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WIDE OPERATORS ON KÖTHE FUNCTION SPACES

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We study operators defined on Köthe function spaces which are uniformly bounded from below at some sign functions supported on any fixed measurable set. Precise definition is a kind of opposite to the definition of narrow operators, so many questions concerning the relationship between narrow and wide operators naturally arise. The main questions are to describe how "large" has to be a wide operator, and how "small" has to be an operator which is "nowhere" wide. Some easy to formulate problems on wide operators turn out to be more involved than their analogues for narrow operators, and most of the results have restrictive assumptions on the domain spaces. We pose some open problems.

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Мы изучаем операторы, определенные на функциональных пространствах Кете, ограниченные снизу на некоторых функциях-знаках, сосредоточенных на произвольном фиксированном измеримом множестве. Точное определение является, в определенном смысле, противоположным к определению узкого оператора, а поэтому возникают естественные вопросы о взаимоотношениях между узкими и широкими операторами. Основними вопросами являются: насколько "большими" обязаны быть широкие операторы и насколько "малыми" должны быть "нигде не широкие" операторы. Некоторые легко формулируемые вопросы о широких операторах оказываются намного сложнее, чем их аналоги для узких операторов, и большинство результатов доказано при существенных ограничениях на пространства. Сформулированы некоторые открытые вопросы.

1. Introduction.

1.1. About the paper. In this paper we introduce and study a new class of operators defined on Köthe Banach spaces, called wide operators. By definition, these operators are in an opposite position to narrow operators. More precisely, narrow operators take arbitrary small values at sign functions supported on any fixed measurable set, and wide operators are bounded from below at suitable sign functions supported on any fixed measurable set. In some particular cases, wide operators are close to operators that are bounded from below on a Haar type system.

The theory of narrow operators becomes more rich when one considers operators defined on $L_p(\mu)$-spaces for $1 \le p < 2$ and especially on $L_1(\mu)$-spaces, see [5], [6, Section 7]. To the contrast, in this paper we obtain more for operators acting on $L_p(\mu)$-spaces for $2 < p < \infty$.

To deal with a narrow operator is much easier than with a wide operator. Indeed, if we have two disjoint sign functions x and y with $\|Tx\| < \varepsilon$ and $\|Ty\| < \varepsilon$ then $\|Tx + Ty\| < 2\varepsilon$

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as well, and so we can reduce a problem on a narrow operator to smaller parts. The same tool fails for the condition $\|Tx\| \geq \delta\|x\|$. However, for a partial case of operators acting on $L_p(\mu)$-spaces with $2 < p < \infty$ we can apply a similar tool which is given by Lemma 1 below. However in general, problems concerning wide operators are more involved, and in this paper we have more questions than results.

1.2. Terminology and notation. We use the standard terminology and notation for the Banach space theory as in [1], [3], [4]. If X and Y are Banach spaces then by $\mathcal{L}(X,Y)$ we denote the Banach space of all linear bounded operators from X to Y; B_X the closed unit ball of X. If (Ω, Σ, μ) is a measure space and $A \in \Sigma$ then we set

$$\Sigma(A) = \{B \in \Sigma\colon B \subseteq A\};\ \Sigma^+(A) = \{B \in \Sigma(A)\colon \mu(B) > 0\};$$

$\mathbf{1}_A$ the characteristic function of A; $C = A \sqcup B$ means that $C = A \cup B$ and $A \cap B = \varnothing$. For the case of $\Omega = [0,1]$ the same symbols Σ, μ stand for the σ-algebra of all Lebesgue measurable subsets of $[0,1]$ and the Lebesgue measure respectively. For elements $x, y \in L_1(\mu)$ the inequality $x \leq y$ means that $x(\omega) \leq y(\omega)$ holds for almost all $\omega \in \Omega$. We say that a Banach space E which is a linear subspace of $L_1(\mu)$ is a *Köthe Banach space* on (Ω, Σ, μ) if $\mathbf{1}_{[0,1]} \in E$, E is solid and the norm of E is order monotone, that is, for every $x \in L_1$ and $y \in E$ the condition $|x| \leq |y|$ implies that $x \in E$ and $\|x\| \leq \|y\|$. If, moreover, $\|\mathbf{1}_\Omega\| = 1$, and for every $x \in L_1$ and $y \in E$ the condition $d_{|x|} = d_{|y|}$ implies that $x \in E$ and $\|x\| = \|y\|$, then the Köthe Banach space E is said to be a rearrangement invariant space (r.i. space, in short) on (Ω, Σ, μ). Here $d_z(t) = \mu\{\omega \in \Omega\colon z(\omega) > t\}$ is the distribution of z. A Köthe Banach space on a finite atomless measure space (Ω, Σ, μ) is said to have an

