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UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO SETS HAVING DEFICIENT VALUES

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With the aid of weighted sharing of sets we study the uniqueness problem of meromorphic functions having the same pole sharing of a finite set. The result of the paper improves, generalizes and extends a recent result of S. S. Bhoosnurmath and R. Davanal ([1]).

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С помощью весовых множеств разделенных значений мы изучаем проблему единственности мероморфных функций с общими полюсами, разделяющих конечное множество. Результат статьи улучшает, обобщает и расширяет недавний результат из [1].

1. Introduction definitions and results. In this paper, by meromorphic functions we will always mean meromorphic functions in the whole complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [6]. For $a \in \mathbb{C} \cup \{\infty\}$, we define

$$\Theta(a; f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r, h)$ any quantity satisfying $S(r, h) = o(T(r, h))$ ($r \to \infty$, $r \notin E$).

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, $r \notin E$.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a subset of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z\colon f(z) = a\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity then the set $\bigcup_{a\in S}\{z\colon f(z) = a\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ then we say that f and g share

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the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say that f and g share the set S IM. Evidently, if S contains only one element then it coincides with the usual definition of CM (respectively, IM) shared values.

During the last few decades the uniqueness theory of entire or meromorphic functions has been grown up as an active subfield of the value distribution theory. The main intention of the uniqueness theory is to determine an entire or meromorphic function uniquely satisfying some prescribed condition. The remarkable five value theorem and four value theorem by Nevanlinna can be considered as the inception of this extensive theory. Later, the study of the relationship between two meromorphic functions via the pre-image sets of several distinct values in the range has also got the priority over value sharing. In this respect one cannot deny the contribution of F. Gross. In 1970 F. Gross and C. C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values.

In [5] F. Gross asked the following question which is known as Gross's question in the literature.

Can one find two finite sets S_j ($j \in \{1,2\}$) such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j \in \{1,2\}$ must be identical?

In course of time researchers became more involved to deal with the above question in case of meromorphic functions under weaker hypothesis. Later the investigations has been shifted to determine a set S of n elements and to make n as small as possible such that any meromorphic functions f and g that share the value ∞ and the set S must be equal. (cf. [1]–[2], [4], [7], [11], [16]–[18]).

In connection with the question of F. Gross, for entire functions, H. X. Yi ([15]) proved the following result.

Theorem A ([15])**.** *Let $S = \{z\colon z^7 - z^6 - 1 = 0\}$. If f and g are non-constant entire functions satisfying $E_f(S) = E_g(S)$ then $f \equiv g$.*

M. Fang and X. Hua ([3]) further extended Theorem A to meromorphic functions imposing on the ramification indexes of f and g. M. Fang and X. Hua proved the following theorem.

Theorem B ([3])**.** *Let $S = \{z\colon z^7 - z^6 - 1 = 0\}$. If meromorphic functions f and g are such that $\Theta(\infty; f) > \frac{11}{12}$, $\Theta(\infty; g) > \frac{11}{12}$ and $E_f(S) = E_g(S)$ then $f \equiv g$.*

In 1995 H. X. Yi proved for meromorphic functions the following result.

Theorem C ([16])**.** *Let $S = \{z\colon z^n + az^{n-m} + b = 0\}$, where n and m are positive integers such that $m \geq 2$, $n \geq 2m + 7$ with n and m having no common factor, a and b be nonzero constants such that $z^n + az^{n-m} + b = 0$ has no multiple root. If f and g are non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

The case where $m = 1$ is also studied by H. X. Yi ([16]) in the same paper. Below we state the result.

Theorem D ([16])**.** *Let $S = \{z\colon z^n + az^{n-1} + b = 0\}$, where $n(\geq 9)$ be an integer and a and b be nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be non-constant meromorphic functions such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then either $f \equiv g$ or $f \equiv \frac{-ah(h^{n-1}-1)}{h^n-1}$ and $g \equiv \frac{-a(h^{n-1}-1)}{h^n-1}$, where h is a non-constant meromorphic function.*

To find under which condition $f \equiv g$ in Theorem D, I. Lahiri obtained the following result.

