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We prove that whenever all differences between zeros of two quasipolynomials form a discrete set, then both quasipolynomials are periodic with the same period. The result is valid for some classes of Dirichlet series and almost periodic holomorphic functions as well.

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Доказано, что если множество разностей корней двух квазиполиномов дискретно, то оба квазиполинома являются периодическими с одним и тем же периодом. Результат остается в силе для некоторых классов рядов Дирихле, а также для голоморфных почти периодических функций.

Consider a quasipolynomial of the form

$$P(z)=\sum_{n=1}^{N} a_n e^{\lambda_n z}, \quad \lambda_n\in\mathbb{R}, \quad a_n\in\mathbb{C}. \tag{1}$$

It can be easily seen that its zero set $Z(P)$ locates in a vertical strip of a finite width. If P is periodic, then it has the form

$$P(z)=C\prod_{k=1}^{N}\cosh(\omega z+b_k), \quad \omega\in\mathbb{R}, \quad C,\, b_k\in\mathbb{C}. \tag{2}$$

In the general case the distribution of $Z(P)$ is rather complicated (see, for example, [5], [4], [2]).

Here we present a simple criterion for periodicity of quasipolynomial (1). Actually, we consider the case of infinite sums

$$Q(z)=\sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad \lambda_n\in\mathbb{R}, \quad a_n\in\mathbb{C}, \tag{3}$$

such that

$$\sum_{n=1}^{\infty}|a_n|<\infty, \quad \lambda_1=\sup\{\lambda_n\}, \quad \lambda_2=\inf\{\lambda_n\}, \quad a_1a_2\neq 0. \tag{4}$$

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Theorem 1. *Let Q_1, Q_2 be functions (3) under conditions (4). If the set of differences $Z(Q_1) - Z(Q_2) = \{z - w : \ z \in Z(Q_1),\, w \in Z(Q_2)\}$ is discrete, then*

$$Q_j(z) = C_j e^{\beta_j z} \prod_{k=1}^{K_j} \cosh(\omega z + b_k^{(j)}), \quad \omega, \beta_j \in \mathbb{R}, \quad C_j,\ b_k^{(j)} \in \mathbb{C}, \quad j \in \{1,2\}, \tag{5}$$

where ω is the same for both functions.

For $Q_2(-z) = Q_1(z) = Q(z)$ we get

Corollary. *If the set $Z(Q) + Z(Q)$ of all sums of two zeros of Q forms a discrete set, then Q is periodic and has form (2) up to a multiplier $\exp(\beta z)$. The same is valid for the case where the set of differences $Z(Q) - Z(Q)$ is discrete.*

In fact, the latter case was investigated earlier [1].

The proof of the Theorem is based on some properties of functions (3) and its zeros.

Proposition 1. *Any function (3) under conditions (4) is uniformly continuous on every strip $|\operatorname{Re} z| < H$.*

Proof. Every term of the sum from (3) is uniformly continuous in this strip, and by (4), the sum uniformly converges on the strip. □

Proposition 2. *For any function (3) under conditions (4), any strip $|\operatorname{Re} z| < H$, and any sequence $\eta_k \in \mathbb{R}$ there is a subsequence $\eta_{k'}$ such that the functions $f(z + i\eta_{k'})$ uniformly converge on the strip.*

Proof. Take a subsequence $\eta_{k'}$ such that $\exp(i\lambda_n \eta_{k'}) \to \exp(i\alpha_n)$ $(k' \to \infty)$, $\alpha_n \in [0, 2\pi)$, for all $n \in \mathbb{N}$. Taking into account (4), for any $\gamma > 0$ we find m such that

$$\sum_{n=m+1}^{\infty} |a_n| \exp(\lambda_n \operatorname{Re} z) < \gamma/3$$

in the strip $|\operatorname{Re} z| < H$. Hence,

$$\Big| f(z + i\eta_{k'}) - \sum_{n=1}^{\infty} a_n e^{i\alpha_n + \lambda_n z} \Big| \le \sum_{n=1}^{m} |a_n| e^{\lambda_n \operatorname{Re} z} \Big| e^{i\lambda_n \eta_{k'}} - e^{i\alpha_n} \Big| + 2 \sum_{n=m+1}^{\infty} |a_n| e^{\lambda_n \operatorname{Re} z} < \gamma$$

for $|\operatorname{Re} z| < H$ and sufficiently large k'. □

We need the following definition

Definition 1. The set $E \subset \mathbb{R}$ is called *relatively dense*, if there exists $L < \infty$ such that $E \cap [a, a + L] \neq \varnothing$ for any $a \in \mathbb{R}$.

