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BOUNDEDNESS OF L-INDEX IN DIRECTION OF FUNCTIONS OF THE FORM $f(\langle z,m\rangle)$ AND EXISTENCE THEOREMS

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We obtain a criterion of boundedness of L-index in direction for functions $f(\langle z,m\rangle)$. Using this criterion we find sufficient conditions of boundedness L-index in direction for some class of entire functions with "plane" zeros. Moreover, we prove some existence theorems of an entire function $f(\langle z,m\rangle)$ of bounded L-index in direction for a given L and of a positive continuous function L for a given entire function $F(z)$ such that F is of bounded L-index in direction.

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Получен критерий ограниченности L-индекса по направлению для функций вида $f(\langle z, m\rangle)$. Используя этот критерий, сформулированы достаточные условия ограниченности L-индекса по направлению для некоторого класса целых функций с "плоскими" нулями. Доказаны теоремы существования целой функции вида $f(\langle z,m\rangle)$ ограниченного L-индекса по направлению для заданной L и существования функции L для заданной целой функции F с ограниченным L-индексом по направлению.

1. Introduction. We introduced a class of entire functions of bounded L-index in direction as an object of study in [1]–[4]. There were investigated properties of these functions. As usually, the investigations have led to new open problems. For example, find conditions of boundedness of L-index in direction for a function $F(z) = f(\langle z,m\rangle)$ and some function L, where $\langle z,m\rangle = \sum_{j=1}^n z_j\overline{m_j}$, $z,m \in \mathbb{C}^n$, and $f(t)$ is of bounded l-index. Especially, this problem is interesting for entire functions with "plane" zeros(definition see in [5]).

We need some standard notation. For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1,\ldots,b_n) \in \mathbb{C}^n \setminus \{0\}$ and a positive continuous function L we define

$$\lambda_1^{\mathbf{b}}(z,\eta) = \inf\left\{\inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}\colon |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}\right\} : t_0\in\mathbb{C}\right\},$$
$$\lambda_2^{\mathbf{b}}(z,\eta) = \sup\left\{\sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}\colon |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}\right\} : t_0\in\mathbb{C}\right\},$$
$$\lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z,\eta)\colon z\in\mathbb{C}^n\},\ \lambda_2^{\mathbf{b}}(\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,\eta)\colon z\in\mathbb{C}^n\}.$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L which satisfy the condition for all $\eta \ge 0$, $0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty$.

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For a positive continuous function $l(t)$, $t \in \mathbb{C}$, $t_0 \in \mathbb{C}$ and $\eta > 0$ we set $\lambda_1(t_0, \eta) \equiv \lambda_1^{\mathbf{b}}(0, t_0, \eta)$ and $\lambda_2(t_0, \eta) \equiv \lambda_2^{\mathbf{b}}(0, t_0, \eta)$ in the case where $z = 0$, $\mathbf{b} = 1$, $n = 1$, $L \equiv l$, and also $\lambda_1(\eta) = \inf\{\lambda_1(t_0, \eta)\colon t_0 \in \mathbb{C}\}$, $\lambda_2(\eta) = \sup\{\lambda_2(t_0, \eta)\colon t_0 \in \mathbb{C}\}$. As in [8], let $Q \equiv Q_1^1$ be the class of positive continuous functions $l(t)$, $t \in \mathbb{C}$, that satisfy the condition for all $\eta > 0$, $0 < \lambda_1(\eta) \le \lambda_2(\eta) < +\infty$.

An entire function of $F(z)$, $z \in \mathbb{C}^n$, is called (see [1]–[3]) a function of *bounded L-index in direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ the following inequality is valid

$$\frac{1}{m! L^m(z)} \left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right| \le \max\left\{ \frac{1}{k! L^k(z)} \left| \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| \colon\ 0 \le k \le m_0 \right\}, \tag{1}$$

where

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z),\ \frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad}\ F,\ \overline{\mathbf{b}} \rangle, \frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}} \Big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}} \Big),\ k \ge 2.$$

Below we formulate assertions that indicate possible ways to construct a function $L(z) \in Q_{\mathbf{b}}^n$ given a function $l(t) \in Q$. Their proofs are based on the definitions of Q and $Q_{\mathbf{b}}^n$. For $l \in Q$ we denote $l_1(z) = l(|z|)$, $z \in \mathbb{C}^n$.