Theorem E ([7]). *Let S be defined as in Theorem D and $n(\geq 8)$ be an integer. If f and g be non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

M. Fang and I. Lahiri ([4]) further reduced the cardinality of the range set and proved the following theorem.

Theorem F ([4]). *Let S be defined as in Theorem D and $n(\geq 7)$ be an integer. If f and g be non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In the mid of 2001 the notion of weighted sharing of values and sets appeared in the uniqueness theory which renders a useful tool for the purpose of relaxation of the nature of sharing the sets.

Definition 1 ([9, 10]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2 ([9]). Let S be a subset of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $E_f(S) = \bigcup_{a \in S}\{z \colon f(z) - a = 0\}$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Next the following definition is required.

Definition 3 ([8]). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f| \leq m)(N(r, a; f| \geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Improving Theorem F, I. Lahiri ([11]) proved the following theorem.

Theorem G ([11]). *Let S be defined as in Theorem D and $n(\geq 7)$ be an integer. If for non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

In 2006 to deal with a question of F. Gross, H. X. Yi and W. C. Lin have proved the following results.

Theorem H ([17]). *Let S be defined as in Theorem D and $n(\geq 7)$ be an integer. If for non-constant meromorphic functions f and g, $\Theta(\infty; f) > \frac{1}{2}$. $E_f(S, \infty) = E_g(S, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

Theorem I ([17]). *Let S be defined as in Theorem D and $n(\geq 8)$ be an integer. If for non-constant meromorphic functions f and g, $\Theta(\infty; f) > \frac{2}{n-1}$. $E_f(S, \infty) = E_g(S, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

It is ought to be noted that I. Lahiri ([11]) obtained the smallest cardinality $n = 7$ corresponding to the set S described so far. The following example establishes the fact that the set S in Theorems D-I cannot be replaced with any arbitrary set containing six elements.

Example 1. Let $f(z) = \sqrt{\alpha\beta\gamma}e^z$, $g(z) = \sqrt{\alpha\beta\gamma}e^{-z}$ and $S = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\alpha}, \beta\sqrt{\gamma}, \gamma\sqrt{\alpha}, \gamma\sqrt{\beta}\}$, where α, β and γ are nonzero distinct complex numbers. Clearly $E_f(S, \infty) = E_g(S, \infty)$ but $f \not\equiv g$.

So it remains an open problem for investigations whether the degree of the equation defining S can be reduced to six and at the same time the conditions over ramification indexes can be weakened further?

In this direction S. S. Bhoosnurmath and R. Dyavanal ([1]) recently proved the following.

Theorem J ([1]). *Let S be defined as in Theorem D and $n(\geq 5)$ be an integer. If for non-constant meromorphic functions f and g, $\Theta(\infty; f) > \frac{2}{n-1}$ and $\Theta(\infty; g) > \frac{2}{n-1}$, $E_f(S, \infty) = E_g(S, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $N(r, 0; f \mid= 1) = S(r, f)$, $N(r, 0; g \mid= 1) = S(r, g)$ then $f \equiv g$.*

In the paper we continue the investigations and obtain the following result which improves, generalizes and extends Theorem J. Our result also supplements Theorem G to a large extent. For $a \in \mathbb{C} \cup \{\infty\}$, we define

$$\delta_{(2}(a; f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, a; f \mid\geq 2)}{T(r, f)}.$$

The following theorem is the main result of the paper.

Theorem 1. *Let S be given as in Theorem D. Suppose that f and g are non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, $N(r, 0; f \mid= 1) = S(r, f)$, $N(r, 0; g \mid= 1) = S(r, g)$ and $\Theta_f + \Theta_g > \frac{4}{n-1}$. If*

(i) $m \geq 2$ and $n \geq 5$;

(ii) or if $m = 1$ and $n \geq 6$;

(iii) or if $m = 0$ and $n \geq 10$

then $f \equiv g$, where $\Theta_f = \delta_{(2}(0; f) + \Theta(-a\frac{n-1}{n}; f) + \Theta(\infty; f)$ and Θ_g is similarly defined.

