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EXTENSIONS OF ORTHOGONALLY ADDITIVE OPERATORS

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We find natural sufficient conditions on a subset D of a vector lattice E under which every orthogonally additive operator $T_0\colon D \to X$, where X is a vector space, can be extended to an orthogonally additive operator $T\colon E \to X$. Two theorems on the extension from lateral ideals and lateral bands, respectively, are obtained.

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Устанавливаются естественные достаточные условия на подмножество D векторной решетки E, при которых каждый ортогонально аддитивный оператор $T_0\colon D \to X$, где X — линейное пространство, продолжается до ортогонально аддитивного оператора $T\colon E \to X$. Доказаны две теоремы о продолжении с латерального идеала и латеральной полосы, соответственно.

1. Introduction. Orthogonally additive operators (OAOs, in short) natural generalize linear operators acting from some general structure to a linear space (see, for instance, [12]). An essential contribution to OAOs on vector lattices were made by Mazón and Segura de León in [6] and [7]. OAOs defined on lattice normed spaces was studied in [4], [5], [10] and on spaces with a mixed norm in [9]. Some known results on narrow linear operators were generalized to OAOs in [11], and new results for both linear and OAOs were obtained in [8]. A new approach to the study of OAOs was recently proposed in [2] based on the lateral order on vector lattices.

The present paper is devoted to a study of extensions of OAOs from reasonable subsets to entire vector lattice.

1.1. Preliminaries on orthogonally additive operators. We use the standard terminology and notation on vector lattices as in [1]. Let E be a vector lattice (which is assumed to be Archimedean). For elements x, y the notation $x \sqsubseteq y$ means that x is a *fragment* of y, that is, $x \perp (y - x)$. The relation $\sqsubseteq$ is a partial order on E. By $\mathfrak{F}_x$ we denote the initial closed segment with respect to this partial order by an element $x \in E$, that is, the set of all fragments of x.

Definition 1. Let E be a vector lattice and X a vector space. A function $T\colon E \to X$ is called an *OAO* if for every $x, y \in E$ the relation $x \perp y$ implies $T(x + y) = T(x) + T(y)$.

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Observe that *the composition of OAOs need not be an OAO*. More precisely, if $T\colon E \to F$ is an OAO between vector lattices and $S\colon F \to X$ is an OAO, where X is a vector space then the composition $S \circ T\colon E \to X$ may fail to be an OAO, even for a linear T. Indeed, consider the linear operator $T\colon L_1[0,1] \to L_1[0,1]$ defined by

$$T(x) = \Big(\int_{[0,1]} x \, d\mu \Big) \cdot \mathbf{1}_{[0,1]}$$

for all $x \in L_1[0,1]$, where $\mathbf{1}_A$ denotes the characteristic function of a set $A \subseteq [0,1]$. Next consider the OAO $S\colon L_1[0,1] \to L_1[0,1]$ defined by $S(x) = \mu(\operatorname{supp} x) \cdot \mathbf{1}_{[0,1]}$ for all $x \in L_1[0,1]$. Then the composition $S \circ T\colon L_1[0,1] \to L_1[0,1]$ is not an OAO, because $\mathbf{1}_{[0,\frac{1}{2})} \perp \mathbf{1}_{[\frac{1}{2},1]}$ and

$$S \circ T(\mathbf{1}_{[0,\frac{1}{2})} + \mathbf{1}_{[\frac{1}{2},1]}) = S \circ T(\mathbf{1}_{[0,1]}) = \mathbf{1}_{[0,1]},$$

however

$$S \circ T(\mathbf{1}_{[0,\frac{1}{2})}) + S \circ T(\mathbf{1}_{[\frac{1}{2},1]}) = \mathbf{1}_{[0,1]} + \mathbf{1}_{[0,1]} = 2 \cdot \mathbf{1}_{[0,1]}.$$

If, in addition, an OAO $\boldsymbol{T}$ preserves disjointness then obviously $\boldsymbol{S} \circ \boldsymbol{T}$ is an OAO for every OAO $\boldsymbol{S}$ such that the composition is well defined (a function $T\colon E \to F$ between vector lattices E, F is said to be disjointness preserving if for all $x, y \in E$ with $x \perp y$ one has that $T(x) \perp T(y)$.

An OAO $T\colon E \to F$ between vector lattices E, F is said to be *positive* provided $T(x) \geq 0$ for all $x \in E$. Remark that the notion of positivity for OAOs is completely different from the notion of positivity for linear operators (it would be much more wise if linear positive operators were called monotone operators). Indeed, the only linear operator which is positive in the sense of positivity for OAOs is zero. To avoid misunderstandings, we write $T\colon E \to F^+$ to introduce a positive OAO $T\colon E \to F$.

