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LEXICOGRAPHICAL ORDERING AND FIELD OPERATIONS IN THE COMPLEX PLANE

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Let $K \in \mathbb{N}, K \geq 3$. Let $\mathbb{C}_K = \mathbb{R}_0^K$ be the Cartesian product of K copies of $\mathbb{R}_0$, where $\mathbb{R}_0$ denotes the set of all nonnegative real numbers. We equip this set with arithmetic operations and show that under the condition of the so-called Cancelation Law, the space $\mathbb{C}_K$ is arithmetically isomorphic with the standard field $\mathbb{C}$ of complex numbers. Distinct $K \in \mathbb{N}$ determine distinct lexicographical orderings on $\mathbb{C}$.

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Пусть $K \in \mathbb{N}, K \geq 3$. Пусть $\mathbb{C}_K = \mathbb{R}_0^K$ является декартовым произведением K копий множества $\mathbb{R}_0$, где $\mathbb{R}_0$ — множество всех неотрицательных вещественных чисел. Мы снабдим это множество арифметическими операциями и покажем, что при условии так называемого "сокращающего закона", пространство $\mathbb{C}_K$ арифметически изоморфно стандартному полю $\mathbb{C}$ комплексных чисел. Разные $K \in \mathbb{N}$ определяют разные лексикографические порядки на $\mathbb{C}$.

1. Introduction. Concerning terminology. We use mixed algebraic-geometrical-analytical terminology similarly as it is commonly accepted in the books on analytical geometry. We hope this will not lead to any misunderstanding or ambiguity. For instance, we use terms: "point — vector — number" or "line — real numbers", etc., as synonyms and the using of a term depends on the situation, where it is the most apt for the best understanding or illumination.

Let $\mathbb{R}_0 = (0, \infty) \cup \{0\} = [0, \infty)$ be the real half-line with all structures heredited from the real line $\mathbb{R}$. The set $\mathbb{R}_0$ will be considered as the *non-polar*, or equivalently, the *1-polar* set of objects. We speak about elements of $\mathbb{R}_0$ also as about the *absolute values*.

The standard geometrical model of the set of all real numbers $\mathbb{R}$ is a line. Among many constructions, $\mathbb{R}$ can be described starting with *two distinct points*. We model the system of (standard) complex numbers $\mathbb{C}$ as a plane starting with *three points not lying on a line*. In the following sections we give this construction for which the three points are apexes of an equally sided triangle, hence we call this structure to be the 3-*polar complex plane*. Then we generalize the construction of the 3-polar complex plane from triangle to the K-equally sided planar polygon, where K is an arbitrary natural number $K \geq 3$. Thus we obtain the K-polar representations $\mathbb{C}_K$ of the field $\mathbb{C}$. All these representations $\mathbb{C}_K$ (and the set $\mathbb{C}$) are isomorphic concerning to arithmetic operations of addition, subtraction, multiplication, and

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division. In other words, each given complex number $a+bi \in \mathbb{C}$ can be written in the same time as a sequence of K-tuples of non-negative numbers, $K \geq 3$, i.e., $a+bi = (x_1', x_2', x_3')_{\mathbb{C}_3} = (x_1'', x_2'', x_3'', x_4'')_{\mathbb{C}_4} = (x_1''', x_2''', x_3''', x_4''', x_5''')_{\mathbb{C}_5} = \ldots$ for suitable x-s in $\mathbb{R}_0$. However stress this already in the introduction, that these K-tuples are representatives of classes of equivalence with a congruence given with the so-called Cancellation law.

The crucial fact for construction of $\mathbb{C}_K$, $K \in \mathbb{N}, K \geq 3$ is that $\mathbb{R}_0$ is a semi-field with zero. A semi-field X is an algebraic structure with binary operations of addition $(+)$ and multiplication $(\cdot)$, where $(X,+)$ is a commutative semi group, $(X,\cdot,1)$ is a multiplicative group with the unit 1, and multiplication is distributive with respect to addition from both sides. An ordered (semi) field is a (semi) field together with a total ordering $\leq$ of its elements that is compatible with the (semi) field operations, i.e., for $a,b,c \in X$,

(i) if $a \leq b$ then $a+c \leq b+c$, and

(ii) if $0 \leq a$ and $b \leq c$ then $a \cdot b \leq a \cdot c$.

The spaces $\mathbb{C}_K$ are lexicographically ordered. The lexicographical ordering of K-tuples is defined as usually. For a review of semi-fields, c.f. ([4]).

2. A tripolar complex space $\mathbb{C}_3$.

2.1. The algebra of polar operators (poles). Let $\mathcal{K} = [1,2,\ldots,K]$ be a sequence of indexes. In Sections 2, 3, and 4, $K=3$. Let $\mathbb{A}_K = [A_K^{(k)} \mid k \in \mathcal{K}]$ be a sequence of three points creating an equal-sided triangle. We introduce three operations on the set $\mathbb{A}_K$.

(a) Nullary operation. We put $I = A_K^{(1)}$.

(b) Orbit. $\mathbb{A}_K$ contains the cycled monounary function (a cycled orbit):

$$\tau : A_K^{(1)} \to A_K^{(2)} \to A_K^{(3)} \to A_K^{(1)}.$$

(c) $\otimes$-multiplication. The binary operation is defined by the following Latin square table.