We now explain some definitions and notation which are used in the paper.

Definition 4. Let f and g be non-constant meromorphic functions such that f and g share $(a, 0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r,a;f)$ the reduced counting function of those a-points of f and g where $p=q\geq 2$. In the same way we can define $\overline{N}_L(r,a;g), N_E^{1)}(r,a;g), \overline{N}_E^{(2}(r,a;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a\in\mathbb{C}\cup\{\infty\}$. If f and g share (a,m), $m\geq 1$ then $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.

Definition 5. We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k\geq 2$ is an integer.

Definition 6 ([9, 10])**.** Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

2. Lemmas. In this section we present some lemmas which will be needed in the sequel. Let F and G be non-constant meromorphic functions defined as follows.

$$F=\frac{f^{n-1}(f+a)}{-b},\quad G=\frac{g^{n-1}(g+a)}{-b}. \tag{1}$$

Henceforth we shall denote by H the following function

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big). \tag{2}$$

Lemma 1 ([14])**.** *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\neq 0$ and $b_m\neq 0$ Then $T(r,R(f))=dT(r,f)+S(r,f)$, where $d=\max\{n,m\}$.

Lemma 2 ([18])**.** *If F, G are non-constant meromorphic functions such that they share (1,0) and $H\not\equiv 0$ then $N_E^{1)}(r,1;F\mid=1)=N_E^{1)}(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G)$.*

Lemma 3. *Let $S=\{z\colon z^n+az^{n-1}+b=0\}$, where a, b are nonzero constants such that $z^n+az^{n-1}+b=0$ has no repeated root, $n(\geq 3)$ be an integer and F, G be given by (1). If for non-constant meromorphic functions f and g one has $E_f(S,0)=E_g(S,0)$, $E_f(\infty,0)=E_g(\infty,0)$ and $H\not\equiv 0$ then*

$$\begin{aligned}N(r,H)\leq{}&\overline{N}(r,0,f)+\overline{N}(r,0;g)+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;nf+a(n-1))+\\&+\overline{N}(r,0;ng+a(n-1))+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of f and $(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. Since $E_f(S,0)=E_g(S,0)$ and $E_f(\infty,0)=E_g(\infty,0)$ it follows that F and G share $(1,0)$ and $(\infty,0)$. We have from (1) that $F'=\frac{[nf+(n-1)a]f^{n-2}f'}{-b}$ and $G'=\frac{[ng+(n-1)a]g^{n-2}g'}{-b}$. We can easily verify that possible poles of H occur at (i) zeros of f and g, (ii) zeros of $nf+a(n-1)$ and $ng+a(n-1)$, (iii) common poles of f and g with different multiplicities, (iv) common 1-points of F and G with different multiplicities, (v) zeros of f' which are not the zeros of $f(F-1)$, (v) zeros of g' which are not the zeros of $g(G-1)$. Since H has only simple poles, the lemma follows from the above. □

Lemma 4 ([11], Lemma 5). *If f, g share $(\infty, 0)$ then for $n(\geq 2)$ $f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2$, where a,b are finite nonzero constants.*

Lemma 5. *Let f, g be non-constant meromorphic functions such that $N(r,0;f \mid= 1) = S(r,f)$, $N(r,0;g \mid= 1) = S(r,g)$ and $\Theta_f + \Theta_g > \frac{4}{n-1}$, where Θ_f and Θ_g have the same meaning as in Theorem 1. Then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where $n(\geq 2)$ is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a) \tag{3}$$

and suppose $f \not\equiv g$. We consider two cases.
Case I. Let $y = \frac{g}{f}$ be a constant. Then from (3) it follows that $y \neq 1$, $y^{n-1} \neq 1$, $y^n \neq 1$ and $f \equiv -a\frac{1-y^{n-1}}{1-y^n}$, a constant, which is impossible.
Case II. Let $y = \frac{g}{f}$ be non-constant. Then

$$f \equiv -a\frac{1-y^{n-1}}{1-y^n} \equiv a\Big(\frac{y^{n-1}}{1+y+y^2+\ldots+y^{n-1}} - 1\Big), \tag{4}$$

$$f + a\frac{(n-1)}{n} \equiv -a\frac{1-y^{n-1}}{1-y^n} + a\frac{(n-1)}{n} \equiv -a\frac{(n-1)y^n - ny^{n-1} + 1}{n(1-y^n)}. \tag{5}$$