- *absolutely continuous norm* if $\lim\limits_{\mu(A)\to 0} \|x \cdot \mathbf{1}_A\| = 0$ for each $x \in E$;
- *absolutely continuous norm on the unit* if $\lim_{\mu(A)\to 0} \|\mathbf{1}_A\| = 0$.

It is clear that if E has an absolutely continuous norm then E has an absolutely continuous norm on the unit, however the converse is not true ([6, Example 1.2]). For a Köthe Banach space E and $A \in \Sigma$ we set

$$E(A) = \{x \in E\colon \operatorname{supp} x \subseteq A\}.$$

We also denote $L_p = L_p[0,1]$.

Let (Ω, Σ, μ) be a finite atomless measure space and $A \in \Sigma$. An element $x \in L_\infty(\mu)$ is called a *sign on* A if $x^2 = \mathbf{1}_A$, that is, $x = \mathbf{1}_B - \mathbf{1}_C$ for a suitable decomposition $A = B \sqcup C$ with $B, C \in \Sigma$. A sign x is said to be of mean zero (or, a mean zero sign) if $\int_\Omega x\, d\mu = 0$, that is, $\mu(B) = \mu(C) = \mu(A)/2$ for the above decomposition. Let E be a Köthe Banach space on (Ω, Σ, μ) and X a Banach space. An operator $T \in \mathcal{L}(E, X)$ is called *narrow* if for each $A \in \Sigma$ and $\varepsilon > 0$ there is a mean zero sign x on A such that $\|Tx\| < \varepsilon$. An operator $T \in \mathcal{L}(E, X)$ is called a *sign-embedding* if there is $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for each sign x.

1.3. Definition, examples and questions.

Definition 1. Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ), X a Banach space. We say that an operator $T \in \mathcal{L}(E, X)$ is *wide* if there is $\delta > 0$ such that for every $A \in \Sigma$ there is a mean zero sign x on A with $\|Tx\| \geq \delta\|x\|$. In this case we say that *T is wide with the constant δ*. An operator $T \in \mathcal{L}(E, X)$ is called *somewhere wide* if there is $A \in \Sigma^+$ such that the restriction $T|_{E(A)}$ is wide; otherwise T is said to be *nowhere wide*. An operator $T \in \mathcal{L}(E, X)$ is called *hereditarily wide* if for every $A \in \Sigma^+$ and every atomless subσ-algebra Σ_1 of $\Sigma(A)$ the restriction $T|_{E(\Sigma_1)}$ is wide.

By definition, a hereditarily wide operator is wide, and a wide operator is somewhere wide. Obviously, a sign embedding is a wide operator. However, no other implication is obvious. The following natural questions are the main subjects of investigation in the present paper.

(i) *What "small" operators are nowhere wide? How "small" has to be a nowhere wide operator?*

(ii) *What "large" operators are wide? How "large" has to be a wide operator?*

(iii) *What is the connection between narrow and wide operators? In particular, is there an operator which is both narrow and wide? Is there a wide hereditarily narrow operator? A narrow hereditarily wide operator? A hereditarily narrow hereditarily wide operator?*

(iv) *Is there an operator which is both nonnarrow and nonwide?*

Question (iv) has an obvious negative answer. Indeed, if $\Omega = A \sqcup B$ with $A, B \in \Sigma^+$ then the operator $P_A : E \to E$ defined by $P_A x = x \cdot \mathbf{1}_A$, $x \in E$, which is well defined and has norm one in any Köthe Banach space E on any finite atomless measure space (Ω, Σ, μ). Obviously, P_A is both nonnarrow and nonwide. So, we modify question (iv) as follows.