$\otimes$	$A_K^{(1)}$	$A_K^{(2)}$	$A_K^{(3)}$
$A_K^{(1)}$	$A_K^{(1)}$	$A_K^{(2)}$	$A_K^{(3)}$
$A_K^{(2)}$	$A_K^{(2)}$	$A_K^{(3)}$	$A_K^{(1)}$
$A_K^{(3)}$	$A_K^{(3)}$	$A_K^{(1)}$	$A_K^{(2)}$

This way the set of poles becomes an algebra, we write $(\mathbb{A}_K, I, \tau, \otimes)$, or simply $\mathbb{A}_K$. Remark that $(\mathbb{A}, \tau)$ is a monounary algebra called a 3-element cycle, cf., e.g., ([1]).

In subsection 2.4 we will extend these structures to the whole plane Π.

Remark 1. No addition-like operation is defined on $\mathbb{A}_K$.

We have introduced the algebra $\mathbb{A}_K$ of poles, a structure over three given points. We "fill in the rest territory" of the plane Π with the help of (a) polarized elements, and (b) operation of addition, i.e. via linear-like envelope [the notion "linear envelope" is not suitable since we are dealing with the semi-field $\mathbb{R}_0$ which is not a vector space (field)].

2.2. Polarized elements.

Definition 1. Denote by θ the center of gravity of the triangle $[A_K^1, A_K^2, A_K^3]$. We say that the point $z \in \Pi, z \neq \theta$, is $A_K^{(k)}$*-polarized*, $k \in \mathcal{K}$, if the points: $\{\theta, z, A_K^{(k)}\}$, lie on a half line with θ as the first point. We write this $z = A_K^{(k)}(x)$, where $x \in \mathbb{R}_0$ is an absolute value measuring the length from the point z to θ. The length x of the point $z = A_K^{(k)}$ is one (the unit length) for every $k \in \mathcal{K}$. For every $k \in \mathcal{K}$, we put $A_K^{(k)}(0) = \theta$ and $\theta \in \Pi$ is a *non-polar element*.

In other words, z is $A_K^{(k)}$*-polarized*, if there exists $x \in \mathbb{R}_0 \setminus \{0\}$ such that $z = A_K^{(k)}(x) = x \cdot A_K^{(k)}(1) = x \cdot A_K^{(k)}$, $k \in \mathcal{K}$.

Note that $A_K^{(k)}$ in $A_K^{(k)}(x)$ is mentioned as an operator (=sign) applied to a non-negative number while, in the expression $x \cdot A_K^{(k)}$, we have the multiplication of a non-negative number and the complex number.

Remark 2. In the case of the real line($K = 2$), the algebra of poles is very simple $\tau : A_K^{(1)} \to A_K^{(2)} \to A_K^{(1)}$,

$\otimes$	$A_K^{(1)}$	$A_K^{(2)}$
$A_K^{(1)}$	$A_K^{(1)}$	$A_K^{(2)}$
$A_K^{(2)}$	$A_K^{(2)}$	$A_K^{(1)}$.

Instead of $A_2^{(1)}$ and $A_2^{(2)}$, we use the terms $(+)$ and $(-)$, respectively, and speak about *bipolarity*. So, the poles in the cases $K \geq 3$ could be called and understand also to be "generalized signs."

2.3. Addition in Π**.** Let $A_K^{(1)}(\mathbb{R}_0) = I(\mathbb{R}_0) \subset \mathbb{C}$ denote the image of $\mathbb{R}_0$ under its natural embedding, i.e. $A_K^{(1)}(\mathbb{R}_0) = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}_0, y = 0\}$. Therefore, every $z = a + bi \in \mathbb{C}$, $z \neq 0$, can be represented as a sum of 3-polarized elements (vectors) in the form (and vice versa):

$$z = a + bi = \mathbf{x}_{\mathbb{A}_K} = A_K^{(1)}(x_1) + A_K^{(2)}(x_2) + A_K^{(3)}(x_3) = \sum_{k \in \mathcal{K}} A_K^{(k)}(x_k), \tag{1}$$

where $a, b \in \mathbb{R}$, i is the imaginary unit; $\mathbf{x}_{\mathbb{A}_K} = (x_1, x_2, x_3)_{\mathbb{A}_K} \in \mathbb{R}_0^3$ is the vector of three projections (three 3-polar coordinates) of the point z along the three half-lines given with the origin point and the second points are $(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

The addition of elements in $\mathbb{C}_3$ is defined coordinatewisely, $\mathbf{x} \oplus \mathbf{y} = A_K^{(1)}(x_1 + y_1) + A_K^{(2)}(x_2 + y_2) + A_K^{(3)}(x_3 + y_3)$.

2.4. Extension of the algebra of poles to the whole plane. For every $z \in \Pi$,
(a) $Iz = A_K^{(1)} z = z$.
(b) The orbit functions can be depicted as the 0, $2\pi/3$, and $4\pi/3$ radian anti-clockwise rotations of any element $z \in \Pi$ around the centre of gravity of our equally-sided triangle, respectively.

Knowing the elementary properties of complex numbers, rotations $A_K^{(1)}$, $A_K^{(2)}$, and $A_K^{(3)}$ are equivalently nothing else than the multiplications of any complex number $z \in \mathbb{C}$ by complex numbers $(1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, respectively.
(c) For every $i, j \in \mathcal{K}$,

$$(A_K^{(i)} \otimes A_K^{(j)})(z) := A_K^{(i)}[A_K^{(j)}(z)].$$

This way, we extended the acting of the the algebra $(\mathbb{A}_K, I, \tau, \otimes)$ from the set of three points to the whole plane Π, we write $(\mathbb{A}_K, \mathbb{R}_0, I, \tau, \otimes, \oplus)$. We preserve for the new algebra the same short denotation, $\mathbb{A}_K$. Our following aim will be to introduce the subtraction and division on this algebra isomorphically satisfying the usual properties of the complex field.

