Математичні Студії. Т.41, №1 Matematychni Studii. V.41, No.1

УДК 515.5

I. V. Protasov

SPARSE AND THIN METRIC SPACES

I. V. Protasov. *Sparse and thin metric spaces*, Mat. Stud. **41** (2014), 92–100.

We define thin and sparse metric spaces as asymptotic counterparts of discrete and very close to discrete metric spaces respectively. We characterize sparse spaces in terms of prohibited subspaces, and classify thin metric spaces up to coarse equivalence. We introduce the types of sparse spaces and construct the spaces of distinct types. We unify the notions of sparse spaces and sparse subsets of a group in context of balleans.

И. В. Протасов. *Разреженные и тонкие метрические пространства* // Мат. Студії. – 2014. – Т.41, №1. – С.92–100.

Тонкие и асимптотически разреженные метрические пространства определяются как асимптотические аналоги дискретных и асимптотически разреженных метрических пространств. Получена характеризация асимптотически разреженных пространств в терминах запрещенных пространств, классифицированы тонкие метрические пространства с точностью до грубой эквивалентности. Введено понятие типа асимптотически разреженных пространств и построены пространства различных типов.

1. Introduction. A metric space (X,d) is called *thin* (or *asymptotically discrete*) if, for every $m\in\omega=\{0,1,\dots\}$, there exists a bounded subset V of X such that $B(x,m)=\{x\}$ for each $x\in X\setminus V$. Here $B(x,m)=\{y\in X\colon d(x,y)\le m\}$. A subset V is bounded if $V\subseteq B(x_0,n)$, for some $x_0\in X$, $n\in\omega$. We say that a subset Y of X is *thin* if the metric space $(Y,d|_Y)$ is thin. Clearly, each bounded subset of a metric space is thin, and each subset of a thin metric space is thin.

Following [2], we say that a subset Y of a metric space (X,d) has *asymptotically isolated m-balls* if there exists a sequence $(x_n)_{n\in\omega}$ in Y and an increasing sequence $(m_n)_{n\in\omega}$ in ω such that $B(x_n,m_n)\setminus B(x_n,m)=\varnothing$ for each $n\in\omega$. If Y has asymptotically isolated m-balls for some $m\in\omega$, we say that Y has *asymptotically isolated balls.*

We say that a metric space (X,d) is *sparse* if each unbounded subspace Y of X has asymptotically isolated balls in X. Equivalently, (X,d) is called sparse if, for every unbounded subset Y of X, there exists $m\in\omega$, such that, for every $n\in\omega$, there is $y\in Y$ such that

$$B(y,n)\setminus B(y,m)=\varnothing.$$

We say that a subset Y of X is *sparse* if the metric space (Y,d_Y) is sparse. Clearly, each thin space is sparse, and each subspace of a sparse space is sparse.

In fact, the notion of a sparse subset was introduced in the context of groups [4] in order to characterize the strongly prime ultrafilters in the Stone-Čech compactification βG of a discrete group G. For sparse subsets of groups see [1], [5], [6], [11].

2010 *Mathematics Subject Classification:* 54E15, 54E35.
Keywords: asymptotically isolated balls; sparse and thin spaces; balleans.

Given two metric spaces (X_1, d_1) and (X_2, d_2), a bijection $f\colon X_1 \to X_2$ is said to be an *asymorphism* if there are two sequences $(c_n)_{n\in\omega}$ and $(c'_n)_{n\in\omega}$ in ω such that, for each $n \in \omega$ and $x, y \in X_1$,

$$d_1(x,y) \le n \Rightarrow d_2(f(x), f(y)) < c_n,\ d_2(f(x), f(y)) \le n \Rightarrow d_2(x,y) < c'_n.$$

We note ([12, Theorem 2.1.1]) that each metric space (X, d) is asymorphic to a metric space (X', d') such that d' takes values in ω. In what follows all metrics under consideration are supposed to be integer valued.

A subset L of a metric space (X, d) is called *large* if there exists $m \in \omega$ such that $B(L, m) = X$.

Metric spaces (X_1, d_1), (X_2, d_2) are called *coarsely equivalent* if there are large subsets $L_1 \subseteq X_1$ and $L_2 \subseteq X_2$ such that the metric spaces (L_1, d_1) and (L_2, d_2) are asymorphic.

We say that a property $\mathcal{P}$ of metric spaces is *asymptotic* (resp. *coarse*) if $\mathcal{P}$ is stable under asymorphisms (resp. coarse equivalence). It is easy to see that “thin” is an asymptotic but not coarse property, and “asymptotic scattered” is a coarse property.