(iv') *Is there an operator which is both nonnarrow and nowhere wide?*

A similar example shows that a sum of two nonwide operators could be wide: $I = P_A + P_B$.

(v) *Is a sum of two nowhere wide operators nowhere wide?*

2. Operators that are bounded from below on a Haar type system. Let (Ω, Σ, μ) be a finite atomless measure space and $G \in \Sigma^+$. A collection $(G_{m,k})_{m=0,k=1}^{\infty\ \ 2^m}$ of sets $G_{m,k} \in \Sigma$ is called a *tree of sets* (or, more precisely, a *tree of sets on the set* G) if $G_{0,1} = G$ and

$$G_{m,k} = G_{m+1,2k-1} \sqcup G_{m+1,2k} \text{ with } \mu(G_{m+1,2k-1}) = \mu(G_{m+1,2k}) = \frac{1}{2}\,\mu(G_{m,k})$$

for $m \in \{0, 1, \dots\}$ and $k \in \{1, \dots, 2^m\}$. The corresponding system of functions $(g_i)_{i=1}^\infty$ defined by $g_1 = \mathbf{1}_{G_{0,1}}$ and $g_{2^m+k} = \mathbf{1}_{G_{m+1,2k-1}} - \mathbf{1}_{G_{m+1,2k}}$ for $m \in \{0, 1, \dots\}$ and $k \in \{1, \dots, 2^m\}$ is called a *Haar type system* (or, more precisely, a *Haar type system supported on the set* G).

A very important partial case appears if we consider the *dyadic tree of sets* I_m^k, $m \in \{0, 1, \dots\}$, $k \in \{1, \dots, 2^m\}$ on $[0, 1]$, that is, $I_m^k = \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)$. The corresponding Haar type system is called the *Haar system* on $[0, 1]$.

Proposition 1. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ), X a Banach space. If an operator $T \in \mathcal{L}(E, X)$ is wide then there exist a Haar type system $(g_n)_{n=1}^\infty$ supported on Ω and $\delta > 0$ such that $\|Tg_n\| \ge \delta \|g_n\|$ for $n \in \{2, 3, \dots\}$.*

Remark that, since a wide operator need not send $\mathbf{1}_\Omega$ to a nonzero element of X, the condition $\|Tg_n\| \ge \delta \|g_n\|$ is claimed only for $n \in \{2, 3, \dots\}$ (we have that $g_1 = \mathbf{1}_\Omega$ for any Haar type system $(g_n)_{n=1}^\infty$ supported on Ω). Indeed, the operator $T \in \mathcal{L}(E)$ defined by $Tx = x - \left(\int_\Omega x\, d\mu\right) \cdot \mathbf{1}_\Omega$ is wide, because $Tx = x$ for any mean zero sign, and sends $\mathbf{1}_\Omega$ to zero.

Proof of Proposition 1. Let $T \in \mathcal{L}(E, X)$ be wide with a constant δ. Surely, we set $g_1 = \mathbf{1}_\Omega$ and $G_{0,1} = G = \Omega$. Choose a mean zero sign g_2 on $[0, 1]$ so that $\|Tg_2\| \ge \delta$. Then for $G_{1,1} = \{t : g_2(t) = 1\}$ and $G_{1,2} = \{t : g_2(t) = -1\}$ choose mean zero signs g_3 on $G_{1,1}$ and g_4 on $G_{1,2}$ so that $\|Tg_i\| \ge \delta \|g_i\|$ for $i \in \{3, 4\}$. Processing like that, we construct the desired Haar type system. □

In a partial case, a kind of converse statement holds.

Theorem 1. *Let (Ω, Σ, μ) be a finite measure space, $2 \le p < \infty$ and $T \in \mathcal{L}\big(L_p, L_p(\mu)\big)$ be an operator for which there are $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\|Th_n\| \ge \delta \|h_n\|$ for all $n \ge n_0$, where (h_n) is the Haar system on $[0,1]$. Then T is wide.*

We need two lemmas, first of which is known.