If we assume $p(z) = (n-1)z^n - nz^{n-1} + 1$, then $p(0) \neq 0$ and $p(1) = p^{'}(1) = 0$. So we observe that the polynomial $p(z)$ has double zero at the point $z = 1$. Consequently it has $n-1$ distinct zeros which we denote by $u_i, i \in \{1, \ldots, n-1\}$. Hence from (5) we see that $\sum_{i=1}^{n-1} \overline{N}(r, u_i; y) \leq \overline{N}(r, -a\frac{n-1}{n}; f)$.

From (4) we see by Lemma 1 that $T(r,f) = (n-1)T(r,y) + S(r,y)$, $T(r,g) = (n-1)T(r,y) + S(r,y)$ and so $S(r,f)$ and $S(r,g)$ can both be replaced with $S(r,y)$.

We first note that the zeros of $1 + y + y^2 + \ldots + y^{n-2}$ contributes to the zeros of both f and g. In addition to this the poles of y contributes to the zeros of f and since $g = fy$ the zeros of y contributes to the zeros of g. So from (4) we see that

$$\sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \overline{N}(r, \infty; y) \leq \overline{N}(r, 0; f), \sum_{k=1}^{n-1} \overline{N}(r, w_k; y) \leq \overline{N}(r, \infty; f)$$

where $v_j = \exp(\frac{2j\pi i}{n-1})$ for $j \in \{1, 2, \ldots, n-2\}$ and $w_k = \exp(\frac{2k\pi i}{n})$ for $k \in \{1, 2, \ldots, n-1\}$.

By the second fundamental theorem we get

$$\begin{aligned}
&(3n-5)T(r,y) \leq \\
&\leq \overline{N}(r,\infty;y) + \sum_{i=1}^{n-1}\overline{N}(r,u_i;y) + \sum_{j=1}^{n-2}\overline{N}(r,v_j;y) + \sum_{k=1}^{n-1}\overline{N}(r,w_k;y) + \overline{N}(r,0;y) + S(r,y) \leq \\
&\leq \overline{N}(r,0;f) + \overline{N}\Big(r, -a\frac{n-1}{n}; f\Big) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + S(r,y) \leq \\
&\leq \Big(\frac{5}{2} - \frac{1}{2}\delta_{(2}(0;f) - \Theta\Big(-a\frac{n-1}{n};f\Big) - \Theta(\infty;f) + \frac{\varepsilon}{2}\Big)T(r,f) + \\
&+\Big(\frac{1}{2} - \frac{1}{2}\delta_{(2}(0;g) + \frac{\varepsilon}{2}\Big)T(r,g) + S(r,y) = \\
&= (n-1)\Big(3 - \frac{1}{2}\delta_{(2}(0;f) - \frac{1}{2}\delta_{(2}(0;g) - \Theta\Big(-a\frac{n-1}{n};f\Big) - \Theta(\infty;f) + \varepsilon\Big)T(r,y) + S(r,y)
\end{aligned}$$

i.e.,

$$\frac{3n-5}{n-1}T(r,y) \le \Big(3-\Theta_f+\frac{1}{2}\delta_{(2}(0;f)-\frac{1}{2}\delta_{(2}(0;g)+\varepsilon\Big)T(r,y)+S(r,y), \tag{6}$$

where $0<2\varepsilon<\Theta_f+\Theta_g-\frac{4}{n-1}$.

Again putting $y_1=\frac{1}{y}$, noting that $T(r,y)=T(r,y_1)+O(1)$ and proceeding as above we get that

$$\frac{3n-5}{n-1}T(r,y) \le \Big(3-\Theta_g+\frac{1}{2}\delta_{(2}(0;g)-\frac{1}{2}\delta_{(2}(0;f)+\varepsilon\Big)T(r,y)+S(r,y). \tag{7}$$

Adding (6) and (7) we get $\left(\frac{6n-10}{n-1}-6+\Theta_f+\Theta_g-2\varepsilon\right)T(r,y)\le S(r,y)$, which is a contradiction.