A metric space (X, d) is of *bounded geometry* if there exists $m \in \omega$ and a function $c\colon \omega \to \omega$ such that the m-capacity of every ball $B(x, n)$ does not exceed $c(n)$. An *m-capacity* of a subset Y of X is supremum of cardinalities of m-discrete subsets of Y. A subset Z is *m-discrete* if $d(x, y) > m$ for all distinct $x, y \in Z$.

A metric space (X, d) is called *uniformly locally finite* if there is a function $c\colon \omega \to \omega$ such that $|B(x, n)| \le c(n)$ for each $x \in X$ and $n \in \omega$. It is easy to see [10, Proposition 2] that (X, d) is of bounded geometry if and only if (X, d) is coarsely equivalent to some uniformly locally finite metric space.

Recall that a metric d on a set X is *ultrametric* if $d(x, y) \le \max\{d(x, z), d(y, z)\}$ for all $x, y, z \in X$. By [3, Theorem 3.11], every ultrametric space of bounded geometry is coarsely equivalent to some subset of the Cantor macro-cube

$$2^{<\mathbb{N}} = \{(x_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \colon \exists n \in \mathbb{N}\ \forall m > n\ x_m = 0\}$$

endowed with the ultrametric $d((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \max\{n \in \mathbb{N} \colon x_n \ne y_n\}$. By [2], an ultrametric space of bounded geometry without asymptotically isolated balls is coarsely equivalent to $2^{<\mathbb{N}}$. However [9], there are $\mathfrak{c}$ pairwise non-asymorphic uniformly locally finite ultrametric spaces.

In section 2 we characterize the sparse spaces in terms of prohibited subspaces. In section 3 we classify thin spaces up to coarse equivalence. In section 4 we introduce the types of sparse spaces and construct the sparse spaces of distinct types. In sections 5 and 6 we unify the notions of sparse spaces and sparse subsets of a group in context of balleans.

2. Characterizations. For a metric space (X, d) and $m \in \omega$, a sequence $x_1, \dots, x_n$ in X is called an *m-chain* if $d(x_i, x_{i+1}) \le m$ for each $i \in \{1, \dots, n-1\}$. We set

$$B^{\square}(x, m) = \{y \in X \colon \text{ there is an } m\text{-chain between } x \text{ and } y\}$$

and say that X is *cellular* if, for each $m \in \omega$, there is $c_m \in \omega$ such that $B^{\square}(x, m) \subseteq B(x, c_m)$ for every $x \in X$. By [12, Theorems 3.1.1 and 3.1.3], the following three conditions are equivalent:

- (X, d) is cellular;

- $\operatorname{asdim}(X) = 0$;
- (X, d) is asymorphic to some ultrametric space.

Let (X, d) be an unbounded metric space. We define a function $h\colon X \times \omega \to \omega$ by

$$h(x, m) = \min\{n\colon B(x, n) \setminus B(x, m) \neq \varnothing\},$$

and note that (X, d) is sparse if and only if, for each unbounded subset Y of X, there exist $m \in \omega$ and an unbounded subset $Z \subseteq Y$ such that the set $\{h(x, m)\colon x \in Z\}$ is infinite.

Theorem 1. *Every sparse metric space (X, d) is cellular.*

Proof. If (X, d) is not cellular then there exist $m \in \omega$ and an unbounded injective sequence $(x_n)_{n\in\omega}$ in X such that $B^{\square}(x_n, m) \setminus B(x_n, n) \neq \varnothing$. It follows that $h(x_n, k) \leq k + m$ for each $k \in \{0, \dots, n\}$. We put $Y = \{x_n\colon n \in \omega\}$ and observe that the set $\{h(x_n, k)\colon n \in \omega\}$ is finite for each $k \in \omega$. Hence, (X, d) is not sparse. □

Theorem 2. *If X_1, X_2 are sparse subspaces of a metric space (X, d) then $X_1 \cup X_2$ is sparse.*

Proof. Let Y be an unbounded subset of $X_1 \cup X_2$. We may suppose that $Y \subseteq X_1$. Since X_1 is sparse there exist $m \in \omega$, a sequence $(y_n)_{n\in\omega}$ in Y and an increasing sequence $(m_n)_{n\in\omega}$ in ω such that for every $n \in \omega$, $(X_1 \cap B(y_n, m_n)) \setminus B(y_n, m) = \varnothing$.