Lemma 1. *If $l \in Q$ then $l_1 \in Q_{\mathbf{b}}^n$ for every $\mathbf{b} \in \mathbb{C}^n$.*

Proof. Since $l \in Q$ we have that for $u \in \mathbb{C}$

$$0 < \inf_{u_0 \in \mathbb{C}} \lambda_1(u_0, \eta) \le \inf\left\{ \frac{l(u)}{l(u_0)} \colon |u - u_0| \le \frac{\eta}{l(u_0)} \right\} \le 1 \le$$
$$\le \sup\left\{ \frac{l(u)}{l(u_0)} \colon |u - u_0| \le \frac{\eta}{l(u_0)} \right\} \le \sup_{u_0 \in \mathbb{C}} \lambda_2(u_0, \eta) < +\infty.$$

Using these inequalities we obtain

$$\inf\left\{ \frac{l_1(z^0 + t\mathbf{b})}{l_1(z^0 + t_0\mathbf{b})} \colon |t - t_0| \le \frac{\eta}{l(|z^0 + t_0\mathbf{b}|)} \right\} = \inf\left\{ \frac{l(|z^0 + t\mathbf{b}|)}{l(|z^0 + t_0\mathbf{b}|)} \colon |t - t_0| \le \frac{\eta}{l(|z^0 + t_0\mathbf{b}|)} \right\} =$$
$$= \inf\left\{ \frac{l(|z^0 + t\mathbf{b}|)}{l(|z^0 + t_0\mathbf{b}|)} \colon |z^0 + t\mathbf{b} - (z^0 + t_0\mathbf{b})| \le \frac{|\mathbf{b}|\eta}{l(|z^0 + t_0\mathbf{b}|)} \right\} \ge$$
$$\ge \inf\left\{ \frac{l(|\widetilde{z}|)}{l(|\widetilde{z}_0|)} \colon \big||\widetilde{z}| - |\widetilde{z}_0|\big| \le \frac{|\mathbf{b}|\eta}{l(|\widetilde{z}_0|)} \right\} \ge \inf\left\{ \frac{l(\widetilde{t})}{l(\widetilde{t}_0)} \colon |\widetilde{t} - \widetilde{t}_0| \le \frac{|\mathbf{b}|\eta}{l(\widetilde{t}_0)} \right\} \ge \lambda_1(|\mathbf{b}\eta) > 0,$$

where $\widetilde{z} = z^0 + t\mathbf{b}$, $\widetilde{z}_0 = z^0 + t_0\mathbf{b}$, $\widetilde{t} = |\widetilde{z}|$, $\widetilde{t}_0 = |\widetilde{z}_0|$.

Using similar considerations we obtain

$$\sup\left\{ \frac{l_1(z^0 + t_0\mathbf{b})}{l_1(z^0 + t_0\mathbf{b})} \colon |t - t_0| \le \frac{\eta}{l(|z^0 + t_0\mathbf{b})} \right\} = \sup\left\{ \frac{l(|z^0 + t_0\mathbf{b}|)}{l(|z^0 + t_0\mathbf{b}|)} \colon |t - t_0| \le \frac{\eta}{l(|z^0 + t_0\mathbf{b})} \right\} =$$
$$= \sup\left\{ \frac{l(|z^0 + t\mathbf{b}|)}{l(|z^0 + t_0\mathbf{b}|)} \colon |z^0 + t\mathbf{b} - (z^0 + t_0\mathbf{b})| \le \frac{|\mathbf{b}|\eta}{l(|z^0 + t_0\mathbf{b}|)} \right\} \le$$
$$\le \sup\left\{ \frac{l(|\widetilde{z}|}{l(|\widetilde{z}_0|)} \colon ||\widetilde{z}| - |\widetilde{z}_0|| \le \frac{|\mathbf{b}|\eta}{l(|\widetilde{z}_0|)} \right\} \le \sup\left\{ \frac{l(\widetilde{t})}{l(\widetilde{t}_0)} \colon |\widetilde{t} - \widetilde{t}_0| \le \frac{|\mathbf{b}|\eta}{l(\widetilde{t}_0)} \right\} \le \lambda_2(|\mathbf{b}\eta) < +\infty.$$