Proposition 3. *For any $\gamma > 0$ and any $H < \infty$ there is a relatively dense set of numbers $\tau \in \mathbb{R}$ such that for $|\operatorname{Re} z| < H$*

$$|Q(z + i\tau) - Q(z)| < \gamma. \tag{6}$$

Moreover, for any finite set of functions $\{Q\}$ there exists a relatively dense set of numbers such that (6) is fulfilled for all Q.

Proof. Taking into account (4), we find m such that

$$\sum_{n=m+1}^{\infty} |a_n| e^{\lambda_n \operatorname{Re} z} < \gamma/3 \tag{7}$$

for all z in the strip $|\operatorname{Re} z| < H$. By the Kronecker Lemma (see, for example, [6], Ch.2, §2), for any $\lambda_1, \dots, \lambda_m$ and $\delta > 0$ there is a relatively dense set of numbers τ such that

$$|\exp(i\lambda_n \tau) - 1| < \delta \tag{8}$$

for all $n \in \{1, \dots, m\}$. Hence for sufficiently small δ we get

$$\Big|\sum_{n=1}^{m} a_n e^{\lambda_n z + i\lambda_n \tau} - \sum_{n=1}^{m} a_n e^{\lambda_n z}\Big| \le \sum_{n=1}^{m} |a_n| e^{\lambda_n \operatorname{Re} z} \sup\left\{\left|e^{i\lambda_n \tau} - 1\right| : \ 1 \le n \le m\right\} < \gamma/3.$$

The later estimates and (7) imply (6). In the case of a finite number of functions $\{Q_j = \sum_{n=1}^{\infty} a_n^{(j)} \exp(\lambda_n^{(j)} z)\}$ we have to find τ such that (8) is valid for a finite subset of $\{\lambda_n^{(j)}\}_{n,j}$. □

In what follows we will assume that all numbers λ_n are distinct.

Proposition 4. *Zeros of (3) under conditions (4) locate in a vertical strip* $\{z : |\operatorname{Re} z| < H(Q)\}$.

Proof. Take $m \in \mathbb{N}$ such that $\sum_{n=m+1}^{\infty} |a_n| < |a_1|/3$. We have

$$|\exp(-\lambda_1 z) Q(z)| > |a_1| - \sum_{n=2}^{m} |a_n| \exp\{(\lambda_n - \lambda_1) \operatorname{Re} z\} - \sum_{n=m+1}^{\infty} |a_n| \exp\{(\lambda_n - \lambda_1) \operatorname{Re} z\}.$$

The last sum on the right-hand side is less than $|a_1|/3$ for $\operatorname{Re} z \ge 0$, the previous one is less than $|a_1|/3$ for $\operatorname{Re} z$ sufficiently large, hence $Q(z) \ne 0$ for $\operatorname{Re} z > H$. Similar arguments work for sufficiently large $-\operatorname{Re} z$. □

Proposition 5. *The numbers of zeros of Q in all horizontal strips of width one are uniformly bounded by a constant $M = M(Q)$.*

Proof. Suppose that there is a sequence h_k such that the numbers of zeros inside the rectangles $|\operatorname{Re} z| < H(Q)$, $h_k < \operatorname{Im} z < h_k + 1$ tend to infinity as $k \to \infty$. Passing to a subsequence as in the proof of Proposition 2, we may suppose that the functions $Q(z + h_k)$ tend to $\widetilde{Q}(z) = \sum_{n=1}^{\infty} a_n e^{i\alpha_n} e^{\lambda_n z}$ as $k \to \infty$ uniformly on the strip $|\operatorname{Re} z| < H(Q) + 1$. Take $\delta > 0$ so that $\widetilde{Q}(z) \ne 0$ on the boundary of the rectangle $\{z : |\operatorname{Re} z| < H(Q) + \delta, -\delta < \operatorname{Im} z < 1 + \delta\}$. Since the number of zeros of $\widetilde{Q}$ inside this rectangle is bounded, we get the contradiction with Gurwitz' Theorem. □

Further, denote by $B(a, r)$ the disc of radius r with center a. Put $Z(Q) = \{z_j\}$.