Hence $f\equiv g$ and this proves the lemma. □

Lemma 6 ([13])**.** *If $N(r,0;f^{(k)}|f\ne 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then $N(r,0;f^{(k)}|f\ne 0)\le k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\ge k)+S(r,f)$.*

3. Proofs of the theorems.

Proof of Theorem 1. We know from the assumption that the zeros of $z^n+az^{n-1}+b$ are simple and we denote them by w_j, $j\in\{1,2,\dots,n\}$. Let F, G be given by (1) and (2). Since $E_f(S,m)=E_g(S,m)$ and $E_f(\infty,\infty)=E_g(\infty,\infty)$ it follows that F, G share $(1,m)$ and (∞,∞) and so $\overline{N}_*(r,\infty;f,g)=0$. Also we note that by the assumption of the theorem $2\overline{N}(r,0;f)\le N(r,0;f\mid\ge 2)+S(r,f)$ and $2\overline{N}(r,0;g)\le N(r,0;g\mid\ge 2)+S(r,g)$.
Case 1. Suppose that $H\not\equiv 0$.
Subcase 1.1. $m\ge 1$. While $m\ge 2$, using Lemma 6 we note that

$$\begin{aligned}\overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\ge 2)+\overline{N}_*(r,1;F,G)\le\overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\ge 2)+\\ +\overline{N}(r,1;G\mid\ge 3)\le\overline{N}_0(r,0;g')+\sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g\mid=2)+2\overline{N}(r,\omega_j;g\mid\ge 3)\}\le\\ \le N(r,0;g'|g\ne 0)+S(r,g)\le\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g).\end{aligned} \tag{8}$$

Hence using (8), Lemmas 1, 2 and 3 we get from the second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned} nT(r,f)\le\\ \le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\ge 2)-N_0(r,0;f')+S(r,f)\le\\ \le 2\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,0;nf+a(n-1))+\\ +\overline{N}(r,0;ng+a(n-1))+\overline{N}(r,1;G\mid\ge 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;g')+\\ +S(r,f)+S(r,g)\le 2\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\Big\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\\ +\overline{N}\Big(r,-a\frac{n-1}{n};f\Big)+\overline{N}\Big(r,-a\frac{n-1}{n};g\Big)+S(r,f)+S(r,g)\le\\ \le N(r,0;f\mid\ge 2)+N(r,0;g\mid\ge 2)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}\Big(r,-a\frac{n-1}{n};f\Big)+\\ +\overline{N}\Big(r,-a\frac{n-1}{n};g\Big)+S(r,f)+S(r,g)\le\Big(6-\Theta_f-\Theta_g+\varepsilon\Big)T(r)+S(r).\end{aligned} \tag{9}$$

In a similar way we can obtain

$$T(r,g) \le (6 - \Theta_f - \Theta_g + \varepsilon)\, T(r) + S(r). \tag{10}$$

Combining (9) and (10) we see that

$$(n - 6 + \Theta_f + \Theta_g + \varepsilon)\, T(r) \le S(r). \tag{11}$$

Since $\varepsilon > 0$ is arbitrary, (11) leads to a contradiction for $n \ge 5$.
While $m = 1$, using Lemma 6, (8) changes to

$$\begin{aligned} &\overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_*(r,1;F,G) \le \\ &\le \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_L(r,1;G) + \overline{N}(r,1;F \mid\ge 3) \le \\ &\le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\sum_{j=1}^{n}\{N(r,\omega_j;f) - \overline{N}(r,\omega_j;f)\} \le \\ &\le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + S(r,f) + S(r,g). \end{aligned} \tag{12}$$