Let E be a vector lattice. Following [2], the partial order $\sqsubseteq$ on E is called the *lateral order* on E. A subset $G \subseteq E$ is said to be *laterally bounded* in E if $G \subseteq \mathfrak{F}_x$ for some $x \in E$. We do not mention here “from above” because every subset is automatically laterally bounded from below by zero. The lateral supremum and infimum are defined as usual in a partially ordered set, using the order $\sqsubseteq$ on E. Given a nonempty subset $G \subseteq E$, the $\sqsubseteq$-supremum and the $\sqsubseteq$-infimum of G in E we denote by $\boldsymbol{\bigcup} G$ and $\boldsymbol{\bigcap} G$ respectively. We also use the notation $x \boldsymbol{\cup} y$ instead of $\boldsymbol{\bigcup}\{x, y\}$, $\boldsymbol{\bigcup}_{k=1}^m x_k$ instead of $\boldsymbol{\bigcup}\{x_1, \ldots, x_m\}$, and similarly for infimums using the symbols $\boldsymbol{\cap}$ and $\boldsymbol{\bigcap}$. This notation is natural due to the following reasons. Firstly, if we deal with a function lattice then the graph of the lateral supremum equals the union of the graphs, in their nonzero parts. Secondly, given an element e of a vector lattice E, the set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with respect to the lateral order ([1, Theorem 3.15]). Then by Stone's representation theorem ([3, Theorem 7.11]), $\mathfrak{F}_e$ is Boolean isomorphic to an algebra of subsets of some set. Such a Boolean isomorphism sends the lateral supremum to the union, and the lateral infimum to the intersection. To distinguish the lateral supremum from the set-theoretical union, the reader just has to check whether the symbol concerns elements or sets of elements of a vector lattice. Another thing which differs the lateral supremum from the set-theoretical union is the bold style $\boldsymbol{\bigcup}$ for lateral suprema against the usual style $\bigcup$ for unions, and the same with $\boldsymbol{\bigcap}$ and $\bigcap$.

For a laterally bounded subset G of E the lateral supremum and infimum are reduced to the usual ones as follows. Set $G^+ = \{f^+ \colon f \in G\}$ and $G^- = \{f^- \colon f \in G\}$. Then $\boldsymbol{\bigcup} G = \sup G^+ - \sup G^-$ and $\boldsymbol{\bigcap} G = \inf G^+ - \inf G^-$ with simultaneous existence of the left and right hand sides of the equalities (in particular, they all exist if E is Dedeking complete), see [2]. Although any two-point subset $\{x, y\} \subseteq E$ is laterally bounded from below by zero, it need not have the lateral infimum $x \Cap y$. However, if E has the principal projection property, then $x \Cap y$ exists for any $x, y \in E$. If for given $x, y \in E$ the lateral infimum $x \Cap y$ exists, we define the difference $x \setminus y$ by setting $x \setminus y = x - (x \Cap y)$. In particular, the set-theoretical difference $x \setminus y$ is well defined for every elements x, y of a vector lattice with the intersection property (e.g., of a vector lattice with the principal projection property).

It is interesting to note that, for any vector lattices E, F an OAO $T\colon E \to F$ is laterally monotone (that is, for all $x, y \in E$ the lateral inequality $x \sqsubseteq y$ implies $T(x) \sqsubseteq T(y)$) if and only if T preserves disjointness.

Definition 2. A nonempty subset G of a vector lattice E is called:

- *finitely laterally closed* if for every laterally bounded two-point subset $\{x, y\}$ of E the condition $\{x, y\} \subseteq G$ implies $x \Cup y \in G$;
- a *lateral field* if it is finitely laterally closed, and for every two-point subset $\{x, y\}$ of G the existence of $x \Cap y$ implies $x \Cap y \in G$ and $x \setminus y \in G$;
- *laterally solid* if for each $x \in P$ and $y \in G$ the relation $x \sqsubseteq y$ implies $x \in G$;
- a *lateral ideal* if it is laterally solid and finitely laterally closed;
- *laterally closed* if for each subset $G_1 \subseteq G$ the existence of $f = \bigcup G_1$ in E implies that $f \in G$;
- a *lateral band* if it is laterally solid and laterally closed.

It is immediate that a lateral band is a lateral ideal, and a lateral ideal is a lateral field, but the converse assertions are not true. It is a technical exercise to check that the kernel of a positive OAO is a lateral ideal (see [2] for the details).