Proposition 6. *For any $\delta > 0$ there is $\nu > 0$ such that*

$$\inf\{|Q(z)| : \ z \in \partial\left(\cup_j B(z_j, \delta)\right)\} > \nu.$$

Proof. Assume that there is a sequence $\zeta_k = \xi_k + i\eta_k$ such that $\mathrm{dist}(\zeta_k, Z(Q)) = \delta$ and $Q(\zeta_k) \to 0$. By propositions 2 and 4, we may pass to a subsequence and suppose that $\xi_k \to \xi'$ and $Q(z + i\eta_k) \to \widetilde{Q}(z) = \sum_{n=1}^{\infty} a_n e^{i\alpha_n} e^{\lambda_n z}$ uniformly on the strip $|\operatorname{Re} z| < H(Q) + 1$. By Proposition 1, $Q(z + \zeta_k) - Q(z + \xi' + i\eta_k) \to 0$, hence,

$$\widetilde{Q}(\xi' + i0) = \lim_{k\to\infty} Q(\zeta_k) = 0.$$

By Gurwitz' Theorem, there are $z_k \to \xi'$ such that $z_k + i\eta_k \in Z(Q)$. Since $\zeta_k - z_k - i\eta_k \to 0$, we obtain the contradiction. □

Proposition 7. *For any $\gamma > 0$ there is $\delta > 0$ such that the diameter of any connected component A of the set $\cup_j B(z_j, \delta)$ is less than γ.*

Proof. If a number of elements of $Z(Q) \cap A$ is m, then the diameter of A is at most $2\delta m$. Therefore, if $2\delta < \min\{1, \gamma\}/M(Q)$, then $\mathrm{diam} A < \gamma$. □

Proposition 8. *For any $\varepsilon > 0$ there are a relatively dense set of numbers τ and bijections $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$|z_j + i\tau - z_{\sigma(j)}| < \varepsilon, \quad \forall z_j \in Z(Q). \tag{9}$$

Moreover, for a finite set of functions $\{Q_\mu\}$ there exists a common relatively dense set of numbers τ such that (9) is fulfilled for all $Z(Q_\mu)$ (bijections σ in (9) may be different).

Proof. By Proposition 6,

$$|Q(z)| \geq \nu > 0 \tag{10}$$

on the boundary of the set $\cup_{z_j \in Z(Q)} B(z_j, \delta)$. Hence if τ satisfies (6) for $\gamma = \nu/2$, then we get

$$|Q(z + i\tau) - Q(z)| \leq (1/2)|Q(z)|. \tag{11}$$

Let A be a connected component of the set $\cup_{z_j \in Z(Q)} B(z_j, \delta)$. By Rouche's Theorem, the number of zeros of Q in $A - i\tau$ equals the number of zeros in A. If $z_{j_1}, \dots, z_{j_l}$ are zeros in A and $z_{m_1}, \dots, z_{m_l}$ are zeros in $A - i\tau$, then define a bijection σ on the set $E = \{j_1, \dots, j_l\}$ such that $\sigma(E) = \{m_1, \dots, m_l\}$. Repeat the construction of σ for the other connected components. By Proposition 7, the diameters of the components are less than ε for sufficiently small δ. Therefore, σ satisfies (9).

If we have a finite number of functions Q_μ, we can take $\nu > 0$ such that (10) is valid for all functions. By Proposition 3, one can take a relatively dense set τ such that (11) satisfies for all Q_μ and $z \in \partial\left(\cup_{z_j \in Z(Q_\mu)} B(z_j, \delta)\right)$. Arguing as above and taking a sufficiently small δ, we obtain that (9) is valid for all $Z(Q_\mu)$ with the appropriate σ. □

In the previous proposition we take into account in (9) multiplicities of zeros, i.e., k points $z_j = a$ correspond to the zero of multiplicity k at the point a. If we do not count multiplicities, we get the following consequence.

Corollary. *For any $\varepsilon > 0$ there is a relatively dense set of numbers τ such that point $z'_\mu, z''_\mu \in Z(Q_\mu)$ corresponds to each $z_\mu \in Z(Q_\mu)$ such that*

$$|z_\mu + i\tau - z'_\mu| < \varepsilon, \tag{12}$$

and

$$|z_\mu - i\tau - z''_\mu| < \varepsilon. \tag{13}$$

In particular, there is a number $R = R(\{Q_\mu\}) < \infty$ such that any closed horizontal strip of width R contains at least one zero of each Q_μ.