Lemma 1 (Lemma 7.63, [6])**.** *Let (Ω, Σ, μ) be a finite atomless measure space and $2 \le p < \infty$. Then for each $n \in \mathbb{N}$ and each vectors $(z_k)_{k=1}^n$ in $L_p(\mu)$ there is a collection of sign numbers $(\theta_k)_{k=1}^n$, $\theta_k \in \{-1, 1\}$ such that*

$$\Big\|\sum_{k=1}^n \theta_k z_k\Big\| \ge \Big(\sum_{k=1}^n \|z_k\|^p\Big)^{1/p}.$$

Lemma 2. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ) with an absolutely continuous norm on the unit. Let X be a Banach space and $T \in \mathcal{L}(E, X)$. Assume that there is $\delta > 0$ such that for every $A \in \Sigma$ and every $\varepsilon > 0$ there is a sign x such that $|\int_\Omega x\, d\mu| < \varepsilon$ and $\|Tx\| \ge \delta \|x\|$. Then T is wide.*

Proof of Lemma 2. Fix any $A \in \Sigma^+$. Using the absolute continuity of the norm on the unit, choose $\eta > 0$ such that for any $B \in \Sigma$ the inequality $\mu(B) < \eta$ implies $\|\mathbf{1}_B\| < \frac{\delta\|\mathbf{1}_A\|}{4\|T\|}$. Then choose a sign $\overline{x}$ on A such that $|\int_\Omega \overline{x}\, d\mu| < 2\eta$ and $\|T\overline{x}\| \ge \delta\|\overline{x}\|$. With no loss of generality we may and do assume that $\int_\Omega \overline{x}\, d\mu \ge 0$, otherwise we consider $-\overline{x}$ instead of $\overline{x}$. Set $A^+ = \{t \colon \overline{x}(t) = 1\}$ and $A^- = \{t \colon \overline{x}(t) = -1\}$. By the above, $A = A^+ \sqcup A^-$ and $\mu(A^+) \ge \mu(A^-)$. Using the atomlessness of μ, we choose $A_0 \in \Sigma(A^+)$ so that

$$\mu(A_0) = \frac{1}{2}\Big(\mu(A^+) - \mu(A^-)\Big) = \frac{1}{2}\int_\Omega \overline{x}\, d\mu < \eta. \tag{1}$$

Then we set $x = \overline{x} - 2\mathbf{1}_{A_0}$. Observe that x is a sign on A, so, $\|x\| = \|\overline{x}\| = \|\mathbf{1}_A\|$. Moreover, $\int_\Omega x\, d\mu = 0$, that is, x is of mean zero. By (1) and the choice of η,

$$\|x - \overline{x}\| = 2\|\mathbf{1}_{A_0}\| < \frac{\delta\|\mathbf{1}_A\|}{2\|T\|} = \frac{\delta\|x\|}{2\|T\|}.$$

Thus we obtain

$$\|Tx\| \ge \|T\overline{x}\| - \|T\|\|x - \overline{x}\| \ge \delta\|x\| - \|T\|\frac{\delta\|x\|}{2\|T\|} = \frac{\delta}{2}\|x\|.$$

□

Proof of Theorem 1. We show that for every $A \in \Sigma^+$ there exists a sign x on A such that $\|Tx\| \ge \frac{\delta}{2}\|x\|$. Fix any $A \in \Sigma^+$ and $\varepsilon > 0$. Choose $m \in \mathbb{N}$ and a subset $J \subseteq \{1, \ldots, 2^m\}$ so that

$$\mu(A \triangle B)^{1/p} < \min\Big\{\frac{\delta}{2(\delta + \|T\|)}, \varepsilon\Big\}, \tag{2}$$

where $B = \bigcup_{j \in J} I_m^j$. Using the obvious property of the Haar system, without loss of generality, we assume that $2^m \ge n_0$. By Lemma 1 choose sign numbers $\theta_j \in \{-1, 1\}$ so that

$$\Big\|\sum_{j \in J} \theta_j T h_{2^m + j}\Big\| \ge \Big(\sum_{j \in J} \|Th_{2^m + j}\|^p\Big)^{1/p}.$$