So using (12), Lemmas 2 and 3 and proceeding as in (9) we get from the second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} nT(r,f) &\le 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + 2\overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \\ &+ \overline{N}\Big(r, -a\frac{n-1}{n}; f\Big) + \overline{N}\Big(r, -a\frac{n-1}{n}; g\Big) + \frac{3}{4}T(r,f) + S(r,f) + S(r,g) \le \\ &\le N(r,0;f \mid\ge 2) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N(r,0;g \mid\ge 2) + \overline{N}\left(r, -a\frac{n-1}{n}; f\right) + \\ &+ \overline{N}\Big(r, -a\frac{n-1}{n}; g\Big) + \frac{3}{4}T(r,f) + S(r,f) + S(r,g) \le \Big(6\frac{3}{4} - \Theta_f - \Theta_g + \varepsilon\Big)T(r) + S(r). \end{aligned} \tag{13}$$

Similarly we can obtain

$$nT(r,g) \le \Big(6\frac{3}{4} - \Theta_f - \Theta_g + \varepsilon\Big)T(r) + S(r). \tag{14}$$

Combining (13) and (14) we see that

$$\Big(n - 6\frac{3}{4} + \Theta_f + \Theta_g - \varepsilon\Big)T(r) \le S(r). \tag{15}$$

Since $\varepsilon > 0$ is arbitrary, (15) leads to a contradiction for $n \ge 6$.
Case 2. $m = 0$. Observe that, $-a\frac{n-1}{n}$ cannot be a member of S. So

$$\begin{aligned} &\overline{N}(r,1;G \mid\ge 2) \le N\Big(r,0;g' \mid g \ne 0, -a\frac{n-1}{n}\Big) \le \overline{N}\Big(r,\infty; \frac{g(g + a\frac{n-1}{n})}{g'}\Big) \le \\ &\le N\Big(r,\infty; \frac{g'}{g(g + a\frac{n-1}{n})}\Big) + S(r,g) \le \overline{N}(r,0;g) + \overline{N}\Big(r, -a\frac{n-1}{n}; g\Big) + S(r,g). \end{aligned}$$

Similar argument holds for F also. Hence using Lemma 6 we note that

$$\begin{aligned}
&\overline{N}_0(r,0;g^{'}) + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \leq \\
&\leq \overline{N}_0(r,0;g^{'}) + \overline{N}_E^{(2}(r,1;G) + \overline{N}_L(r,1;G) + \overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \leq \\
&\leq \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \leq \\
&\leq N(r,0;g^{'} \mid g \neq 0) + \overline{N}(r,1;G \mid\geq 2) + 2\overline{N}(r,1;F \mid\geq 2) \leq \\
&\leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + N\Big(r,0;g^{'} \mid g \neq 0, -a\frac{n-1}{n}\Big) + \overline{N}(r,0;f) + \\
&+\overline{N}(r,\infty;f) + N\Big(r,0;f^{'} \mid f \neq 0, -a\frac{n-1}{n}\Big) + S(r,f) + S(r,g) \leq \\
&\leq 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + N\Big(r,-a\frac{n-1}{n};f\Big) + 2\overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \\
&+N\Big(r,-a\frac{n-1}{n};g\Big) + S(r,f) + S(r,g).
\end{aligned} \tag{16}$$

Hence using (16), Lemmas 2 and 3 we get from the second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned}
&nT(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + N_E^{1)}(r,1;F) + \\
&+\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) - N_0(r,0;f^{'}) + S(r,f) \leq \\
&\leq 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}\Big(r,-a\frac{n-1}{n};f\Big) + \overline{N}\Big(r,-a\frac{n-1}{n};g\Big) + \\
&+\overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g) \leq \\
&\leq 4\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + 3\overline{N}(r,0;g) + \overline{N}(r,\infty;g) + 2\overline{N}\Big(r,-a\frac{n-1}{n};f\Big) + \\
&+2\overline{N}\Big(r,-a\frac{n-1}{n};g\Big) + S(r,f) + S(r,g) \leq \frac{3}{2}\Big\{N(r,0;f \mid\geq 2) + \overline{N}(r,\infty;f) + \\
&+\overline{N}\Big(r,-a\frac{n-1}{n};f\Big) + N(r,0;g \mid\geq 2) + \overline{N}(r,\infty;g) + \overline{N}\Big(r,-a\frac{n-1}{n};g\Big)\Big\} + \\
&+T(r,f) + \frac{1}{2}T(r,g) + S(r,f) + S(r,g) \leq \Big\{10\frac{1}{2} - \frac{3}{2}\Big(\Theta_f + \Theta_g\Big) + \varepsilon\Big\}T(r) + S(r).
\end{aligned} \tag{17}$$