3. The 3-cancelation law, classes of equivalence. Instead of the numbers we will deal with classes of equivalences. As the congruence we will use the so-called 3-*cancelation law*. The 3-cancelation law enables us to introduce all field (arithmetic) operations (addition, subtraction, multiplication, and division), c.f. Section 4.

Definition 2. (The 3-cancelation law) For every $z \in \Pi$, $A_K^{(1)}z + A_K^{(2)}z + A_K^{(3)}z = \theta = (0,0,0)$.

According to the 3-cancelation law, the triple of coordinates $(x_1, x_2, x_3) \in \mathbb{R}_0^3$ is ambiguous in (1) since for any arbitrary $y \in \mathbb{R}_0$, $\sum_{k\in\mathcal{K}} A_K^{(k)}(x_k + y) = \sum_{k\in\mathcal{K}} A_K^{(k)} x_k$. To make representation of elements in (1) and operations with them unique, it is needed the 3-cancellation law. In short, we are working with classes of equivalences (representatives of these classes). The congruence is given with the 3-cancellation law.

In short, we are working with classes of equivalences (representatives of these classes). The congruence is given with the 3-cancelation law.

Lemma 1. *If the 3-cancelation law holds, then the results of the operations of addition and multiplication over theses equivalence classes does not depend on the choice of representatives of the equivalence classes.*

Remark 3. The lemma holds also for subtraction and division.

Proof. Let $x_1, x_2, x_3 \in \mathbb{R}_0$, $a, b, c \in \mathbb{R}_0$.

Then

$$\begin{aligned} A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3 &= A_K^{(1)}(x_1+a) + A_K^{(2)}(x_2+a) + A_K^{(3)}(x_3+a) = \\ &= [A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3] \oplus [A_K^{(1)}a + A_K^{(2)}a + A_K^{(3)}a]. \end{aligned}$$

Also,

$$\begin{aligned} &[A_K^{(1)}(x_1+a) + A_K^{(2)}(x_2+a) + A_K^{(3)}(x_3+a)] \oplus [A_K^{(1)}(y_1+b) + A_K^{(2)}(y_2+b) + A_K^{(3)}(y_3+b)] = \\ &\quad = A_K^{(1)}(x_1+y_1+c) + A_K^{(2)}(x_1+y_2+c) + A_K^{(3)}(x_1+y_3+b). \end{aligned}$$

For the multiplication,

$$\begin{aligned} &[A_K^{(1)}(x_1+a) + A_K^{(2)}(x_2+a) + A_K^{(3)}(x_3+a)] \odot [A_K^{(1)}(y_1+b) + A_K^{(2)}(y_2+b) + A_K^{(3)}(y_3+b)] = \\ &\quad = [A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3] \odot [A_K^{(1)}y_1 + A_K^{(2)}y_2 + A_K^{(3)}y_3]. \end{aligned}$$

□

3.1. The bijection between the standard $\mathbb{C}$ and $\mathbb{C}_3$. In this section we prove the existence of a bijection between the standard $\mathbb{C}$ and $\mathbb{C}_3$ of equivalence classes with the 3-cancellation law. We choose representatives of equivalence classes with the condition that at least one 3-polar coordinate is equal to zero.

The 3-polarity axes be given with vectors $A_K^{(1)} = (1,0)$, $A_K^{(2)} = (-\frac{1}{2}, +\frac{\sqrt{3}}{2})$, and $A_K^{(3)} = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, i.e.

$$z = (1,0)x_1 + \Big(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\Big)x_2 + \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big)x_3 = (x_1, x_2, x_3)_{\mathbb{A}_K}.$$

Let us split the complex plane $\mathbb{C}$ into three parts:

$$P_1 = \{(a,b) \in \mathbb{C} \mid (a,b) = (0, x_2, x_3)_{\mathbb{A}_K}, x_2 > 0, x_3 \geq 0\},$$

$$P_2 = \{(a,b) \in \mathbb{C} \mid (a,b) = (x_1, 0, x_3)_{\mathbb{A}_K}, x_3 > 0, x_1 \geq 0\},$$
$$P_3 = \{(a,b) \in \mathbb{C} \mid (a,b) = (x_1, x_2, 0)_{\mathbb{A}_K}, x_1 > 0, x_2 \geq 0\}.$$

Here P_1, P_2, P_3 are written when $z \in \mathbb{C}$ are expressed in the trigonometric form:

$$P_3 = \{(x,\varphi) \in \mathbb{C} \mid (x,\varphi) = (x_1,\varphi), \varphi \in [0, 2\pi/3)\},$$
$$P_1 = \{(x,\varphi) \in \mathbb{C} \mid (x,\varphi) = (x_2,\varphi), \varphi \in [2\pi/3, 4\pi/3)\},$$
$$P_2 = \{(x,\varphi) \in \mathbb{C} \mid (x,\varphi) = (x_3,\varphi), \varphi \in [4\pi/3, 6\pi/3 = 2\pi)\}.$$

(A) *Direction* $(a+bi) \to (x_1, x_2, x_3)_{\mathbb{A}_K}$.

