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ON SUBHARMONIC FUNCTIONS IN THE HALF-PLANE OF INFINITE ORDER WITH RADIALLY DISTRIBUTED MEASURE

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If a proper subharmonic function of infinite order has the full measure at the finite system of rays in the upper half-plane, then its lower order also equals infinity.

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Если полная мера истинно субгармонической функции бесконечного порядка распределена на конечной системе лучей в верхней полуплоскости, то ее нижний порядок также равен бесконечности.

1. In this paper we use the Fourier series method to study the properties of subharmonic functions. This method was introduced by L. A. Rubel and B. A. Taylor ([1]). Further the Fourier series method was used by J. B. Miles ([2]), A. A. Kondratyuk ([3, 4, 5]) and others.

Let v be a subharmonic function in the complex plane $\mathbb{C}$, $M(v,r) = \max_{0\le\theta\le 2\pi} v(re^{i\theta})$. *The order and lower order* of the function v are defined to be the values

$$\beta[\gamma] = \varlimsup_{r\to\infty} \frac{\ln M(v,r)}{\ln r}, \quad \alpha[\gamma] = \varliminf_{r\to\infty} \frac{\ln M(v,r)}{\ln r}.$$

The order and lower order of an entire function f are defined as the order and lower order of the subharmonic function $\ln|f|$, respectively.

In [6] the author considered the entire functions which zeros lie on the finite system of rays. In particular, it was proved that if f is an entire function of infinite order with positive zeros then its lower order equals infinity as well. This result is easily generalized to the subharmonic functions in the complex plane: if the Riesz measure of a subharmonic function in the entire complex plane v of infinite order is located on a positive half-axis then its lower order also equals infinity. We prove a similar result for functions which are subharmonic in the half-plane. The special case, where the measure is distributed on the imaginary axis, was considered in [7].

2. Let $\mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$ be the upper half-plane of the complex variable z. We denote by $C(a,r)$ the open disc of radius r with center at a, and by Ω_+ the intersection of a set Ω with the half-plane $\mathbb{C}_+\colon \Omega_+ = \Omega \cap \mathbb{C}_+$; $\overline{G}$ means closure of a set G. If $0 < r_1 < r_2$ then $D_+(r_1,r_2) = \overline{C_+(0,r_2)\setminus C_+(0,r_1)}$ means a closed half-ring.

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Let SK be the class of subharmonic functions in $\mathbb{C}_+$ prossessing a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$. Functions $v(z)$ from SK have the following properties ([8]):

a) $v(z)$ has non-tangential limits $v(t)$ almost everywhere on the real axis and $v(t) \in L^1_{\mathrm{loc}}(-\infty, \infty)$;

b) there exists a measure of variable sign ν on the real axis such that

$$\lim_{y\to+0} \int_a^b v(t+iy)dt = \nu([a,b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\}).$$

The measure ν is called the *boundary measure* of v;

c) $d\nu(t) = v(t)dt + d\sigma(t)$, where σ is a singular measure with respect to the Lebesgue measure.

For a function $v \in SK$, following [8], we define the *full measure* λ by setting

$$\lambda(K) = 2\pi \int_{\mathbb{C}_+\cap K} \operatorname{Im}\zeta d\mu(\zeta) - \nu(K),$$

where μ is the Riesz measure of v.

A subharmonic in $\mathbb{C}_+$ function v is said to be *proper subharmonic* if $\limsup_{z\to t} v(z) \le 0$ for all real numbers $t \in \mathbb{R}$. Denote the class of proper subharmonic functions by JS. The full measure of the function $v \in JS$ is a positive measure, which explains the term "proper subharmonic function".

The class of delta-subharmonic functions $J\delta$ is defined to be the difference $J\delta = JS - JS$.

For a function $v \in J\delta$ the representation in a disc $z \in C_+(0,R)$ is well defined

$$v(z) = -\frac{1}{2\pi} \iint\limits_{\overline{C_+(0,R)}} \frac{G(z,\zeta)}{\operatorname{Im}\zeta} d\lambda(\zeta) + \frac{R}{2\pi}\int_0^\pi \frac{\partial G(z, Re^{i\varphi})}{\partial\tau} v(Re^{i\varphi})d\varphi, \tag{1}$$

where $G(z,\zeta)$ is the Green function of the half-disc, $\frac{\partial G}{\partial \tau}$ means the derivative in the inward normal direction, and the kernel of double integral is extended by continuity to the real axis for $|t| \le R$.

For the measure λ denote $\lambda(t) = \lambda(\overline{C(0,t)})$. Let $v \in J\delta$, $v = v_+ - v_-$, λ be the full measure of v, $\lambda = \lambda_+ - \lambda_-$ be the Jordan decomposition of measure λ. Let us introduce the following characteristics of the function v

$$m(r,v) := \frac{1}{r}\int_0^\pi v_+(re^{i\varphi})\sin\varphi d\varphi, \quad N(r,v,r_0) := \int_{r_0}^r \frac{\lambda_-(t)}{t^3}dt,$$
$$T(r,v,r_0) := m(r,v) + N(r,v,r_0) + m(r_0,-v), \quad r > r_0,$$

where r_0 is an arbitrary fixed positive number (one may as well take $r_0 = 1$) which in designations (if it does not cause a misunderstanding) we will not write (for example, instead of $T(r,v,r_0)$ will write $T(r,v)$ and so on).