Then, by the assumptions that $2^m \geq n_0$ and $\|Th_n\| \geq \delta \|h_n\|$ for $n \geq n_0$ for

$$\overline{x} = \sum_{j \in J} \theta_j h_{2^m + j}$$

we obtain

$$\|T\overline{x}\| = \Big\| \sum_{j \in J} \theta_j T h_{2^m+j} \Big\| \geq \delta \Big(\sum_{j \in J} 2^{-m} \Big)^{1/p} = 2^{-m/p} \cdot \delta |J|^{1/p} = \delta \mu(B)^{1/p} = \delta \|\overline{x}\|.$$

Since $\operatorname{supp} h_{2^m+j} = I_m^j$ and $|\overline{x}(t)| = 1$ for $t \in I_m^j$, we have that $\overline{x}$ is a sign on B. Now we set $x = \overline{x} \cdot \mathbf{1}_A + \mathbf{1}_{A \setminus B}$ and observe that $x^2 = \mathbf{1}_A$ and $\|x - \overline{x}\|^p \leq \mu(A \triangle B)$. Then

$$\|Tx\| \geq \|T\overline{x}\| - \|T\| \|x - \overline{x}\| \overset{\text{by (3)}}{=} \delta \|\overline{x}\| - \|T\| \|x - \overline{x}\| \geq$$
$$\geq \delta \|x\| - \delta \|x - \overline{x}\| - \|T\| \|x - \overline{x}\| \geq \|x\| \Big(\delta - \mu(A \triangle B)^{1/p} \big(\delta + \|T\| \big) \Big) \overset{\text{by (2)}}{\geq} \frac{\delta}{2} \|x\|.$$

It remains to observe that

$$\Big| \int_\Omega x \, d\mu \Big| \leq \Big| \int_\Omega \overline{x} \, d\mu \Big| + \Big| \int_\Omega (x - \overline{x}) \, d\mu \Big| \leq 0 + \mu(A \triangle B) \overset{\text{by (2)}}{<} \varepsilon.$$

By Lemma 2, T is wide. □

We do not know if Theorem 1 is true for $1 \leq p < 2$.

Problem 1. *Let $1 \leq p < 2$ and $T \in \mathcal{L}(L_p)$. Assume that $\|Th_n\| \geq \delta \|h_n\|$ for some $\delta > 0$ and all n. Is then T wide?*

3. Operators that are narrow and wide. We show that such operators exist on r.i. spaces on $[0, 1]$ with an unconditional basis.

Definition 2. Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ). We say that an operator $T \in \mathcal{L}(E)$ is *double-sided narrow* if both operators T and $Id - T$ are narrow, where Id is the identity operator on E.

In other words, if Id is represented as a sum of two narrow operators $Id = S + T$ then both S and T are defined to be double-sided narrow. The existence of double-sided narrow projections is proved for any r.i. Banach space E on $[0, 1]$ with an unconditional basis ([6, Theorem 5.2]).

Proposition 2. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ). Then an operator $T \in \mathcal{L}(E)$ is wide whenever $Id - T$ is narrow. In particular, every double-sided narrow operator $T \in \mathcal{L}(E)$ is wide.*

Proof. Let $T \in \mathcal{L}(E)$ be any operator with $Id - T$ narrow. We show that T is wide with any fixed number $\delta \in (0, 1)$. Given any $A \in \Sigma$, we choose a mean zero sign x on A so that

$$\|x - Tx\| = \|(Id - T)\, x\| < (1 - \delta) \|x\|.$$

Then

$$\|Tx\| \geq \|x\| - \|x - Tx\| > \|x\| \big(1 - (1 - \delta) \big) = \delta \|x\|.$$

□

As a consequence of Proposition 2 and [6, Theorem 5.2] we obtain the following assertion.