After doing some exercises in the elementary analytical geometry, we obtain the following explicit expressions between the standard algebraic and the 3-polarity vector representation (1) of a number $(a,b) \in \mathbb{C}$:

- if $z = (a,b) \in P_1$, then

$$\mathbf{x}_{\mathbb{A}_K} = A_K^{(2)}\Big|\Big(\frac{a}{2} - \frac{b}{2\sqrt{3}}, \frac{b}{2} - \frac{a\sqrt{3}}{2}\Big)\Big| + A_K^{(3)}\Big|\Big(\frac{a}{2} + \frac{b}{2\sqrt{3}}, \frac{b}{2} + \frac{a\sqrt{3}}{2}\Big)\Big| =$$
$$= A_K^{(2)}\Big(-a + \frac{b}{\sqrt{3}}\Big) + A_K^{(3)}\Big(-a - \frac{b}{\sqrt{3}}\Big);$$

- if $z = (a,b) \in P_2$, then

$$\mathbf{x}_{\mathbb{A}_K} = A_K^{(1)}\Big|\Big(a - \frac{b}{\sqrt{3}}, 0\Big)\Big| + A_K^{(3)}\Big|\Big(\frac{b}{\sqrt{3}}, b\Big)\Big| = A_K^{(1)}\Big(a - \frac{b}{\sqrt{3}}\Big) + A_K^{(3)}\Big(-\frac{2b}{\sqrt{3}}\Big);$$

- if $z = (a,b) \in P_3$, then

$$\mathbf{x}_{\mathbb{A}_K} = A_K^{(1)}\Big|\Big(a + \frac{b}{\sqrt{3}}, 0\Big)\Big| + A_K^{(2)}\Big|\Big(-\frac{b}{\sqrt{3}}, b\Big)\Big| = A_K^{(1)}\Big(a + \frac{b}{\sqrt{3}}\Big) + A_K^{(2)}\Big(\frac{2b}{\sqrt{3}}\Big).$$

where within the absolute value signs, there are projection points of (a,b) to the corresponding coordinate 3-polarity coordinate axes described above.

(B) *Direction* $(x_1, x_2, x_3)_{\mathbb{A}_K} \to (a+bi)$.

$$(x_1,x_2,x_3)_{\mathbb{A}_K} = (1,0)x_1 + \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\Big)x_2 + \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big)x_3 = \Big(x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3, \frac{\sqrt{3}}{2}x_2 - \frac{\sqrt{3}}{2}x_3\Big).$$

4. Operations in $\mathbb{C}_3$ in detail. Again, in addition to the 3-cancelation law, we suppose that at least one of the 3-polar coordinates is zero.

4.1. Addition $\oplus$. This operation is coordinate-wise like in Euclidean spaces. We define

$$\mathbf{x}_{\mathbb{A}_K} \oplus \mathbf{y}_{\mathbb{A}_K} = (A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3) \oplus (A_K^{(1)}y_1 + A_K^{(2)}y_2 + A_K^{(3)}y_3) :=$$
$$:= A_K^{(1)}(x_1 + y_2) + A_K^{(2)}(x_2 + y_2) + A_K^{(3)}(x_3 + y_3),$$

where $\mathbf{x}_{\mathbb{A}_K} = (x_1,x_2,x_3)_{\mathbb{A}_K}; x_1,x_2,x_3 \in \mathbb{R}_0$; $\mathbf{y}_{\mathbb{A}_K} = (y_1,y_2,y_3)_{\mathbb{A}_K}; y_1,y_2,y_3 \in \mathbb{R}_0$.

4.2. Subtraction $\ominus$. For every $x \in \mathbb{R}_0$ we have by the 3-cancelation law

$$-A_K^{(1)}x = A_K^{(2)}x + A_K^{(3)}x,\ -A_K^{(2)}x = A_K^{(1)}x + A_K^{(3)}x,\ -A_K^{(3)}x = A_K^{(1)}x + A_K^{(2)}x.$$

We define the operation of subtraction in $\mathbb{C}_3$ as follows:

$$(A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3) \ominus (A_K^{(1)}y_1 + A_K^{(2)}y_2 + A_K^{(3)}y_3) :=$$
$$:= (A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3) \oplus (A_K^{(2)}y_1 + A_K^{(3)}y_1) \oplus (A_K^{(1)}y_2 + A_K^{(3)}y_2) \oplus (A_K^{(1)}y_3 + A_K^{(2)}y_3) =$$
$$= A_K^{(1)}(x_1 + y_2 + y_3) + A_K^{(2)}(x_2 + y_1 + y_3) + A_K^{(3)}(x_3 + y_1 + y_2).$$

4.3. Multiplication $\odot$. This operation of multiplication on $\mathbb{C}_3$ is defined in accordance with both the table of the operation $\otimes$ for the algebra $\mathbb{A}_K$ of polarities, and the multiplication in $\mathbb{R}_0$. Namely,

$$\mathbf{x}_{\mathbb{A}_K} \odot \mathbf{y}_{\mathbb{A}_K} = \Big(\sum_{i\in\mathcal{K}} A_K^{(i)}x_i\Big) \odot \Big(\sum_{j\in\mathcal{K}} A_K^{(j)}y_j\Big) := \sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} \Big[A_K^{(i)} \otimes A_K^{(j)}\Big](x_i \cdot y_j),$$

where $A_K^{(i)}, A_K^{(j)} \in \mathbb{A}_K; x_i, x_j = \mathbb{R}_0; i, j \in \mathcal{K}$. Taking into the account the distributive law, we can say that the behavior of the multiplication $\mathbf{x}_{\mathbb{A}_K} \otimes \mathbf{y}_{\mathbb{A}_K}$ is "polynomial-like."

As a particular case we pick out the multiplication of elements $a \in \mathbb{R}_0$ by $\mathbf{x}_{\mathbb{A}_K}$

$$a \odot \mathbf{x}_{\mathbb{A}_K} = (A^{(1)}a) \cdot \Big(\sum_{k\in\mathcal{K}} A_K^{(k)}x_k\Big) = \sum_{k\in\mathcal{K}} A_K^{(k)}(a \cdot x_k).$$

4.4. Conjugation *. The unary operation of conjugation enables us to introduce the operation of division on Let $\mathbf{x} = A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3$, where $(x_1, x_2, x_3) \in \mathbb{R}_0^3$. We define $\mathbf{x}^* := A_K^{(1)}x_1 + A_K^{(2)}x_3 + A_K^{(3)}x_2$.