Thus we proved that if $l \in Q$ then for any $\mathbf{b} \in \mathbb{C}^n$ one has $l_1 \in Q_{\mathbf{b}}^n$. □

Lemma 2. *If $l(|t|)\in Q$ then for all $m\in\mathbb{C}^n$ and every $\mathbf{b}\in\mathbb{C}^n$ we have $l(|\langle z,m\rangle|)\in Q^n_{\mathbf{b}}$.*

Proof. Since $l(|t|)\in Q$ we have that for any $q>0$

$$\sup\left\{\frac{l(|t|)}{l(|t_0|)}:|t-t_0|\le\frac{q}{l(t_0)}\right\}\le\lambda_2(q)<+\infty.$$

We substitute $t=\langle z,m\rangle$, $t_0=\langle z_0,m\rangle$ and obtain

$$\sup\left\{\frac{l(|\langle z,m\rangle|)}{l(|\langle z_0,m\rangle|)}:|\langle z,m\rangle-\langle z^0,m\rangle|\le\frac{q}{l(|\langle z^0,m\rangle|)}\right\}\le\lambda_2(q)<+\infty.$$

Let $z=\widetilde{z}+t\mathbf{b}$, $z^0=\widetilde{z}+t_0\mathbf{b}$. Then we have

$$|\langle z,m\rangle-\langle z^0,m\rangle|=|\langle\mathbf{b},m\rangle|\,|t-t_0|\le\frac{q}{l(|\langle z^0,m\rangle|)}.$$

Hence

$$\sup\left\{\frac{l(|\langle\widetilde{z}+t\mathbf{b},m\rangle|)}{l(|\langle\widetilde{z}+t_0\mathbf{b},m\rangle|)}:|t-t_0|\le\frac{q}{|\langle\mathbf{b},m\rangle|l(|\widetilde{z}+t_0\mathbf{b},m|)}\right\}\le\lambda_2(q)<+\infty.$$

We denote $q^*=\frac{q}{|\langle\mathbf{b},m\rangle|}$. Since the number q is arbitrary, we obtain that for every $q^*>0$ the following inequality is valid

$$\sup\left\{\frac{l(|\langle\widetilde{z}+t\mathbf{b},m\rangle|)}{l(|\langle\widetilde{z}+t_0\mathbf{b},m\rangle|)}:|t-t_0|\le\frac{q^*}{l(|\langle\widetilde{z}+t_0\mathbf{b},m\rangle|)}\right\}\le\lambda_2(q^*|\langle\mathbf{b},m\rangle|)<\infty. \tag{2}$$

A similar inequality can be deduced for inf. Indeed, the condition $l(t)\in Q$ implies the inequality

$$\inf\left\{\frac{l(|t|)}{l(|t_0|)}:|t-t_0|\le\frac{q}{l(t_0)}\right\}\ge\lambda_1(q)>0.$$

As above we substitute $t=\langle\widetilde{z}+t\mathbf{b},m\rangle$ and $t_0=\langle\widetilde{z}+t_0\mathbf{b},m\rangle$ and obtain

$$\inf\left\{\frac{l(|\langle\widetilde{z}+t\mathbf{b},m\rangle|)}{l(|\langle\widetilde{z}+t_0\mathbf{b},m\rangle|)}:|t-t_0|\le\frac{q}{|\langle\mathbf{b},m\rangle|l(|\langle\widetilde{z}+t_0\mathbf{b},m\rangle|)}\right\}\ge\lambda_1(q)>0. \tag{3}$$

Therefore from (2) and (3) we have that $l(|\langle z,m\rangle|)\in Q^n_{\mathbf{b}}$. □

We need an analogue of Hayman's theorem for entire functions of bounded l-index.