Following [6], a net (x_α) in a vector lattice *laterally converges* to an element $x \in E$ if $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ for all indices $\alpha \le \beta$ and (x_α) order converges to x. In this case we write $x_\alpha \xrightarrow{\text{lat}} x$. Obviously, $x_\alpha \xrightarrow{\text{lat}} x$ holds if and only if (x_α) is laterally increasing and $\bigcup_\alpha x_\alpha = x$. A function $f\colon E \to F$ between vector lattices is called *laterally continuous at a point* $x \in E$ if for every net (x_α) in E with $x_\alpha \xrightarrow{\text{lat}} x$ in E one has $f(x_\alpha) \xrightarrow{\text{lat}} f(x)$ in F. f is said to be *laterally continuous* provided f is laterally continuous at every point $x \in E$.

1.2. From what sets to extend? One can consider an OAO defined on an arbitrary subset of a vector lattice. More precisely, let E be a vector lattice, X a vector space and $D \subseteq E$. A function $T\colon D \to X$ is called an *OAO* if for any $x, y \in D$ with $x \perp y$ and $x + y \in D$ one has $T(x + y) = T(x) + T(y)$.

Observe that the very general extension problem of whether every OAO defined on an arbitrary subset D of a vector lattice E has an extension to an OAO on E, has a negative answer, even for "good lattices" E. Indeed, let E denote the vector lattice $\mathbb{R}^5$ with the usual coordinate-wise order. Denote by $e_1, \ldots, e_5$ be the unit vector basis of E. Let D consists of all sums $\sum_{k \in A} e_k$ over three-point subsets $A \subset \{1, \ldots, 5\}$. Since D consists of $C_5^3 = 10$ elements, the set F of all functions $T\colon D \to \mathbb{R}$ is a 10-dimensional vector space with respect to the coordinate-wise operations. Since D contains no orthogonal elements, every element of F is an OAO. On the other hand, the set of all OAOs defined on the set $\mathfrak{F}_e$ of all fragments of $e = e_1 + \ldots + e_5$ is a 5-dimensional vector space. Hence, not every OAO defined on D

can be extended to an OAO on E. Clearly, such an example exists for every vector lattice E with $\dim E \geq 5$.

Next is the main question which motivates our investigation.

Problem 1. *Let E be a vector lattice and X a vector space. For what subsets D of E every OAO $T_0\colon D \to X$ can be extended to an OAO $T\colon E \to X$?*

Our results here concern extensions from lateral fields, lateral ideals and lateral bands.

2. Extensions from lateral ideals. Recall that an order bounded OAO $T\colon E \to F$ between vector lattices E and F is called an *abstract Uryson operator*. The set of all abstract Uryson operators from E to F is denoted by $\mathcal{U}(E, F)$. Remark that $\mathcal{U}(E, F)$ is an ordered vector space with respect to the point-wise order. Moreover, if F is Dedekind complete then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice ([6]). The following theorem concerns extensions of positive abstract Uryson operators.

Theorem 1. *Let E, F be vector lattices with F Dedekind complete, D a lateral ideal in E and $T_0\colon D \to F^+$ an OAO so that $T_0(D)$ is an order bounded set in F. Then there exists $T \in \mathcal{U}^+(E, F)$ which extends T_0.*

Proof. Define a map $T\colon E \to F$ by setting $T(x) = \sup T_0(D \cap \mathfrak{F}_x)$ for every $x \in E$. We show that T is a positive abstract Uryson operator from E to F. Fix any elements $x, y \in E$ with $x \perp y$. Then for each $f \in D \cap \mathfrak{F}_{x+y}$, by the Riesz decomposition property, there exist $f_1, f_2 \in E$ such that $f_1 + f_2 = f$, $f_1 \sqsubseteq x$ and $f_2 \sqsubseteq y$. Since D is a lateral ideal, $f_1, f_2 \in D$. Thus,

$$T_0(f) = T_0(f_1 + f_2) = T_0(f_1) + T_0(f_2) \leq T(x) + T(y).$$

By the arbitrariness of $f \in D \cap \mathfrak{F}_{x+y}$,

$$T(x + y) \leq T(x) + T(y).$$

On the other hand, if $f_1 \in D \cap \mathfrak{F}_x$ and $f_2 \in D \cap \mathfrak{F}_y$ then $f_1 + f_2 \in D \cap \mathfrak{F}_{x+y}$. Therefore,

$$T_0(f_1) + T_0(f_2) = T_0(f_1 + f_2) \leq \sup(D \cap \mathfrak{F}_{x+y}) = T(x + y).$$