Lemma 2. *Let* $\mathbf{x} = A_K^{(1)}x_1 + A_K^{(2)}x_2 + A_K^{(3)}x_3$, $(x_1, x_2, x_3) \in \mathbb{R}_0^3$. *Let* $\mathbf{y} = A_K^{(1)}y_1 + A_K^{(2)}y_2 + A_K^{(3)}y_3$, $(y_1, y_2, y_3) \in \mathbb{R}_0^3$. *Then*

1. $(\mathbf{x}^*)^* = \mathbf{x}$,
2. $(\mathbf{x} \oplus \mathbf{y})^* = \mathbf{x}^* \oplus \mathbf{y}^*$,
3. $(\mathbf{x} \odot \mathbf{y})^* = \mathbf{x}^* \odot \mathbf{y}^*$,
4. $\mathbf{y} \odot \mathbf{y}^* = A_K^{(1)}\left[\frac{(y_1-y_2)^2+(y_1-y_3)^2+(y_2-y_3)^2}{2}\right]$.

Proof. The proofs of items 1, 2, 3 are exercises in algebra, we let them to the reader. Prove the last statement 4. Suppose without loss of generality that $y_1 \geq y_2 \geq y_3$ (the other possibilities can be proved similarly). We have

$$\mathbf{y}_{\mathbb{A}_K} \odot (A_K^{(1)}y_1 + A_K^{(2)}y_3 + A_K^{(3)}y_2) = (A_K^{(1)}y_1 + A_K^{(2)}y_2 + A_K^{(3)}y_3) \odot (A_K^{(1)}y_1 + A_K^{(2)}y_3 + A_K^{(3)}y_2) =$$

using the multiplication table and the distributive law, we obtain after some elementary calculation

$$= A_K^{(1)}(y_1^2 + y_2^2 + y_3^2) + A_K^{(2)}(y_1y_2 + y_1y_3 + y_2y_3) + A_K^{(3)}(y_1y_2 + y_1y_3 + y_2y_3).$$

If

$$0 \leq y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3 - y_3y_1, \tag{2}$$

then by the cancellation law,

$$\mathbf{y}_{\mathbb{A}_K} \odot (A_K^{(1)}y_1 + A_K^{(2)}y_3 + A_K^{(3)}y_2) = A_K^{(1)}\left[(y_1^2 + y_2^2 + y_3^2) - (y_1y_2 + y_1y_3 + y_2y_3)\right].$$

Indeed,

$$
\begin{gathered}
y_1^2+y_2^2+y_3^2-y_1y_2-y_2y_3-y_3y_1=\\
=\left(\frac{1}{2}y_1^2-y_1y_2+\frac{1}{2}y_2^2\right)+\left(\frac{1}{2}y_1^2-y_1y_3+\frac{1}{2}y_3^2\right)+\left(\frac{1}{2}y_2^2-y_2y_3+\frac{1}{2}y_3^2\right)=\\
=\frac{1}{2}\left[(y_1-y_2)^2+(y_1-y_3)^2+(y_2-y_3)^2\right]\in\mathbb{R}_0. \qquad (3)
\end{gathered}
$$

□

4.5. Division $\oslash$. Let $\mathbf{x}_{\mathbb{A}_K}\in\mathbb{C}_3,\mathbf{y}_{\mathbb{A}_K}\in\mathbb{C}_3,\mathbf{x}_{\mathbb{A}_K}\neq\theta$. Division is defined by $\mathbf{x}_{\mathbb{A}_K}\oslash\mathbf{y}_{\mathbb{A}_K}:=(\mathbf{x}_{\mathbb{A}_K}\odot A_K^{(1)}(1))\oslash\mathbf{y}_{\mathbb{A}_K}=\mathbf{x}_{\mathbb{A}_K}\odot(A_K^{(1)}(1)\oslash\mathbf{y}_{\mathbb{A}_K})$. We find this element.

Theorem 1. *Let* $\mathbf{y}_{\mathbb{A}_K}=A_K^{(1)}y_1+A_K^{(2)}y_2+A_K^{(3)}y_3\neq(0,0,0)$. *Then*

$$
(A_K^{(1)}(1)\oslash\mathbf{y}_{\mathbb{A}_K})=\frac{\mathbf{y^*}_{\mathbb{A}_K}}{\mathbf{y}_{\mathbb{A}_K}\odot\mathbf{y^*}_{\mathbb{A}_K}}=\frac{A_K^{(1)}y_1+A_K^{(2)}y_3+A_K^{(3)}y_2}{y_1^2+y_2^2+y_3^2-y_1y_2-y_2y_3-y_3y_1}.
$$

Proof. Applying the point 4. of the previous lemma, we obtain the assertion.

The condition $\mathbf{y}_{\mathbb{A}_K}\neq(0,0,0)_{\mathbb{A}_K}$ means that one of the elements y_1,y_2,y_3 is positive, what is equivalent to the condition $y_1^2+y_2^2+y_3^2>0$. The equality $y_1^2+y_2^2+y_3^2-y_1y_2-y_2y_3-y_3y_1=0$ is possible if and only if $y_1=y_2=y_3$. Using the cancelation law, we obtain that $\mathbf{y}_{\mathbb{A}_K}=(0,0,0)$, which is in the contradiction with the assumption $\mathbf{y}_{\mathbb{A}_K}\neq(0,0,0)_{\mathbb{A}_K}$. □

Definition 3. Let the plane Π be equipped with the operation of addition and multiplication as above. The system of all equivalence classes $(\mathbb{A}_3,\mathbb{R}_0,I,\tau,\oplus,\odot)$ with the congruence given with the 3-cancellation law is called the *3-polar complex plane*, we write $\mathbb{C}_3$.