If the set $\{n \in \omega\colon B(y_n, m_n) \setminus B(y_n, m) \neq \varnothing\}$ is finite then $Y' = \{y_n\colon n \in \omega\}$ has asymptotically isolated m-balls in $X_1 \cup X_2$. Otherwise, we may suppose that $B(y_n, m_n) \setminus B(y_n, m) \neq \varnothing$ for each $n \in \omega$. We pick $z_n \in B(y_n, m_n) \setminus B(y_n, m)$ such that

$$d(z_n, y_n) = \min\{d(z, y_n)\colon z \in B(y_n, m_n) \setminus B(y_n, m)\}.$$

If the sequence $(d(z_n, y_n)_{n\in\omega})$ is unbounded in ω, Y' has an asymptotically isolated m-balls in $X_1 \cup X_2$. Otherwise, we take $t \in \omega$ such that $d(z_n, y_n) \leq t$ for each $n \in \omega$. Since $Z = \{z_n\colon n \in \omega\}$ is an unbounded subset of X_2, Z has an asymptotically isolated m'-balls in X_2 for some $m' \in \omega$. Then Y' has asymptotically isolated $(m + m' + t)$-balls in $X_1 \cup X_2$. □

We define a metric ρ' on ω by the rule: $\rho'(n, n) = 0$ and $\rho'(n, m) = \max\{n, m\}$ if $m \neq n$. Then we define a metric ρ on $W = \omega \times \omega$ by the rule $\rho((n, m), (n', m')) = \max\{\rho'(n, n'), \rho'(m, m')\}$.

We consider two subspaces W_1 and W_2 of (W, ρ)

$$W_1 = \{(n, 0)\colon n \in \omega\},\ W_2 = W_1 \cup \{(n, m)\colon n > m > 0\},$$

and observe that W_1 is isometric to the subspace $\{x \in 2^{<\mathbb{N}}\colon \operatorname{supt}(x) \leq 1\}$, W_2 is isometric to the subspace $\{x \in 2^{<\mathbb{N}}\colon \operatorname{supt}(x) \leq 2\}$ of the Cantor macro-cube $2^{<\mathbb{N}}$. Here $\operatorname{supt}(x)$ is the number of non-zero coordinates of $x = (x_n)_{n\in\mathbb{N}}$.

Theorem 3. *A metric space (X, d) is sparse if and only if (X, d) has no subspaces asymorphic to W_2.*

Proof. To show that W_2 is not sparse, we use the function $h\colon W_2 \times \omega \to \omega$. For $n > m \geq 0$, we have $h((n, 0), m) = m + 1$. Hence the subspace W_1 of W_2 has no asymptotically isolated balls.

Now assume that X is not sparse and find a subset W' of X asymorphic to W_2. We take a subset $X' = \{x_n \colon n \in \omega\}$ without isolated balls. Passing to subsequences of $(x_n)_{n\in\omega}$ ω times, we can choose a sequence $(y_n)_{n\in\omega}$ in X' and an increasing sequence $(m_n)_{n\in\omega}$ in ω such that $m_0 = 0$ and

(1) $(X \cap B(y_n, m_i)) \setminus B(y_n, m_{i-1}) \neq \varnothing$, $n > 0$, $i \in \{1, \dots, n\}$;

(2) $B(y_n, m_n) \cap B(y_i, m_i) = \varnothing$, $0 \leq i < n < \omega$.

We use (1), (2) to choose a subsequence $(z_{n0})_{n\in\omega}$ of $(y_n)_{n\in\omega}$, a subsequence $(k_n)_{n\in\omega}$ and, for each $n > 0$, the elements $z_{n1}, \dots, z_{nn}$ such that

(3) $z_{ni} \in (X \cap B(z_{n0}, k_i)) \setminus B(z_{n0}, k_{i-1})$, $i \in \{1, \dots, n\}$;

(4) $d(z_{ni}, z_{nj}) > j$, $0 \leq i < j \leq n$;

(5) $B(z_{n0}, k_n) \cap B(z_{i0}, k_i) = \varnothing$, $0 \leq i < n < \omega$.

Then we consider a set $W' = \{z_{00}\} \cup \{z_{ni} \colon n > i \geq 0\}$ and define a mapping $f \colon W_2 \to W'$ by the rule: $f(0,0) = z_{00}$ and $f(n,i) = z_{ni}$, $n > i \geq 0$. By (3) and (4), f is a bijection.

If f is not an asymorphism, we get the following two cases.

Case 1. There exist $t > 0$ and two sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ in W_2 such that $\rho(a_n, b_n) = t$ but $d(f(a_n), f(b_n)) \to \infty$. We may suppose that $a_n = (c_n, t)$, $b_n = (c_n, s)$, $s < t$. But $d(z_{c_n t}, z_{c_n s}) \leq 2k_t$, a contradiction.

Case 2. There exist $t > 0$ and two sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ in W_2 such that $d(f(a_n), f(b_n)) = t$ but $\rho(a_n, b_n) \to \infty$. In view of (4), (5), we may suppose that $f(a_n) = z_{c_n i_n}$, $f(b_n) = z_{c_n j_n}$ and $0 \leq i_n < j_n < t$. But then $\rho(a_n, b_n) < t$, a contradiction. ☐

3. Thin spaces.

The following three theorems are from [7].