Proposition 9. *If $Z(Q)$ has a period iT, then Q has an representation (5).*

Proof. Taking into account that $Z(Q)$ is located in a vertical strip of a finite width, we see that only a finite number of points $c_1, \dots, c_K \in Z(Q)$ sites in $\{z\colon 0 \le \operatorname{Im} z < T\}$. Therefore, $Z(Q) = \{c_1, \dots, c_K\} + iT\mathbb{Z}$. Put $\omega = \pi/T, b_k = -\omega c_k + i\pi/2,\ k \in \{1, 2, \dots, K\}$. The function

$$Q(z)\Big/\prod_{k=1}^{K} \cosh(\omega z + b_k) \tag{14}$$

is an entire one without zeros. Note that the denominator is uniformly bounded away from zero outside the discs of a small radius with a center belonging to $Z(Q)$. On the other hand,

$$|Q(z)| \le \sum_n |a_n| e^{\max\{|\lambda_1|,|\lambda_2|\}|\operatorname{Re} z|}.$$

Therefore the function Q has exponential growth and is bounded on any vertical strip of a finite width. Hence (14) also is a function of exponential growth on the plane and bounded on the imaginary axis. Thus, (14) is equal to $Ce^{\beta z},\ \beta \in \mathbb{R},\ C \in \mathbb{C}$. □

Proof of Theorem 1. Put $R = R(\{Q_1, Q_2\})$. Since $Z(Q_1)$ and $Z(Q_2)$ are located in a vertical strip of a finite width, we see that for any points $z_n, z_{n'} \in Z(Q_1),\ w_m, w_{m'} \in Z(Q_2)$ such that

$$|\operatorname{Im}(z_n - w_m)| < 2R + 3, \quad |\operatorname{Im}(z_{n'} - w_{m'})| < 2R + 3, \quad z_n - w_m \ne z_{n'} - w_{m'},$$

we have $\gamma < |(z_{n'} - w_{m'}) - (z_n - w_m)|$ for some $\gamma \in (0, 1)$. In particular, if we put $w_m = w_{m'}$, then we get $\gamma < |z_n - z_{n'}|$ for any $z_n, z_{n'} \in Z(Q_1),\ z_n \ne z_{n'}$.

Fix $z_n \in Z(Q_1)$, and let a real number $\tau > 1$ satisfy (12) for $Z(Q_1)$ and $Z(Q_2)$ with $\varepsilon = \gamma/2$. By (9), there is a unique $z_{n'} \in Z(Q_1)$ such that $|z_n + i\tau - z_{n'}| < \gamma/2$. Indeed, otherwise we would obtain $|z_{n'} - z_{n''}| \le |z_n + i\tau - z_{n'}| + |z_n + i\tau - z_{n''}| < \gamma$, But this is impossible due to the choice of γ.

Let us show that $iT := (z_{n'} - z_n)$ is a common period of $Z(Q_1)$ and $Z(Q_2)$.

Put $w_m \in Z(Q_2)$ such that $|\operatorname{Im}(w_m - z_n)| < 2R+2$. By (12), there is a point $w_{m'} \in Z(Q_2)$ such that $|w_m + i\tau - w_{m'}| < \gamma/2$. Therefore,

$$|(z_n - w_m) - (z_{n'} - w_{m'})| \le |w_m + i\tau - w_{m'}| + |z_{n'} - z_n - i\tau| < \gamma.$$

Since

$$|\operatorname{Im}(z_{n'} - w_{m'})| \le |\operatorname{Im}(z_n - w_m)| + |z_n - z_{n'} + i\tau| + |w_m - w_{m'} + i\tau| < 2R + 3,$$

we get $z_n - w_m = z_{n'} - w_{m'}$ due to the choice of γ. Therefore, $w_{m'} = w_m + iT$.