Theorem 2. *For elements* $(a,b)\to(x_1,x_2,x_3)\in\mathbb{C}_3,(c,d)\to(y_1,y_2,y_3)\in\mathbb{C}_3$, *where* $(a,b)\in\mathbb{C},(c,d)\in\mathbb{C}$, *there is an isomorphism* $(\mathbb{C},\cdot)\to(\mathbb{C}_3,\odot)$: $(a,b)\cdot(c,d)\to(x_1,x_2,x_3)_{\mathbb{A}_3}\odot(y_1,y_2,y_3)_{\mathbb{A}_3}$.

Proof. Let $(a,b)\in P_3,(c,d)\in P_3$ and $(a,b)=(x_1,x_2,x_3)\in\mathbb{C}_3,(c,d)=(y_1,y_2,y_3)\in\mathbb{C}_3$. We have

$$
\begin{gathered}
(x_1,x_2,x_3)_{\mathbb{A}_K}\odot(y_1,y_2,y_3)_{\mathbb{A}_K}=\\
=\left[A_K^{(1)}\Big(a+\frac{b}{\sqrt{3}}\Big)+A_K^{(2)}\Big(\frac{2b}{\sqrt{3}}\Big)\right]\cdot\left[A_K^{(1)}\Big(c+\frac{d}{\sqrt{3}}\Big)+A_K^{(2)}\Big(\frac{2d}{\sqrt{3}}\Big)\right]=\\
=A_K^{(1)}\otimes A_K^{(1)}\Big(a+\frac{b}{\sqrt{3}}\Big)\Big(c+\frac{d}{\sqrt{3}}\Big)+A_K^{(1)}\otimes A_K^{(2)}\Big(a+\frac{b}{\sqrt{3}}\Big)\Big(\frac{2d}{\sqrt{3}}\Big)+\\
+A_K^{(2)}\otimes A_K^{(1)}\Big(\frac{2b}{\sqrt{3}}\Big)\Big(c+\frac{d}{\sqrt{3}}\Big)+A_K^{(2)}\otimes A_K^{(2)}\Big(\frac{2b}{\sqrt{3}}\Big)\Big(\frac{2d}{\sqrt{3}}\Big)=\\
=A_K^{(1)}\Big(ac+\frac{cb}{\sqrt{3}}+\frac{ad}{\sqrt{3}}+\frac{bd}{\sqrt{3}^2}\Big)+A_K^{(2)}\Big(\frac{2ad}{\sqrt{3}}+\frac{2bd}{\sqrt{3}^2}+\frac{2bc}{\sqrt{3}}+\frac{2bd}{\sqrt{3}^2}\Big)+\\
+A_K^{(3)}\Big(\frac{4bd}{\sqrt{3}^2}\Big)=(1,0)\Big(ac+\frac{cb+ad}{\sqrt{3}}+\frac{bd}{3}\Big)+(-1/2,\sqrt{3}/2)\Big(\frac{2ad+2bc}{\sqrt{3}}+\frac{4bd}{3}\Big)+\\
+(-1/2,-\sqrt{3}/2)\Big(\frac{4bd}{3}\Big)=(ac-bd,ad+bc)=(a,b)\cdot(b,d).
\end{gathered}
$$

The other combinations (P_1, P_1), (P_1, P_2), (P_2, P_1), (P_1, P_3), (P_3, P_1), (P_2, P_2), (P_2, P_3), (P_3, P_2) can be verified analogously. □

The proof of the previous theorem was based on the fact that the operators $A_K^{(k)}, k \in \mathcal{K}$, can be explicitly expressed as a multiplication of complex numbers. In other words, the ("classical") complex numbers are "translators" between various "dimensions:" $K \to 2 \to H, K \neq H; K, H \in \mathbb{N}$.

The isomorphisms of operations $\oplus \to +, \ominus \to -, \oslash \to :$ can be verified similarly.

Subsumed, we proved the following theorem.

Theorem 3. *The 3-polarized space of complex numbers $\mathbb{C}_3$ is equipped with the algebraic operations of addition, subtraction, multiplication, and division. The operations of the standard complex field $\mathbb{C}$ and these operations over $\mathbb{C}_3$ are isomorphic.*

5. The space $\mathbb{C}_4$. Passing to the case $K = 4$ (the square), the $\otimes$-multiplication table is defined by the following Latin square

$\otimes$	$A_4^{(1)}$	$A_4^{(2)}$	$A_4^{(3)}$	$A_4^{(4)}$
$A_4^{(1)}$	$A_4^{(1)}$	$A_4^{(2)}$	$A_4^{(3)}$	$A_4^{(4)}$
$A_4^{(2)}$	$A_4^{(2)}$	$A_4^{(3)}$	$A_4^{(4)}$	$A_4^{(1)}$
$A_4^{(3)}$	$A_4^{(3)}$	$A_4^{(4)}$	$A_4^{(1)}$	$A_4^{(2)}$
$A_4^{(4)}$	$A_4^{(4)}$	$A_4^{(1)}$	$A_4^{(2)}$	$A_4^{(3)}$

So, every $z = a + bi \in \mathbb{C}, z \neq 0$, can be represented in the form

$$z = a + bi = A_4^{(1)} x_1 + A_4^{(2)} x_2 + A_4^{(3)} x_3 + A_4^{(4)} x_4, \tag{4}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}_0$ are obtained as parallel planar projection similarly as in the 3-polarity model $\mathbb{C}_3$ of $\mathbb{C}$. However, the situation is analogous to the case $K = 3$ only for the planar case. The spacial (tetrahedral) case for $K = 4$ is not considered in this paper.