Theorem 4. *A metric space X is thin if and only if each unbounded subset of X has asymptotically isolated 0-balls.*

We say that a metric space X is *coarsely thin* if X is coarsely equivalent to some thin space.

Theorem 5. *For a metric space X, the following statements are equivalent:*

(i) *X is coarsely thin;*

(ii) *X contains large thin subset;*

(iii) *there exists $m \in \omega$ such that each unbounded subset of X has asymptotically isolated m-balls.*

A subset Y of a metric space X is called *asymptotically isolated* if, for each $m \in \omega$, there is a bounded subset V of X such that $B(y, m) \subseteq Y$ for each $y \in Y \setminus V$.

Theorem 6. *A metric space X is sparse if and only if each unbounded subset of X has an asymptotically isolated coarsely thin subset.*

Given a sequence of cardinals $(\varkappa_n)_{n\in\mathbb{N}}$, we consider the space $T(\varkappa_n)_{n\in\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \varkappa_n \times \{n\}$ endowed with the ultrametric ρ defined by $\rho(x, y) = \max\{m, n\}$ for any distinct points $x \in \varkappa_n \times \{n\}$, $y \in \varkappa_m \times \{m\}$.

Lemma 1. *Every thin metric space (X, d) is asymorphic to the space $T(\varkappa_n)_{n\in\mathbb{N}}$ for some sequence of cardinals $(\varkappa_n)_{n\in\mathbb{N}}$.*

Proof. We partition $X = \bigcup_{i\in\omega} X_{i1}$ by the equivalence $\sim_1$ defined by $x \sim_1 y \Leftrightarrow x = y \vee d(x,y) = 1$. Since X is thin, the set $X_1 = \bigcup\{X_{i1}\colon |X_{i1}| > 1\}$ is bounded. We partition $X \setminus X_1 = \bigcup_{i\in\omega} X_{i2}$ by the equivalence $\sim_2$ defined by $x \sim_2 y \Leftrightarrow x = y \vee d(x,y) = 2$. Since $X \setminus X_1$ is thin, the set $X_2 = \cup\{X_{i2}\colon |X_{i2}| > 1\}$ is bounded. We partition $X \setminus (X_1 \cup X_2)$ by the equivalence $\sim_3$ defined by $x \sim_3 y \Leftrightarrow x = y \vee d(x,y) = 3$, and so on.

After ω steps, we get a partition $X = \bigcup_{n\in\mathbb{N}} X_n$, $|X_n| = \varkappa_n$. Then we partition $\varkappa = \bigcup K_n$ so that $|K_n| = \varkappa_n$. For each $n \in \mathbb{N}$, let $f_n\colon X_n \to K_n$ be an arbitrary bijection. It is easy to see that $f = \bigcup_{n\in\mathbb{N}} f_n$ is an asymorphism between X and $T(\kappa_n)_{n\in\mathbb{N}}$. □

For a metric space X, the minimal cardinality $\operatorname{asden}(X)$ of large subsets of X is called an *asymptotic density* of X. Clearly $\operatorname{asden}(X) = 1$ if and only if X is bounded. We note also that asymptotic density is invariant under coarse equivalence, and each metric space is coarsely equivalent to a metric space X such that $|X| = \operatorname{asden}(X)$.

Theorem 7. *Let X be an unbounded thin metric space such that $|X| = \operatorname{asden}(X) = \varkappa$. Then the following statements hold*

- *(i) if $\varkappa = \aleph_0$ then X is coarsely equivalent either to $T(1,1,\dots)$ or to $T(\aleph_0,\aleph_0,\dots)$;*
- *(ii) if $cf(\varkappa) > \aleph_0$ then X is coarsely equivalent to $T(\varkappa,\varkappa,\dots)$;*
- *(iii) if $\varkappa > \aleph_0$, $cf(\varkappa) = \aleph_0$ and $(\varkappa'_n)_{n\in\mathbb{N}}$ is a sequence of cardinals such that $\varkappa'_n < \varkappa'_{n+1}$ and $\sup\{\varkappa'_n\colon n \in \mathbb{N}\} = \varkappa$ then X is coarsely equivalent either to $T(\varkappa'_n)_{n\in\mathbb{N}}$ or to $T(\varkappa,\varkappa,\dots)$.*

Proof. In view of Theorem 1, we may suppose that X is ultrametric. By Lemma 1, there is a partition $\varkappa = \bigcup_{n\in\mathbb{N}} K_n$, $|K_n| = \varkappa_n$ such that X is asymorphic to $T(\kappa_n)_{n\in\mathbb{N}}$.

(i) We consider two cases.