Corollary 1. *Let E be an r.i. space on $[0,1]$ with an unconditional basis. Then there is a projection on E which is both narrow and wide.*

Now consider the case $E = L_1$. Since the sum of two narrow operators on L_1 is narrow ([6, Theorem 7.46]) and the identity operator on L_1 is not narrow, there is no double-sided narrow operator $T \in \mathcal{L}(L_1)$. So, the above method of constructing operators that are both narrow and wide fails for $E = L_1$. In return, the following tool works only for $E = L_1$.

Proposition 3. *Consider the integration operator with respect to the second variable on $L_1[0,1]^2$ defined by*

$$(Px)(s,t) = \int_{[0,1]} x(s,t')\,dt' \tag{3}$$

for each $x \in L_1[0,1]^2$. Then P is both narrow and wide.

Observe that P is the conditional expectation operator with respect to the sub-σ-algebra $\Sigma \times \{[0,1]\}$ of the Lebesgue σ-algebra on $[0,1]^2$.

Proof. By [6, Theorem 4.10], P is narrow. Moreover, P is strictly narrow, that is, for every measurable subset $A \subseteq [0,1]^2$ there is a mean zero sign x on A such that $Px = 0$. Now we show that P is wide with constant 1. Given any measurable subset $A \subseteq [0,1]^2$, choose a number $\tau \in (0,1)$ so that $\mu\{(x,y) \in A \colon x < \tau\} = \mu(A)/2$ (this is possible because the function $f : [0,1] \to [0,\mu(A)]$ given by $f(t) = \mu\{(x,y) \in A \colon x < t\}$ is continuous, $f(0) = 0$ and $f(1) = \mu(A)$). Then set $B = \{(x,y) \in A \colon x < \tau\}$, $C = A \setminus B$, $x = \mathbf{1}_B - \mathbf{1}_C$. Then

$$x^2 = \mathbf{1}_A, \quad \iint_{[0,1]^2} x\,d\mu = 0 \quad \text{and} \quad \big|(Px)(s,t)\big| = \int_{[0,1]} \mathbf{1}_A(s,t')\,dt'$$

and hence, by the Fubini Theorem,

$$\|Px\| = \int_{[0,1]} ds \int_{[0,1]} dt \int_{[0,1]} \mathbf{1}_A(s,t')\,dt' = \int_{[0,1]} ds \int_{[0,1]} \mathbf{1}_A(s,t')\,dt' = \mu(A) = \|x\|.$$

□

It is interesting to note that for $p > 1$ this is not the case. More precisely, the operator defined by (3) is well defined in $L_p[0,1]^2$ for $1 < p \le \infty$ and is strictly narrow (this immediately follows from the continuity of the inclusion embedding $L_p \subseteq L_1$ and the strict narrowness of P in L_1; see [6, Chapter 4.2]), however P is not wide.

Proposition 4. *The integration operator with respect to the second variable on $L_p[0,1]^2$ defined by (3) is nowhere wide if $1 < p \le \infty$.*

Proof. Let $A \subseteq [0,1]^2$ be any measurable subset of positive measure. Given any $\varepsilon > 0$, we choose $n \in \mathbb{N}$ so that

$$\mu(A)^{1/p}\, n^{1-\frac{1}{p}} > \frac{1}{\varepsilon} \ \text{ if } p < \infty \quad \text{and} \quad \frac{1}{n} < \varepsilon \ \text{ if } p = \infty. \tag{4}$$

Then choose $i \in \{1, \dots, n\}$ so that $\mu(A_i) \ge \mu(A)/n$, where $A_i = A \cap \big([0,1] \times \big[\frac{i-1}{n}, \frac{i}{n}\big)\big)$ (such a number i exists because otherwise $\mu(A) = \sum_{k=1}^n \mu(A_k) < \mu(A)$, a contradiction). Let x be any sign on A_i. Then, on the one hand,

$$\|x\| = \mu(A_i)^{1/p} \ge \frac{\mu(A)^{1/p}}{n^{1/p}} \text{ if } p < \infty \quad \text{and} \quad \|x\| = 1 \text{ if } p = \infty.$$