The latter equality takes place for all points of $Z(Q_2) \cap \{w : \operatorname{Im} z_n - 2R < \operatorname{Im} w < \operatorname{Im} z_n + 2R\}$, in particular, for some w_l such that $\operatorname{Im} z_n + R < \operatorname{Im} w_l < \operatorname{Im} z_n + 2R$. Namely, there is $w_{l'} \in Z(Q_2)$ such that $w_{l'} = w_l + iT$. Let $\zeta \in Z(Q_1)$ be any point of the set

$$\{z : \operatorname{Im} z_n \le \operatorname{Im} z < \operatorname{Im} z_n + 3R\} \subset \{z : \operatorname{Im} w_l - 2R < \operatorname{Im} z < \operatorname{Im} w_l + 2R\}.$$

By (12), there is a point $\zeta' \in Z(Q_1)$ such that $|\zeta + i\tau - \zeta'| < \gamma/2$. Therefore,

$$|(\zeta - w_l) - (\zeta' - w_{l'})| \le |\zeta + i\tau - \zeta'| + |iT - i\tau| < \gamma.$$

Since $|\operatorname{Im}\zeta - \operatorname{Im} w_l| < 2R$ and $|\operatorname{Im}(\zeta' - w_{l'})| \le |\operatorname{Im}(\zeta - w_l)| + |\zeta + i\tau - \zeta'| + |iT - i\tau| < 2R+3$, we get $\zeta - w_l = \zeta' - w_{l'}$ due to the choice of γ. Therefore, $\zeta' = \zeta + iT$.

In particular, there is a point $z_s \in Z(Q_1) \cap \{z : \operatorname{Im} z_n + 2R \le \operatorname{Im} z < \operatorname{Im} z_n + 3R\}$ such that $z_s + iT \in Z(Q_1)$. Continuing the line of reasoning, we obtain that for all $z \in Z(Q_1)$

such that $\operatorname{Im} z \geq \operatorname{Im} z_n$ we get $z + iT \in Z(Q_1)$ and for all $w \in Z(Q_2)$ such that $\operatorname{Im} w \geq \operatorname{Im} z_n$ we get $w + iT \in Z(Q_2)$.

If we take $w'_l \in Z(Q_2)$ such that $\operatorname{Im} z_n - 2R < \operatorname{Im} w'_l < \operatorname{Im} z_n - R$, we also can find $w'_{l'} \in Z(Q_2)$ such that $w'_{l'} = w'_l + iT$. Next, we prove that for any point $\widetilde{\zeta} \in Z(Q_1) \cap \{z : \operatorname{Im} z_n - 3R \leq \operatorname{Im} z < \operatorname{Im} z_n\}$ there is $\widetilde{\zeta}' \in Z(Q_1)$ such that $\widetilde{\zeta}' = \widetilde{\zeta} + iT$. Arguing as above, we show that for all $z \in Z(Q_1)$ such that $\operatorname{Im} z \leq \operatorname{Im} z_n$ we get $z + iT \in Z(Q_1)$ and for all $w \in Z(Q_2)$ such that $\operatorname{Im} w \leq \operatorname{Im} z_n$ we get $w + iT \in Z(Q_2)$.

Next, by (13), take for any $z \in Z(Q_1)$ a point $z'' \in Z(Q_1)$ such that $|z'' + i\tau - z| < \gamma/2$. Then $z'' + iT \in Z(Q_1)$ and $|(z'' + iT) - z| \leq |z'' + i\tau - z| + |i\tau - iT| < \gamma$. Therefore, $z'' + iT = z$ and $z - iT \in Z(Q_1)$ for all $z \in Z(Q_1)$. By the same arguments, $w - iT \in Z(Q_2)$ for all $w \in Z(Q_2)$.

Thus, $z \pm iNT \in Z(Q_1)$ for all $z \in Z(Q_1)$ and $w \pm iNT \in Z(Q_2)$ for all $w \in Z(Q_2)$. Since $Z(Q_1)$, $Z(Q_2)$ are located in a vertical strip of a finite width, we see that T is real. By Proposition 9, we obtain the statement of Theorem 1. □

If the assertion of Proposition 3 is fulfilled for an entire function f, then this function is called *almost periodic*. Entire almost periodic functions satisfy Propositions 1 and 2 as well.

In 1949 M. G. Krein and B. Ja. Levin (see [3], also [4], ch.6, p.2, and Appendix 6) introduced and studied the class Δ of entire almost periodic functions of an exponential growth with zeros in a horizontal strip of a finite width. In particular, all functions $Q(iz)$ for Q of form (3) under conditions (4) belong to Δ. There was proved that $F \in \Delta$ satisfy all the assertions of propositions 5 – 8 (of course, it should be changed horizontal strips to vertical ones and, conversely, vertical strips to horizontal ones). Therefore our criterion of periodicity (with real periods) is valid for functions $F_1, F_2 \in \Delta$ as well. In the case $F_1 = F_2$ this result was obtained in [1].

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