Case 1. There exists $n_0 \in \mathbb{N}$ such that $\kappa_n < \aleph_0$ for each $n > n_0$. The subset $\aleph_0 \setminus \bigcup_{i\le n_0} K_i$ is large in $T(\kappa_n)_{n\in\mathbb{N}}$ and hence coarsely equivalence to $T(\varkappa_n)_{n\in\mathbb{N}}$. We take an arbitrary bijection $f\colon \aleph_0 \setminus \bigcup_{i\le n_0} K_i \to T(1,1,\dots)$ and note that f is an asymorphism.

Case 2. There exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\varkappa_{n_k} = \aleph_0$ for each $k \in \mathbb{N}$. We partition $\aleph_0 = \bigcup_{k\in\mathbb{N}} K'_k$ so that $|K'_k| = \aleph_0$ for each $k \in \mathbb{N}$. Then we take an arbitrary bijection $f\colon \aleph_0 \to \aleph_0$ such that $f(\bigcup_{i\le n_1} K_i) = K'_1$, $f(\bigcup_{n_1 < i \le n_2} K_i) = K_2, \dots$. It is easy to see that f is an asymorphism between $T(\kappa_n)_{n\in\mathbb{N}}$ and $T(\aleph_0,\aleph_0,\dots)$.

(ii) Asume that there exists $n_0 \in \mathbb{N}$ such that $\varkappa_n < \varkappa$ for each $n > n_0$. On one hand, the subspace $\varkappa \setminus \bigcup_{i\le n_0} K_i$ is coarsely equivalent to $T(\varkappa_n)_{n\in\mathbb{N}}$. On the other hand, $|\varkappa \setminus \bigcup_{i\le n_0} K_i| < \varkappa$ because $cf(\varkappa) > \aleph_0$. Thus $\operatorname{asden}(X) < \varkappa$ contradicting the assumption.

Hence there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $K_{n_k} = \varkappa$ for each $k \in \mathbb{N}$. We partition $\varkappa = \bigcup_{k\in\mathbb{N}} K'_k$, $|K'_k| = \varkappa$ and define an asymorphism $f\colon T(\kappa_n)_{n\in\mathbb{N}} \to T(\kappa,\kappa,\dots)$ as in the Case 2 of (i).

(iii) We consider two cases.

Case 1. There is $m \in \mathbb{N}$ such that $\varkappa_n < \varkappa$ for each $n \ge m$. We may suppose that $m = 1$. We partition $\varkappa = \bigcup_{n\in\mathbb{N}} K'_n$ such that $K'_n = \kappa'_n$ and choose two increasing sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\varkappa'_{m_1} \le |\bigcup_{i\le n_1} K_i| < \varkappa'_{m_2}$ and, for each $k > 1$, $\varkappa'_{m_k} \le |\bigcup_{n_k < i \le n_{k+1}} K_i| < \varkappa'_{m_{k+1}}$. Then we choose a bijection $f\colon \varkappa \to \varkappa$ such that for each $k \in \mathbb{N}$,

$$\bigcup_{i\le m_k} K'_i \subseteq f\Big(\bigcup_{i\le n_k} K_i\Big) \subseteq \bigcup_{i\le m_{k+1}} K'_i.$$

Then f is a desired asymorphism between $T(\varkappa_n)_{n\in\mathbb{N}}$ and $T(\varkappa'_n)_{n\in\mathbb{N}}$.

Case 2. There is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\varkappa_{n_k} = \varkappa$ for each $k \in \mathbb{N}$. Then an asymorphism $f\colon T(\varkappa_n)_{n\in\mathbb{N}} \to T(\varkappa, \varkappa, \dots)$ can be defined as in the Case 2 of (*i*). $\square$

Remark 1. The metric spaces $T(1, 1, \dots)$ and $T(\aleph_0, \aleph_0, \dots)$ are not coarsely equivalent because $T(1, 1, \dots)$ is uniformly locally finite but $T(\aleph_0, \aleph_0, \dots)$ is not of bounded geometry. We show that the metric spaces $T(\varkappa, \varkappa, \dots)$ and $T(\varkappa'_n)_{n\in\mathbb{N}}$ from (*iii*) of Theorem 7 are not coarsely equivalent. We note that each large subset of $T(\varkappa, \varkappa, \dots)$ (resp. $T(\varkappa'_n)_{n\in\mathbb{N}}$) is asymorphic to $T(\varkappa, \varkappa, \dots)$ (resp. $T(\varkappa'_n)_{n\in\mathbb{N}}$). Let $f\colon T(\varkappa, \varkappa, \dots) \to T(\varkappa'_n)_{n\in\mathbb{N}}$ is a bijection, and let $\varkappa = \bigcup_{n\in\mathbb{N}} K_n$ be a partition which determine $T(\varkappa, \varkappa, \dots)$. On one hand K_1 is bounded in $T(\varkappa, \varkappa, \dots)$. On the other hand $|f(K_1)| = \varkappa$ so $f(K_1)$ is unbounded in $T(\varkappa'_n)_{n\in\mathbb{N}}$. Hence f is not an asymorphism.