On the other hand,

$$\|Px\|^p = \int_{[0,1]} ds \int_{[0,1]} dt \Big(\int_{[0,1]} \mathbf{1}_{A_i}(s,t')\, dt'\Big)^p \le$$
$$\le \int_{[0,1]} ds \int_{[0,1]} dt \Big(\int_{[0,1]} \mathbf{1}_{\left[\frac{i-1}{n}, \frac{i}{n}\right)}(s,t')\, dt'\Big)^p = \int_{[0,1]} ds \int_{[0,1]} dt \, \frac{1}{n^p} = \frac{1}{n^p}$$

if $p < \infty$ and

$$\|Px\|_\infty = \sup_s \int_{[0,1]} \mathbf{1}_{A_i}(s,t')\, dt' \le \sup_s \int_{[0,1]} \mathbf{1}_{\left[\frac{i-1}{n}, \frac{i}{n}\right)}(s,t')\, dt' = \frac{1}{n}\,.$$

Thus, in view of (4), we obtain that

$$\frac{\|Px\|}{\|x\|} \le \frac{\frac{1}{n}}{\frac{\mu(A)^{1/p}}{n^{1/p}}} = \frac{1}{\mu(A)^{1/p}\, n^{1-\frac{1}{p}}} < \varepsilon$$

if $p < \infty$ and

$$\frac{\|Px\|_\infty}{\|x\|_\infty} \le \frac{1}{n} < \varepsilon.$$

□

We do not have such examples for Köthe Banach spaces on $[0,1]$ that are not r.i.

Problem 2. *Does in every Köthe Banach space E on $[0,1]$ there exist an operator $T \in \mathcal{L}(E)$ which is narrow and wide?*

For $E = L_p$ with $p > 2$ we obtain much more.

Proposition 5. *Let $2 < p < \infty$. Then there is an operator $T \in \mathcal{L}(L_p)$ which is hereditarily narrow and wide.*

Proof. Since L_p contains a subspace isometrically isomorphic to ℓ_p, it is enough to construct a hereditarily narrow wide operator $T \in \mathcal{L}(L_p, \ell_p)$. Let $(\overline{h}_n)_{n=1}^\infty$ be the normalized Haar system in L_p, that is, $\overline{h}_1 = \mathbf{1}_{[0,1]}$, $\overline{h}_{2^m+k} = 2^{m/p} h_{2^m+k}$ for $m \in \{0,1,2,\dots\}$ and $k \in \{1,2,\dots,2^m\}$. Denote by $(e_n)_{n=1}^\infty$ the unit vector basis of ℓ_p. By the Orlicz theorem [2, p. 101], there exists an operator $T \in \mathcal{L}(L_p, \ell_p)$ such that $T\overline{h}_n = e_n$ for all $n \in \mathbb{N}$. Thus, we have that $\|Th_{2^m+k}\| = 2^{-m/p} = \|h_{2^m+k}\|$. By Theorem 1, T is wide. By [6, Corollary 11.4], every linear bounded operator from L_p to ℓ_p is hereditarily narrow, and so is T. □

Problem 3. *Let $1 \le p \le 2$. Does there exist an operator $T \in \mathcal{L}(L_p)$ which is hereditarily narrow and wide?*

4. Nowhere wide operators. The results of this section assert that nowhere wide operators are "small". Recall that an operator $T \in (X,Y)$ between Banach spaces X and Y is called:

- *compact* if TB_X is a relatively compact subset of Y;
- a *Dunford-Pettis operator* if T sends weakly null sequences to norm null sequences.

It is not very hard to show that every compact operator is a Dunford-Pettis operator. For some pairs of Banach spaces (X, Y) the converse is also true. However, for example, for $X = Y = L_1$ the converse is not true.