In the following section 6, we bring the formulae for arbitrary planar models $\mathbb{C}_K, K \geq 3$.

In the usual denotation, the table of multiplication of polarities for $\mathbb{C}_4$ (Latin square) in the planar case is as follows:

$\otimes$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

which well coincide with the table of multiplication for the classical model of the complex plane $\mathbb{C}$. We see here that the poles $\{1, i, -1, -i\}$ form a square in a plane.

In the (planar) case of $\mathbb{C}_4$, the condition of uniqueness may sound as follows: at least two of elements $x_1, x_2, x_3, x_4 \in \mathbb{R}_0$ are zeros in the expression (4). It cannot be obtained with the 4-Cancellation Law, so this rule is replaced with the following two conditions: for every $x, y \in \mathbb{R}_0$, $A_4^{(1)} x + A_4^{(3)} x = 0$, and $A_4^{(2)} y + A_4^{(4)} y = 0$, which both, of course, implies the 4-Cancellation Law: for every $x \in \mathbb{R}_0$,

$$A_4^{(1)} x + A_4^{(2)} x + A_4^{(3)} x + A_4^{(4)} x = (0, 0, 0, 0).$$

In short, we write $z = (x_1, x_2, x_3, x_4)_{\mathbb{A}_4} = \mathbf{x}_{\mathbb{A}_4}$ and (4) is a 4-polarity expression of $z \in \mathbb{C}, z \neq (0, 0, 0, 0)_{\mathbb{A}_K}$ as an element of $\mathbb{C}_4$.

We can rewrite all the results from the case for $K = 3$ to the case $K = 4$, in particular, that $\mathbb{C}_4$ is also isomorphic to $\mathbb{C}$. The conjugate to the element $z = (x_1, x_2, x_3, x_4)_{\mathbb{A}_4}$ is $z^* = (x_1, x_4, x_3, x_2)_{\mathbb{A}_4}$ and

$$z \odot z^* = A^{(1)}(x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_3 - 2x_2x_4) = A^{(1)}[(x_1 - x_3)^2 + (x_2 - x_4)^2],$$

where we choose those $(x_1 - x_3)$ or $(x_3 - x_1)$ which are in $\mathbb{R}_0$. Then the operation of division is well defined and the analogue of Theorem 3 holds for $\mathbb{C}_4$, i.e. the set of all equivalence classes $\mathbb{C}_4$ is arithmetically isomorphic to $\mathbb{C}$.

Definition 4. Let the plane Π be equipped with the operation of addition and multiplication as above. Let the 4-cancelation law hold. The system $(\mathbb{A}_4, \mathbb{R}_0, \oplus, \odot)$ is called the *4-polar complex plane*, we write $\mathbb{C}_4$.

6. Multi-polar complex space.

6.1. The notion of multi polar complex space. This process can be extended for $K = 4, 5, \ldots$, and not immersing ourself into details the following definition has a sense.

Definition 5. Let a plane Π be equipped with the operation of multiplication given by the table below. Let the K-cancelation law hold. The system $(\mathbb{A}_K, \mathbb{R}_0, \oplus, \odot)$ is called the *K-polar complex plane*, we write $\mathbb{C}_K$, $K \in \mathbb{N}, K \geq 3$.

Definition 6. Let $\mathbb{R}_0$ be the space of 1-polar elements for all spaces $\mathbb{C}_3, \mathbb{C}_4, \ldots, \mathbb{C}_K, \ldots$. Let the multiplication tables of polarity operators are Latin squares

$$\begin{array}{c|cccccc}
\otimes & A_K^{(1)} & A_K^{(2)} & A_K^{(3)} & \ldots & A_K^{(K)} \\
\hline
A_K^{(1)} & A_K^{(1)} & A_K^{(2)} & A_K^{(3)} & \ldots & A_K^{(K)} \\
A_K^{(2)} & A_K^{(2)} & A_K^{(3)} & \ldots & A_K^{(K)} & A_K^{(1)} \\
A_K^{(3)} & A_K^{(3)} & \ldots & A_K^{(K)} & A_K^{(1)} & A_K^{(2)} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_K^{(K)} & A_K^{(K)} & A_K^{(1)} & A_K^{(2)} & \ldots & A_K^{(K-1)},
\end{array}$$

for every $K \geq 3$.

Then the sequence of spaces $[\mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_5, \ldots, \mathbb{C}_K, \ldots]$ each isomorphic to $\mathbb{C}$, is said to be the *multipolar complex space* and is denoted by $\mathcal{M}\mathbb{C}$.

For $z = a + bi \in \mathbb{C}$, an element of $\mathcal{M}\mathbb{C}$ ($\mathcal{M}$ from the word "multipolar") is a sequence $[\mathbf{x}_{\mathbb{A}_3}, \mathbf{x}_{\mathbb{A}_4}, \ldots, \mathbf{x}_{\mathbb{A}_K}, \ldots] \in \mathcal{M}\mathbb{C}$, where $\mathbf{x}_{\mathbb{A}_K} \in \mathbb{C}_K$ and $\mathbf{x}_{\mathbb{A}_3} = \mathbf{x}_{\mathbb{A}_4} = \cdots = \mathbf{x}_{\mathbb{A}_K} = \cdots = a+bi \in \mathbb{C}$ in the sense of isomorphisms of spaces $\mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_5, \ldots, \mathbb{C}_K, \ldots$.