4. Sparse types. Let X be a sparse metric space. We say that X is of *type* 0 if X is bounded, and X is of *type* 1 if X is unbounded and coarsely thin. For $m > 1$, X is of *type* m if and only if X can be partitioned in m coarsely thin subsets, but X is not of type less then m. If X is not of type m for each $m \in \omega$, we say that X is of *infinite type.* Clearly, the types are invariant under coarse equivalence.

For $m \in \omega$, we say that a metric space X is m-*thin* if, for every $n \in \omega$, there exists a bounded subset V of X such that $|B(x, n)| \le m$ for each $x \in X \setminus V$. Clearly, X is 0-thin if and only if X is bounded. By [5], every unbounded m-thin space can be partitioned in $\le m$ thin subsets. Applying Theorem 5, we conclude that an unbounded metric space X is of type m if and only if m is the minimal number such that X contains a large m-thin subsets. Thus, a classification of sparse metric spaces of finite types is reduced to the case of m-thin spaces.

Now we consider some construction of sparse spaces. Let $(X_n)_{n\in\omega}$ be a sequence of subsets of ω such that $\min X_n > n$ for each $n \in \omega$. We denote by $W(X_n)_{n\in\omega}$ the subspace $\bigcup\{X_n \times \{n\}\colon n \in \omega\}$ of W_2. Applying Theorem 3, we conclude that $W(X_n)_{n\in\omega}$ is sparse if and only if, for each infinite subset I of ω, there exists a finite subset $F \subset I$ such that $\bigcap_{n\in F} X_n$ is finite.

By Theorem 7 (*i*), each metric space of bounded geometry of type 1 is coarsely equivalent to W_1.

Example 1. For each $i \geqslant 2$, we construct a subset of W_2 of type i. To this end, we take a sequence $(X_n)_{n\in\omega}$ of infinite subsets of ω such that $\min X_n > n, n \in \omega$ and

(1) $\bigcap_{n\in F} X_n$ is finite for each $F \subset \omega$, $|F| = i + 1$;

(2) $\bigcap_{n\in H} X_n$ is infinite for each $H \subset \omega$, $|H| = i$.

By (1), $W(X_n)_{n\in\omega}$ is i-thin. By (2), each large subset of $W(X_n)$ is not $(i-1)$-thin. Hence, $W(X_n)_{n\in\omega}$ is of type i.

Example 2. We put $X_0 = \omega \setminus \{0\}$ and choose a sequence $(X_n)_{0<n<\omega}$ of infinite pairwise disjoint subsets of $\omega \setminus \{0, 1\}$ such that $\min X_n > n$ and $\omega \setminus \cup_{0<n<\omega} X_n$ is infinite. Clearly, the space $W(X_n)_{n\in\omega}$ is of type 2. Moreover, each metric space of bounded geometry of type 2 is coarsely equivalent to $W(X_n)_{n\in\omega}$. Thus, up to the coarse equivalence, there is only one space of bounded geometry of type 2.

Question 1. *Let X be a metric space of bounded geometry of type $i \geqslant 3$. Does there exist a family $\{X_n\colon n \in \omega\}$ of subsets of ω such that X is coarsely equivalent to $W(X_n)_{n\in\omega}$?*

Question 2. *For each $i \geqslant 3$, classify the metric spaces of bounded geometry of type i up to the coarse equivalence.*

5. Ballean context. Following [12], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius α around x*. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the set of *radii.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x.\alpha) = \{y \in X\colon x \in B(y, \alpha)\},\ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

A ballean $\mathcal{B}$ on X can also be defined in terms of entourages of diagonal Δ_X of $X \times X$, in this case it is called a *coarse structure* ([13]). For our “scattered” goal, we prefer the ball language.

We suppose that all ballean under consideration are *connected*, i.e. for any $x, y \in X$ there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f\colon X_1 \to X_2$ is called a *$\prec$-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X$, $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$. If there exists a bijection $f\colon X_1 \to X_2$ such that f and f^{-1} are $\prec$-mappings, $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *asymorphic.*

For a ballean $\mathcal{B} = (X, P, B)$, a subset $Y \subseteq X$ is called *large* if there is $\alpha \in P$ such that $X = B(Y, \alpha)$. A subset $V \subseteq X$ is called *bounded* if $V \subseteq B(x, \alpha)$ for some $x \in X$ and $\alpha \in P$. Each non-empty subset $Y \subseteq X$ defines a *subballean* $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(Y, \alpha) = Y \cap B(y, \alpha)$.

