-2)}|f_{n-2}|$ for $n \ge 3$. Using this relations we prove the following theorem.

Theorem 4. *Let $\alpha \in [0,1)$. If $\gamma_2 = -2, \beta_1 = \beta_0 = \gamma_1 = 0$ and $|\gamma_0| < 4/9$ then there exists a solution (4) of differential equation (3), which by the condition*

$$\frac{(1+\alpha)|\gamma_0|}{2} \le (1-\alpha)\left(1-\frac{3|\gamma_0|}{4}\right), \tag{19}$$

is meromorphically starlike of order α and by the condition

$$\frac{(1+\alpha)|\gamma_0|}{2} \le (1-\alpha)\left(1-\frac{9|\gamma_0|}{4}\right) \tag{20}$$

is meromorphically convex of order α.

Proof. Since f_2 may be arbitrary, we set $f_2 = 0$. Then

$$\sum_{n=1}^{\infty}(n+\alpha)|f_n| = (1+\alpha)|f_1| + \sum_{n=3}^{\infty}\frac{(n+\alpha)|\gamma_0|}{(n+1)(n-2)}|f_{n-2}| =$$
$$= (1+\alpha)|f_1| + \sum_{n=1}^{\infty}\frac{(n+2+\alpha)|\gamma_0|}{n(n+3)}|f_n|$$

and, thus,

$$\sum_{n=1}^{\infty}\left(1-\frac{(n+2+\alpha)|\gamma_0|}{(n+\alpha)n(n+3)}\right)(n+\alpha)|f_n| = (1+\alpha)|f_1|.$$

But

$$\frac{(n+2+\alpha)|\gamma_0|}{(n+\alpha)n(n+3)} \le \frac{(n+2)|\gamma_0|}{n^2(n+3)} \le \frac{3|\gamma_0|}{4} < 1.$$

Therefore,

$$\left(1-\frac{3|\gamma_0|}{4}\right)\sum_{n=1}^{\infty}(n+\alpha)|f_n| \le \frac{(1+\alpha)|\gamma_0|}{2},$$

whence in view on (19) we obtain the inequality $\sum_{n=1}^{\infty}(n+\alpha)|f_n| \le 1-\alpha$, and by Lemma 1 function (4) is meromorphically starlike of order α. The first part of Theorem 4 is proved.

The proof of the second part is similar. As above, we have

$$\sum_{n=1}^{\infty}n(n+\alpha)|f_n| = (1+\alpha)|f_1| + \sum_{n=3}^{\infty}\frac{n(n+\alpha)|\gamma_0|}{(n+1)(n-2)}|f_{n-2}| =$$
$$= (1+\alpha)|f_1| + \sum_{n=1}^{\infty}\frac{(n+2)(n+2+\alpha)|\gamma_0|}{n(n+3)}|f_n|,$$

whence

$$\sum_{n=1}^{\infty}\left(1-\frac{(n+2)(n+2+\alpha)|\gamma_0|}{n(n+\alpha)n(n+3)}\right)n(n+\alpha)|f_n| = (1+\alpha)|f_1|.$$

But

$$\frac{(n+2)(n+2+\alpha)|\gamma_0|}{n^2(n+\alpha)(n+3)} \le \frac{(n+2)^2|\gamma_0|}{n^3(n+3)} \le \frac{9|\gamma_0|}{4} < 1$$

and, therefore,

$$\left(1-\frac{9|\gamma_0|}{4}\right)\sum_{n=1}^{\infty}n(n+\alpha)|f_n| \le \frac{(1+\alpha)|\gamma_0|}{2},$$

whence in view on (20) we obtain the inequality $\sum_{n=1}^{\infty}n(n+\alpha)|f_n| \le 1-\alpha$, and by Lemma 1 function (4) is meromorphically convex of order α. □

For $\alpha = 0$ Theorem 4 implies the following statement.

Corollary 2. *If $\gamma_2 = -2, \beta_1 = \beta_0 = \gamma_1 = 0$ then differential equation (3) has a solution*

$$f(z) = \frac{1}{z} + \frac{\gamma_0}{2}z + \sum_{n=3}^{\infty}f_n z^n,$$

where $f_n = -\frac{\gamma_0}{(n+1)(n-2)}f_{n-2}$ for $n \ge 3$, which by the condition $|\gamma_0| \le 4/5$ is meromorphically starlike and by the condition $|\gamma_0| \le 4/11$ is meromorphically convex.

By the assumptions of Corollary 2 differential equation (3) has the form

$$zw'' + (\gamma_0 z^2 - 2)w = 0.$$

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