Theorem 1 ([7])**.** *An entire function f is of bounded l-index if and only if there exist numbers $p\in\mathbb{Z}_+$ and $C>0$ such, that for every $z\in\mathbb{C}$*

$$\frac{|f^{(p+1)}(z)|}{l^{p+1}(z)}\le C\max\left\{\frac{|f^{(k)}(z)|}{l^k(z)}:0\le k\le p\right\}.$$

This theorem was proved M. M. Sheremeta in [7].

In [1] we proved a proposition, which is a multidimensional analogue of Hayman's theorem for functions of bounded L-index in direction.

Theorem 2 ([1])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F(z), z \in \mathbb{C}^n$, is of bounded L-index in the direction $\mathbf{b}$ if and only if there exist numbers $p \in \mathbb{Z}_+$ and $c > 0$ such that for every $z \in \mathbb{C}^n$*

$$\left|\frac{1}{L^{p+1}(z)}\frac{\partial^{p+1}F(z)}{\partial \mathbf{b}^{p+1}}\right| \le C\max\left\{\left|\frac{1}{L^k(z)}\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| :\ 0\le k\le p\right\}. \tag{4}$$

As a consequence, we obtain the following result.

Theorem 3. *Let $l(|t|) \in Q$. An entire function $f(t)$, $t \in \mathbb{C}$, is of bounded l-index if and only if the entire function $f(\langle z, m\rangle)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$, where $L(z) = l(|\langle z, m\rangle|)$, $z \in \mathbb{C}^n$, $m \in \mathbb{C}^n$, $\langle \mathbf{b}, m\rangle \ne 0$.*

Proof. At first we calculate the directional derivative

$$\frac{\partial^s f(\langle z,m\rangle)}{\partial \mathbf{b}^s} = f^{(s)}(\langle z,m\rangle)\langle \mathbf{b},m\rangle^s \text{ for } s\ge 1. \tag{5}$$

Since the function $f(t)$ is of bounded l-index, by Theorem 1 there exist $p \in \mathbb{Z}_+$ and $C^* > 0$ such that for all $t \in \mathbb{C}$

$$\frac{|f^{(p+1)}(t)|}{l^{p+1}(|t|)} \le C^*\max\left\{\frac{|f^{(k)}(z)|}{l^k(|t|)} : 0\le k\le p\right\}.$$

In other words, for $t = \langle z, m\rangle$ the following estimation holds

$$\frac{1}{l^{p+1}(|\langle z,m\rangle|)}\left|\frac{\partial^{p+1}f(\langle z,m\rangle)}{\partial\mathbf{b}^{p+1}}\right| = \frac{|f^{(p+1)}(\langle z,m\rangle)|}{l^{p+1}(|\langle z,m\rangle|)}\cdot|\langle\mathbf{b},m\rangle|^{p+1}\le$$
$$\le C^*|\langle\mathbf{b},m\rangle|^{p+1}\max\left\{\frac{|f^{(k)}(\langle z,m\rangle)|}{l^k(|\langle z,m\rangle|)} : 0\le k\le p\right\} =$$
$$= C^*|\langle\mathbf{b},m\rangle|^{p+1}\max\left\{\frac{1}{l^k(|\langle z,m\rangle|)|\langle\mathbf{b},m\rangle|^k}\left|\frac{\partial^k f(\langle z,m\rangle)}{\partial\mathbf{b}^k}\right| : 0\le k\le p\right\}\le$$
$$\le C^*\max\{|\langle\mathbf{b},m\rangle|^{p+1-k} : 0\le k\le p\}\max\left\{\frac{1}{l^k(|\langle z,m\rangle|)}\left|\frac{\partial^k f(\langle z,m\rangle)}{\partial\mathbf{b}^k}\right| : 0\le k\le p\right\}$$

Hence there exist $p \in \mathbb{Z}_+$ and $C = C^*\max\{|\langle \mathbf{b}, m\rangle|^{p+1-k} : 0 \le k \le p\}$, that for all $z \in \mathbb{C}^n$ inequality (4) holds. Therefore by Theorem 2 the function $f(\langle z, m\rangle)$ is of bounded L-index in the direction $\mathbf{b}$ ($L(z) = l(|\langle z, m\rangle|) \in Q^n_{\mathbf{b}}$ by Lemma 2).