Passing to the supremum first over $f_1 \in D \cap \mathfrak{F}_x$ and then over $f_2 \in D \cap \mathfrak{F}_y$, one gets

$$T(x) + T(y) \leq T(x + y).$$

So, $T\colon E \to F$ is an OAO. If $x \in D$ then $D \cap \mathfrak{F}_x = \mathfrak{F}_x$ by the definition of a lateral ideal and hence, $T(x) = \sup T_0(\mathfrak{F}_x) = T_0(x)$ by the positivity and orthogonal additivity of T_0. So, T is an extension of T_0. Since $T_0(D)$ is order bounded in F, we have that $T \in \mathcal{U}^+(E, F)$. □

3. Extensions from lateral bands.

Theorem 2. *Let E, F be vector lattices with E Dedekind complete, E_0 a lateral band of E and $T_0\colon E_0 \to F$ an OAO. Then there is an OAO extension $T\colon E \to F$ of T_0. If, moreover, T_0 is positive (laterally bounded, preserves disjointness or laterally continuous) then so is T.*

Theorem 2 is a consequence of the following result.

Theorem 3. *Let E_0 be a lateral band of a Dedeking complete vector lattice E. Then the function $P_{E_0}\colon E \to E$ defined by setting for every $x \in E$*

$$P_{E_0}(x) = \bigsqcup (\mathfrak{F}_x \cap E_0),$$

is a disjointness preserving laterally continuous projection of E onto E_0.

Proof of Theorem 3. Since the set $\mathfrak{F}_x \cap E_0$ is laterally bounded by x and E is Dedekind complete, the function is well defined. It is immediate that P_{E_0} preserves disjointness. Given any $x \in E_0$, one has $\mathfrak{F}_x \cap E_0 = \mathfrak{F}_x$ and hence $P_{E_0}(x) = x$.

Assume $P_{E_0}(x) = x$. Since the lateral supremum of the set $\mathfrak{F}_x \cap E_0$ is x, there is a net in $\mathfrak{F}_x \cap E_0$ laterally converging to x. And since E_0 is laterally closed, $x \in \mathfrak{F}_x \cap E_0$ and hence $x \in E_0$.

It remains to prove the lateral continuity of P_{E_0}. Fix any $x \in E$ and assume (x_α) is a net in E with $x_\alpha \xrightarrow{\text{lat}} x$. Our goal is to prove that $P_{E_0}(x_\alpha) \xrightarrow{\text{lat}} P_{E_0}(x)$, that is,

$$\bigsqcup\nolimits_\alpha P_{E_0}(x_\alpha) = P_{E_0}(x).$$

Observe that if $x \in E_0$ then $x_\alpha \in E_0$, and by the above,

$$P_{E_0}(x_\alpha) = x_\alpha \xrightarrow{\text{lat}} x = P_{E_0}(x).$$

So, assume now $x \notin E_0$. First we obtain an upper lateral estimate

$$\bigsqcup_\alpha P_{E_0}(x_\alpha) = \bigsqcup_\alpha \bigsqcup (\mathfrak{F}_{x_\alpha} \cap E_0) = \bigsqcup \bigl\{ z \in E_0 \colon (\exists \alpha)(\exists y \sqsubseteq x_\alpha)\, (z \sqsubseteq y) \bigr\}. \tag{1}$$

Since

$$\bigl\{ z \in E_0 \colon (\exists \alpha)(\exists y \sqsubseteq x_\alpha)\, (z \sqsubseteq y) \bigr\} \subseteq \bigl\{ z \in E_0 \colon (\exists y \sqsubseteq x)\, (z \sqsubseteq y) \bigr\} = \mathfrak{F}_x \cap E_0,$$

we can continue (1) as follows

$$\sqsubseteq \bigsqcup (\mathfrak{F}_x \cap E_0) = P_{E_0}(x).$$

To prove the lower lateral estimate, fix any $y \in \mathfrak{F}_x \cap E_0$. Then

$$y = x \sqcap y = \bigsqcup_\alpha (x_\alpha \sqcap y) \sqsubseteq \bigsqcup_\alpha P_{E_0}(x_\alpha).$$

Passing to the supremum over y, we obtain $P_{E_0}(x) \sqsubseteq \bigsqcup_\alpha P_{E_0}(x_\alpha)$. □

Proof of Theorem 2. Obviously, the composition operator $T = T_0 \circ P_{E_0}$ of the lateral band projection P_{E_0} by T_0 possesses the desired properties. □

Problem 2. *Let F be a lateral field in a vector lattice E, X a linear space. Whether every OAO $T_0\colon F \to X$ can be extended to an OAO $T\colon E \to X$?*

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