By the definition, two balleans $\mathcal{B} = (X, P, B)$ and $\mathcal{B}' = (X', P', B')$ are *coarsely equivalent* if there exist large subsets $Y \subseteq X$, $Y' \subseteq X'$ such that the subballeans $\mathcal{B}_Y$ and $\mathcal{B}_{Y'}$ are asymorphic.

Let G be a group, $\mathcal{I}$ be an ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G, i.e. $\varnothing \in \mathcal{I}$ and if $A, B \in \mathcal{I}$ and $A' \subseteq A$, then $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a *group ideal* if $\mathcal{I}$ contains all finite subsets of G and, for all $A, B \in \mathcal{I}$, we have $AB \in \mathcal{I}$ and $A^{-1} \in \mathcal{I}$.

Now let X be a transitive G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$, and let $\mathcal{I}$ be a group ideal in $\mathcal{P}_G$. We define a ballean $\mathcal{B}(G, X, \mathcal{I})$ as a triple $(X, \mathcal{I}, B)$, where $B(x, A) = Ax \cup \{x\}$ for all $x \in X$, $A \in \mathcal{I}$. By [8, Theorem 1] every ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, \mathcal{I})$ for some group G of permutation of X and some group ideal $\mathcal{I}$ in $\mathcal{P}_G$.

For a group G, we denote by $\mathfrak{F}_G$ the ideal of all finite subsets of G. Each metric space (X, d) can be considered as the ballean (X, ω, B_d).

The following two theorems are from [12, Theorem 2.1.1] and [10, Theorem 1].

Theorem 8. *Let G be a countable transitive group of permutations of a set X. Then there exists a uniformly locally finite metric d on X such that the ballean $\mathcal{B}(G, \mathfrak{F}_G, X)$ is asymorphic to (X, d).*

Theorem 9. *Let (X, d) be a uniformly locally finite metric space. Then there exists a countable group G of permutations of X such that (X, d) is asymorphic to $\mathcal{B}(G, \mathfrak{F}_G, X)$.*

We say that a ballean $\mathcal{B} = (X, P, B)$ is *thin* if, for every $\alpha \in P$, there exists a bounded subset V of X such that $B(x, \alpha) = \{x\}$ for each $x \in X \setminus V$. A subset $Y \subseteq X$ is called thin if the subballean $\mathcal{B}_Y$ is thin.

We use the natural preordering $<$ on P: $\alpha < \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for each $x \in X$. We say that a subset $Y \subseteq X$ has *asymptotically α-isolated balls* if, for every $\beta > \alpha$, there exists $y \in Y$ such that $B(y, \beta) \setminus B(y, \alpha) = \varnothing$. If Y has asymptotically α-isolated balls for some $\alpha \in P$, we say that Y has *asymptotically isolated balls.*

A ballean $\mathcal{B} = (X, P, B)$ is called *sparse* if each unbounded subset of X has asymptotically isolated balls in X. A subset $Y \subseteq X$ is called *sparse* if the subballean $\mathcal{B}_Y$ is sparse.

6. Sparse subsets of a group. In this section, we consider a group G as a ballean $\mathcal{B}(G, \mathfrak{F}_G, X)$ where $X = G$ and G acts on X by the left translations. We remind that, for $g \in G$ and $F \in \mathfrak{F}_G$, $B(g, F) = Fg \cup \{g\}$, and say that a subset A of a group G is *sparse* if A is a sparse subset of the ballean $\mathcal{B}(G, \mathfrak{F}_G, X)$.

Theorem 10. *For a subset A of a group G, the following statements are equivalent*

(i) A is sparse;

(ii) for every infinite subset Y of A, there exists a finite subset F such that the set $\bigcap_{g \in F} gA$ is finite;

(iii) for each free ultrafilter $\mathcal{U}$ on G with $A \in \mathcal{U}$, the set $\{g \in G \colon A \in g\mathcal{U}\}$ is finite, where $g\mathcal{U} = \{gU \colon U \in \mathcal{U}\}$.

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ has been proven in [11, Proposition 5].

$(i) \Rightarrow (ii)$. Suppose that there exists an injective sequence $(g_n)_{n \in \omega}$ in G such that $\bigcap_{i \leqslant n} g_i A$ is infinite for each $n \in \omega$. Then we choose an injective sequence $(y_n)_{n \in \omega}$ in A such that $g_n^{-1}\{y_n, y_{n+1}, \ldots\} \subseteq A$ for every $n \in \omega$. We put $Y = \{y_n \colon n \in \omega\}$ and note that Y has no asymptotically isolated balls in A. Hence, A is not sparse.