Let $\lambda_k(r) = \lambda_k(\overline{C(0,r)})$ where $d\lambda_k(\tau e^{i\varphi}) = \frac{\sin k\varphi}{\sin\varphi}\tau^{k-1}d\lambda(\tau e^{i\varphi})$, $k \in \mathbb{N}$ (the function $\sin k\varphi/\sin\varphi$ is defined for $\varphi = 0$, π, by continuity).

Note the Carleman's formula in Grishin's notation

$$\frac{1}{r^k}\int_0^\pi v(re^{i\varphi})\sin k\varphi d\varphi=\int_{r_0}^r\frac{\lambda_k(t)}{t^{2k+1}}dt+\frac{1}{r_0^k}\int_0^\pi v(r_0e^{i\varphi})\sin k\varphi d\varphi,\quad k\in\mathbb{N},$$

In particular for $k=1$

$$\frac{1}{r}\int_0^\pi v(re^{i\varphi})\sin\varphi d\varphi=\int_{r_0}^r\frac{\lambda(t)}{t^3}dt+\frac{1}{r_0}\int_0^\pi v(r_0e^{i\varphi})\sin\varphi d\varphi. \tag{2}$$

Formula (2) can be written as

$$T(r,v)=T(r,-v). \tag{3}$$

Definition 1. The *order* and *lower order* of a growth function γ are defined to be the values:

$$\beta[\gamma]=\varlimsup_{r\to\infty}\frac{\ln\gamma(r)}{\ln r},\quad \alpha[\gamma]=\varliminf_{r\to\infty}\frac{\ln\gamma(r)}{\ln r}.$$

Definition 2. The *order* and *lower order* of a function $v\in J\delta$ are defined to be the values $\beta[rT(r,v)]$ and $\alpha[rT(r,v)]$.

The *Fourier coefficients* of a function $v\in J\delta$ are defined by the formula ([9]) $c_k(r,v)=\frac{2}{\pi}\int_0^\pi v(re^{i\theta})\sin k\theta d\theta,\ k\in\mathbb{N}$.

Let λ be the full measure of $v\in J\delta$, then ([9])

$$c_k(r,v)=\alpha_kr^k+\frac{2r^k}{\pi}\int_{r_0}^r\frac{\lambda_k(t)}{t^{2k+1}}dt,\quad k\in\mathbb{N}, \tag{4}$$

where $\alpha_k=r_0^{-k}c_k(r_0,v)$, and

$$c_k(r,v)=\alpha_kr^k+$$
$$+\frac{r^k}{\pi kr_0^{2k}}\iint\limits_{C_+(0,r_0)}\frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^kd\lambda(\zeta)+\frac{r^k}{\pi k}\iint\limits_{D_+(r_0,r)}\frac{\sin k\varphi}{\tau^k\operatorname{Im}\zeta}d\lambda(\zeta)-\frac{1}{r^k\pi k}\iint\limits_{C_+(0,r)}\frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^kd\lambda(\zeta), \tag{5}$$

where $\zeta=\tau e^{i\varphi}$.

By the definition of $c_k(r,v)$ one has $|c_k(r,v)|\le\frac{2k}{\pi}\int_0^\pi|v(re^{i\varphi})|\sin\varphi d\varphi$, $k\in\mathbb{N}$. Taking into account (3) we obtain

$$rT(r,v)\ge\frac{\pi}{2k}|c_k(r,v)|,\quad k\in\mathbb{N}. \tag{6}$$

3. The main result of this paper is the following theorem.

Theorem 1. *If $v\in SK$ is a subharmonic function on $\mathbb{C}_+$ of infinite order with the full measure λ on the finite system of rays $\mathbb{L}_k=\{z\colon \arg z=e^{i\theta_k},\ \theta_k=\frac{\pi p_k}{q_k}\}$; $k\in\overline{1,N_0}$; $p_k,q_k,N_0\in\mathbb{N}$; $p_k<q_k$; then its lower order equals infinity.*

Proof. We assume that $0\notin\operatorname{supp}v$. By formulae (5) for Fourier coefficients of the function v we obtain

$$c_n(r,v)=\alpha_nr^n+\sum_{k=1}^{N_0}\frac{r^n\sin(\theta_kn)}{\pi nr_0^{2n}}\int_0^{r_0}t^{n-1}d\lambda(t)+$$
$$+\sum_{k=1}^{N_0}\frac{r^n\sin(\theta_kn)}{\pi n}\int_{r_0}^r\frac{d\lambda(t)}{t^{n+1}}-\sum_{k=1}^{N_0}\frac{\sin(\theta_kn)}{r^n\pi n}\int_0^rt^{n-1}d\lambda(t),\ n\in\mathbb{N}.$$