In particular, in this sense $(0,0,0)_{\mathbb{C}_3} = (0,0,0,0)_{\mathbb{C}_4} = \cdots = (0,0,\ldots,0)_{\mathbb{C}_K} = \ldots$,

$$\Theta = [(0,0,0)_{\mathbb{C}_3}, (0,0,0,0)_{\mathbb{C}_4}, \ldots, (0,0,\ldots,0)_{\mathbb{C}_K}, \ldots] \in \mathcal{M}\mathbb{C}$$

and we will call this object the $\mathcal{M}\mathbb{C}$-*null*. According to the K-cancelation laws,

$$\Theta = [(r_3, r_3, r_3)_{\mathbb{C}_3}, (r_4, r_4, r_4, r_4)_{\mathbb{C}_4}, \ldots, (r_K, r_K, \ldots, r_K)_{\mathbb{C}_K}, \ldots],$$

$r_k \in \mathbb{R}_0, k \geq 3, k \in \mathbb{N}$.

6.2. Evaluation of coordinates, $K \geq 3$. Now we evaluate the coordinates for $K \geq 3$ in the planar case. Supposing that all but two K-polar coordinates are zero, an arbitrary complex number $a + bi = (\rho, \varphi) \in \mathbb{C}$, (algebraic and polar forms, respectively), $\varphi = \arctan \frac{b}{a}$, $\rho =$

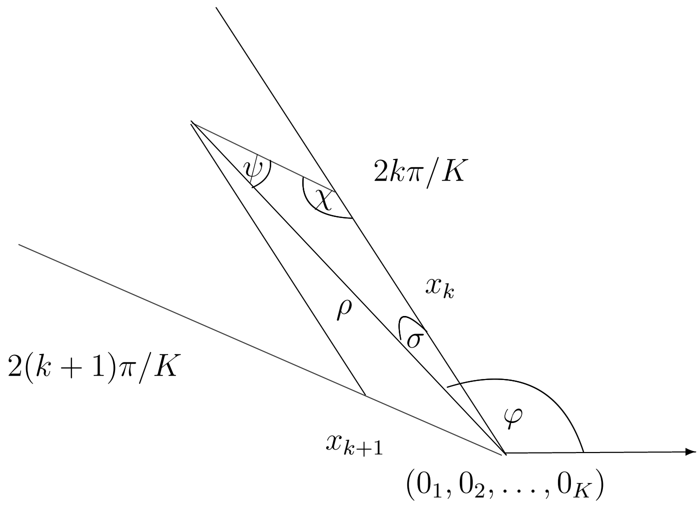

Fig. 1: Evaluation of coordinates.

$\sqrt[2]{a^2+b^2}$, can be unambiguously expressed (a representative of the equivalence class) in the form $(0, 0, \dots, 0, x_k, x_{k+1}, 0, \dots, 0)$ or $(x_1, 0, 0, \dots, 0, 0, x_K)$, where $x_k \in \mathbb{R}_0, k = 1, 2, \dots K$, $K = 3, 4, \dots$, see Fig. 1.

Here the angles satisfy $\sigma + \psi = \frac{2\pi}{K}$ and $\chi = \pi - (\sigma + \psi)$.

Clearly, for a given complex number (ρ, φ), there exists a natural $k = 0, 1, 2, \dots, K-1$, such that $\frac{2k\pi}{K} \le \varphi < \frac{2(k+1)\pi}{K}$ and $\varphi = \frac{2k\pi}{K} + \sigma, \sigma = \varphi - \frac{2k\pi}{K}, \psi = \frac{2\pi(k+1)}{K} - \varphi$.

Using the sine theorem we obtain $\frac{\rho}{\sin\psi} = 2R$, where R is the radius of the circle inscribed into the triangle (not important for our consideration now). By the sine theorem again we obtain $2R = \frac{x_k}{\sin\psi} = \frac{x_{k+1}}{\sin\sigma}$, which implies

$$x_k = \frac{\rho\sin\psi}{2\sin\left(\pi - \frac{2\pi}{K}\right)} = \frac{\sqrt[2]{a^2+b^2}\sin\left(\varphi - \frac{2k\pi}{K}\right)}{2\sin\pi\left(1-\frac{2}{K}\right)},$$

$$x_{k+1} = \frac{\rho\sin\sigma}{2\sin\left(\pi - \frac{2\pi}{K}\right)} = \frac{\sqrt[2]{a^2+b^2}\sin\left(\frac{2(k+1)\pi}{K} - \varphi\right)}{2\sin\pi\left(1-\frac{2}{K}\right)},$$

(and $x_h = 0$ when $h \ne k, h \ne k+1,\ k, h, = 0, 1, 2, \dots K-1, K = 3, 4, \dots$).

7. Applications. As far as the authors know, the idea about polarity (sets equipped with the arithmetic operations and lexicographical orderings with arbitrary number of "poles") goes from V. Lenski, cf. [2]. However, up to the present days, there is no explanation of the multi polarity idea in the mathematical literature, i.e., a description of a vector-space-like structure with a chosen number of poles $K \in \mathbb{N}$ (the term "dimension" is not apt since we deal with semi-fields). However, there are material application of K-polar vector spaces to electricity and magnetism, e.g. we know the 3-phase, K-phase electric streams, K-polar oscillators, receivers, magnets, microscopes, telescopes, etc., ([3]).

Other possible applications are implied from the using of semi-fields in general, i.e. to automata theory, optimization theory, discrete event dynamical systems, algebra of formal processes, generalized fuzzy computation, combinatorial optimizations, Bayesian networks and belief propagation (including turbo decoding), algebraic geometry over the optimization algebra, dequantizations and amoebas, cf. [5]. Following the aim of this paper, we dealt only with the semi-field $\mathbb{R}_0$.

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