$(ii) \Rightarrow (ii)$. Suppose that A is not sparse. Then there is an infinite subset X of A with no asymptotically isolated balls in A. We choose inductively an injective sequence $(x_n)_{n \in \omega}$ in X and an injective sequence $(y_n)_{n \in \omega}$ in G such that $y_n\{x_n, x_{n+1}, \ldots\} \subseteq A$ for every $n \in \omega$. We put $Y = \{y_n \colon n \in \omega\}$ and note that $\bigcap_{g \in F} gA$ is infinite for every finite subset $F \subset Y^{-1}$. □

Theorem 11. *Every countable group G contains a sparse subset of infinite type and, for each $m \in \omega$, a sparse subset of type m.*

Proof. We write G as a union $G = \bigcup_{n \in \omega} K_n$, $K_0 = \{e\}$ of an increasing chain of finite symmetric subsets.

In the proof of Theorem 2.1 from [4], we constructed a sparse subset A of the form $A = \bigcup_{n \in \omega} F_n x_n$ where $(F_n)_{n \in \omega}$ is an appropriate sequence in $\mathfrak{F}_G$. By the construction, A has the following property

(1) for all $n > 0$ and $m > 0$, there exists $F \in \mathfrak{F}_G$ and an infinite subset $I \subset \omega$ such that $|F| = m + 1$, $F = F_n$ for each $n \in I$ and $K_n x x_i \cap A = \{x x_i\}$ for all $x \in F$ and $i \in I$.

We fix $m > 0$, take an arbitrary large subset $L \subseteq A$ and pick $n \in \omega$ such that $A \subseteq K_n L$. By (1), $F_i x_i \subset L$ for each $i \in I$. Since $F = F_i$ and $|F| = m + 1$, we conclude that L is not m-thin. Hence A is of infinite type.

In the proof of Theorem 1.1 ([4]), for each $m > 0$, we constructed a sparse subset $A = \bigcup_{n\in\omega} F_n x_n$ with following properties

(2) $|F_i| = m$, $K_i F_i x_i \cap K_n F_n x_n = \varnothing$, $0 \le i < n < \omega$;

(3) for each $n \in \omega$, there exist $F \in \mathfrak{F}_G$ and an infinite subset $I \subset \omega$ such that $F = F_i$ and $K_n x x_i \cap A = \{x x_i\}$ for all $i \in I$ and $x \in F$.

By (2), A is m-thin. Let L be a large subset of A. We pick $n \in \omega$ such that $A \subseteq K_n L$. By (3), $F_i x_i \subset L$ for each $i \in I$. Since $F = F_i$ and $|F| = m$, we conclude that L is not $(m-1)$-thin. Hence A is of type m. □

REFERENCES

1. T. Banakh, I. Protasov, S. Slobodianiuk, *Scattered subsets of groups*, preprint available at arxiv: submit/0877498.
2. T. Banakh, I. Zarichnyi, *Characterizing the Cantor bi-cube in asymptotic categories*, Groups, Geometry and Dynamics, **5** (2011), 691–728.
3. A. Dranishnikov, M. Zarichnyi, *Universal spaces for asymptotic dimention*, Topology Appl., **140** (2004), 203–225.
4. Ie. Lutsenko, I.V. Protasov, *Sparse, thin and other subsets of groups*, Intern. J. Algebra Computation, **19** (2009), 491–510.
5. Ie. Lutsenko, I.V. Protasov, *Thin subsets of balleans*, Appl. Gen. Topology, **1** (2010), 89–93.
6. Ie. Lutsenko, I.V. Protasov, *Relatively thin and sparse subsets of groups*, Ukr. Math. J., **63** (2011), 216–225.
7. O.V. Petrenko, I.V. Protasov, *Balleans and filters*, Mat. Stud., **38** (2012), 3–11.
8. O.V. Petrenko, I.V. Protasov, *Balleans and G-spaces*, Ukr. Math. J., **64** (2012), 344–350.
9. I.V. Protasov, *Morphisms of ball structures of groups and graphs*, Ukr. Math. J., **54** (2002), 847–855.
10. I.V. Protasov, *Balleans of bounded geometry and G-spaces*, Mat. Stud., **30** (2008), 61–66.
11. I.V. Protasov, S. Slobodianiuk *Ultracompanions of subsets of groups*, preprint available at arxiv: 1308.1497.
12. I. Protasov, M. Zarichnyi, General Asymptology, Math. Stud. Monogr. Ser., V.12, VNTL Publishers, Lviv, 2007.
13. J. Roe, Lectures on Coarse Geometry, Amer. Math. Soc., Providence, R.I., 2003.

Department of Cybernetics, Kyiv University
i.v.protasov@gmail.com

Received 14.08.2013