Assume r_0 satisfies $C(0,r_0)\notin \operatorname{supp} v$. Then we obtain

$$c_n(r,v)=\alpha_n r^n+\sum_{k=1}^{N_0}\frac{\sin(\theta_k n)}{\pi n}\int_{r_0}^{r}\frac{1}{t}\left[\left(\frac{r}{t}\right)^n-\left(\frac{t}{r}\right)^n\right]d\lambda(t),\ n\in\mathbb{N}. \tag{7}$$

Applying twice the integration by parts in (7), we obtain

$$\begin{aligned} c_n(r,v)=\alpha_n r^n+\frac{2}{\pi}\sum_{k=1}^{N_0}\sin(\theta_k n)\left(\tilde N(r)+r^n\int_{r_0}^{r}\frac{\tilde N(r)}{t^{n+1}}dt\right)+\\ +\frac{n-1}{\pi}\sum_{k=1}^{N_0}\sin(\theta_k n)\int_{r_0}^{r}\frac{1}{t}\left[\left(\frac{r}{t}\right)^n-\left(\frac{t}{r}\right)^n\right]\tilde N(r)dt,\ n\in\mathbb{N}, \end{aligned} \tag{8}$$

where $\tilde N(r)=\int_{r_0}^{r}\frac{\lambda(t)}{t^2}dt$.

Denote by $C=\sum_{k=1}^{N_0}\sin\theta_k$. It is clear that $C>0$. From (8) with $n=n_l=1+2l\prod_{k=1}^{N_0}q_k$, $l\in\mathbb{N}$, we obtain

$$\frac{|c_n(r,v)|}{r^n}\geq\frac{2C}{\pi}\Big(\frac{\tilde N(r)}{r^n}+\int_{r_0}^{r}\frac{\tilde N(r)}{t^{n+1}}dt\Big)-|\alpha_n|,\ n\in\mathbb{N}. \tag{9}$$

If the function $\tilde N(r)$ has infinite order then the integral from the right hand side of the latter inequality is unbounded as $r\to\infty$ because $\int_r^\infty\frac{\tilde N(t)}{t^{n+1}}dt\geq\frac{\tilde N(r)}{nr^n}$, $n\in\mathbb{N}$, and the right-hand side of this inequality can be made arbitrarily large by a suitable choice of r. By this inequality and inequality (6), from (9) we obtain the required statement.

If $\tilde N(r)$ has finite order then there exist positive numbers $K>0$ and $\rho>0$ such that $\tilde N(r)\leq Kr^\rho$ for all $r>0$. It is possible to consider non-integer ρ. Then

$$K2^\rho r^\rho\geq\tilde N(2r)\geq\int_r^{2r}\frac{\lambda(t)}{t^2}dt\geq\lambda(r)\int_r^{2r}\frac{dt}{t^2}=\frac{\lambda(r)}{2r},$$

i.e. $\lambda(r)\leq K2^{\rho+1}r^{\rho+1}$.

In this case one can deduce from [8] that there exists a function $g\in JS$ of order ρ and with full measure λ. Then $G=v-g\in J\delta$ and $\lambda_G\equiv 0$. Further we need the following lemma.

Lemma 1. *If $G\in JS$ and $\lambda_G\equiv 0$, then $G(z)=\operatorname{Im} f(z)$, where $f(z)$ is an entire real function.*

Proof. Remind [10] that an entire function is said to be real if $f(\mathbb{R})\subset\mathbb{R}$.

As the full measure of the function G equals zero then from (1) it follows that for any $R>0$ $G(z)=\frac{R}{2\pi}\int_0^\pi\frac{\partial G(z,Re^{i\varphi})}{\partial n}G(Re^{i\varphi})d\varphi, z\in C_+(0,R)$.

The right-hand side is a harmonic function in the half-disc $C_+(0,R)$, which is extended by the continuity as zero on the interval $(-R,R)$. Since R is an arbitrary positive number, the function $G(z)$ is harmonic on the half-plane $\mathbb{C}_+$, which is extended by continuity as zero on the real axis. By the symmetry principle, this function is extended as a harmonic function to the bottom half-plane. Then there exists a harmonic function $h(z)$ on the complex plane such that $f(\mathbb{R})=0$ and $G(z)=h(z)$ for $\operatorname{Im} z>0$.

Let $-h_1(z)$ be a function which is harmoniously conjugated to the function $h(z)$. Then $f(z)=h_1(z)+ih(z)$ is an entire function, real on the real axis and $h(z)=\operatorname{Im} f(z)$. $\square$

According to Lemma 1, $G(z) = \operatorname{Im} f(z)$, where $f(z)$ is an entire real function $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If only a finite number $a_n \neq 0$, then $f(z)$ is a polynomial, hence G and v have a finite order, which contradicts the assumption.

As $c_n(r, G) = a_n r^n$, $n \in \mathbb{N}$, the inequality

$$rT(r, v) \geq rT(r, G) - rT(r, g) \geq \frac{\pi}{2n} \left| c_n(r, G) \right| + O(r^\rho) \geq \frac{1}{2} \left| a_n \right| r^n + O(r^\rho), \; r \to \infty, \; n \in \mathbb{N},$$

implies that $\alpha[rT(r, v)] = \infty$. $\square$

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