In a similar manner we can obtain

$$nT(r,g) \leq \left\{10\frac{1}{2} - \frac{3}{2}\left(\Theta_f + \Theta_g\right) + \varepsilon\right\} T(r) + S(r). \tag{18}$$

Combining (17) and (18) we see that

$$\Big\{n - 10\frac{1}{2} + \frac{3}{2}\Big(\Theta_f + \Theta_g\Big) - \varepsilon\Big\}T(r) \leq S(r). \tag{19}$$

Since $\varepsilon > 0$ is arbitrary, (19) leads to a contradiction for $n \geq 10$.
Case 2. $H \equiv 0$. On integration we get from (2)

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B, \tag{20}$$

where A, B are constants and $A \neq 0$. From (20) we obtain

$$F \equiv \frac{(B+1)G + A - B - 1}{BG + A - B}. \tag{21}$$

Clearly (21) together with Lemma 1 yields

$$T(r,f) = T(r,g) + O(1). \tag{22}$$

Subcase 2.1. Suppose that $B \neq 0, -1$. Since F and G share (∞, ∞), it follows from (20) and (21) that ∞ is a Picard exceptional value of both f and g.

If $A - B - 1 \neq 0$, from (21) we obtain $\overline{N}(r, \frac{B+1-A}{B+1}; G) = \overline{N}(r, 0; F)$. From the above, Lemma 1 and the second fundamental theorem we obtain

$$nT(r,g) < \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{B+1-A}{B+1}; G\Big) + S(r,g) \leq \overline{N}(r,0;g)+$$
$$+\overline{N}(r,0;g+a) + \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + S(r,g) \leq 2T(r,f) + 2T(r,g) + S(r,g),$$

which in view of (22) implies a contradiction as $n \geq 5$. Thus $A - B - 1 = 0$ and hence (22) reduces to $F \equiv \frac{(B+1)G}{BG+1}$. From this we have $\overline{N}(r, \frac{-1}{B}; G) = \overline{N}(r, \infty; f)$.

Again by Lemma 1, (22) and the second fundamental theorem we have

$$nT(r,g) < \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{-1}{B}; G\Big) + S(r,g) \leq$$
$$\leq \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + S(r,g) \leq 2T(r,g) + S(r,g),$$

which in view of (22) leads to a contradiction since $n \geq 5$.

Subcase 2.2. Suppose that $B = -1$.

From (21) we obtain $F \equiv \frac{A}{-G+A+1}$. If $A+1 \neq 0$, then we obtain $\overline{N}(r, A+1; G) = \overline{N}(r, \infty; f)$. So using the same argument as used in the above subcase we can again obtain a contradiction. Hence $A + 1 = 0$ and we have $FG \equiv 1$ that means $f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2$, which is impossible by Lemma 4.

Subcase 2.3. Suppose that $B = 0$.

From (21) we obtain $F \equiv \frac{G+A-1}{A}$.

If $A - 1 \neq 0$, then we obtain $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Using (22), Lemma 1 and the second fundamental theorem we obtain

$$nT(r,g) < \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1-A;G) + S(r,g) \leq$$
$$\leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,0;g+a) + \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + S(r,g) \leq$$
$$\leq \frac{1}{2}\{N(r,0;f) + N(r,0;g)\} + \overline{N}(r,0;g+a) + \overline{N}(r,0;f+a) + S(r,f) + S(r,g) \leq$$
$$\leq \frac{3}{2}T(r,f) + \frac{3}{2}T(r,g) + S(r,g) \leq 3T(r,g) + S(r,g),$$

which is a contradiction for $n \geq 5$. So $A = 1$ and hence $F \equiv G$ that is $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$. Now the theorem follows from Lemma 5. □

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