Theorem 2. *Let (Ω, Σ, μ) be a finite atomless measure space, $(\Omega_1, \Sigma_1, \nu)$ a finite measure space, $2 \le p < \infty$, X a Banach space. Then every Dunford-Pettis operator $T \in \mathcal{L}\big(L_p(\mu), L_p(\nu)\big)$ is nowhere wide.*

Proof. Let $T \in \mathcal{L}\big(L_p(\mu), L_p(\nu)\big)$ be a Dunford-Pettis operator. Assume, on the contrary, that $T|_{L_p(A)}$ is wide with a constant $\delta > 0$ for some $A \in \Sigma^+$. Our goal is to construct a sequence $(r_n)_{n=1}^\infty$ of probabilistically independent mean zero signs on A with $\|Tr_n\| \ge \delta \mu(A)^{1/p}$, which is impossible because $r_n \xrightarrow{\mathrm{w}} 0$ in $L_p(\mu)$. We start with picking any mean zero sign r_1 on A with $\|Tr_1\| \ge \delta\mu(A)^{1/p}$. Then set $A_2 = \{t \colon r_1(t) = 1\}$ and $A_3 = \{t \colon r_1(t) = -1\}$. Since r_1 is of mean zero, $\mu(A_2) = \mu(A_3) = \mu(A)/2$. Choose mean zero signs $r_{2,1}$ on A_2 and $r_{2,2}$ on A_3 so that $\|Tr_{2,1}\| \ge \delta(\mu(A)/2)^{1/p}$ and $\|Tr_{2,2}\| \ge \delta(\mu(A)/2)^{1/p}$. Then by Lemma 1 we choose sign numbers $\theta_{2,1}, \theta_{2,2} \in \{-1, 1\}$ so that for $r_2 = \theta_{2,1} r_{2,1} + \theta_{2,2} r_{2,2}$ one has

$$\|Tr_2\| \ge \Big(\|Tr_{2,1}\|^p + \|Tr_{2,2}\|^p\Big)^{1/p} \ge \delta\Big(\frac{\mu(A)}{2} + \frac{\mu(A)}{2}\Big)^{1/p} = \delta\mu(A)^{1/p}.$$

On the third step, we define A_4, A_5, A_6, A_7 by

$$A_4 = \{t \colon r_{2,1} = 1\},\ A_5 = \{t \colon r_{2,1} = -1\},\ A_6 = \{t \colon r_{2,2} = 1\},\ A_7 = \{t \colon r_{2,2} = -1\},$$

and choose mean zero signs $r_{3,i}$ on A_{3+i}, $i \in \{1, 2, 3, 4\}$ with $\|Tr_{3,i}\| \ge \delta(\mu(A)/4)^{1/p}$. Using Lemma 1, we choose sign numbers $\theta_{3,i}$ so that for $r_3 = \sum_{i=1}^4 \theta_{3,i} r_{3,i}$ one has

$$\|Tr_3\| \ge \Big(\sum_{i=1}^4 \|Tr_{3,i}\|^p\Big)^{1/p} \ge \delta\Big(\sum_{i=1}^4 \frac{\mu(A)}{4}\Big)^{1/p} = \delta\mu(A)^{1/p}.$$

Processing like above, we construct the desired sequence. □

Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ), and let X be a Banach space. Following [6, Definition 7.58], an operator $T \in \mathcal{L}(E, X)$ we call *somewhat narrow* if for each $A \in \Sigma$ and each $\varepsilon > 0$ there exists a set $B \in \Sigma(A)$ and a sign x on B such that $\|Tx\| < \varepsilon\|x\|$. The next proposition immediately follows from the definitions.

Proposition 6. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ), and let X be a Banach space. Then every nowhere wide operator $T \in \mathcal{L}(E, X)$ is somewhat narrow.*

By [6, Theorem 7.59], if $1 \le p \le 2$ then every somewhat narrow operator $T \in \mathcal{L}(L_p)$ is narrow. So, as a consequence, we obtain the following assertion.

Corollary 2. *Let $1 \le p \le 2$. Then every nowhere wide operator $T \in \mathcal{L}(L_p)$ is narrow.*

Problem 4. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ) with an absolutely continuous norm on the unit, and let X be a Banach space. Is every Dunford-Pettis operator $T \in \mathcal{L}(E, X)$ nowhere wide? What about $E = L_p(\mu)$ for $1 \le p \le 2$?*

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