The proof of sufficiency is similar and uses (5). □

This theorem is useful in the study of boundedness of L-index in direction for some infinite products.

Let π be an entire function in $\mathbb{C}^n$ *of genus p with "plane" zeros*

$$\pi(z) = \prod_{k=1}^{\infty} g(\langle z, a^k|a^k|^{-2}\rangle, p), \tag{6}$$

$$p\ne 0\quad g(u,p) = (1-u)\exp\left\{u+\frac{u^2}{2}+\cdots+\frac{u^p}{p}\right\},\ p=0\quad g(u,0)=(1-u),$$

where $a^k\in\mathbb{C}^n$ is a sequence of genus p, i.e.

$$\sum_{k=1}^{\infty} 1/|a^k|^{p+1} < +\infty,\ \sum_{k=1}^{\infty} 1/|a^k|^{p} = +\infty. \tag{7}$$

We assume that the sequence (a^k) is ordered in such a way that $|a^k|\le|a^{k+1}|$ $(k\ge 1)$. Moreover, we suppose that elements of sequence (a^k) are located on some ray

$$a_j^k = m_j|a^k| \text{ for all } k\ge 1, \tag{8}$$

$m=(m_1,m_2,\ldots,m_n)$. If condition (8) holds then $\pi(z)$ is a function of $\langle z,m\rangle$. For the class of such functions we obtained some conditions (see [4] and [1]) on the sequence a^k, under which $\pi(z)$ is a function of bounded L-index in direction.

We note that for these conditions the proof of sufficiency is similar to that for one-dimensional case ([9], [10]). But in view of Theorem 3 and Lemma 2 now we can apply the corresponding propositions for infinite products from [9], [10] to obtain sufficient conditions of boundedness L-index in direction for functions $\pi(z)$. Thus the next corollaries of Propositions 2–4 from [9] are true. Let $n(r)=\sum_{|a^k|<r}1$.

Corollary 1. *If* $\frac{|a^k|^{p+1}}{k}\nearrow\infty$ $(k\to\infty)$, *(a^k) satisfies condition (8),* $L(z)=l(|\langle z,m\rangle|)$, $l\in Q$, $n(r)\ln n(r)=O(rl(r))$ *and*

$$r^{p-1}\sum_{l=1}^{n(r)}\frac{1}{|a^k|^p}+r^p\sum_{k=n(r)+1}^{\infty}\frac{1}{|a^k|^{p+1}}=O(l(r)),\ r\to+\infty,$$

then a function $\pi(z)$ defined by (6) is a function of bounded L-index in the direction $\mathbf{b}$.

Corollary 2. *Let* $\frac{|a^k|^{p+1}}{k}\nearrow\infty$ $(k\to\infty)$, *(a^k) satisfy condition (8),* $L_1(z)=l_1(|\langle z,m\rangle|)$, $l_1\in Q$ *and* $l_1(r)\asymp r^p\sum_{k=1}^{n(r)}\frac{1}{|a^k|^p}$ $(r_0\le r\to+\infty)$. *If* $\frac{n(r)\ln n(r)}{r}=O(L_1(r))$ $(r\to+\infty)$, *then the function $\pi(z)$ defined by (6) is a function of bounded L_1-index in the direction* $\mathbf{b}$.

Corollary 3. *Let* $\frac{|a^k|^{p+1}}{k}\nearrow\infty$ $(k\to\infty)$, *(a^k) satisfy condition (8),* $L_2(z)=l_2(|\langle z,m\rangle|)$, $l_2(z)\in Q$ *and* $l_2(r)\asymp r^p\sum_{k=n(r)+1}^{\infty}\frac{1}{|a^k|^p}$ $(r_0\le r\to+\infty)$. *If* $\frac{n(r)\ln n(r)}{r}=O(l_2(r))$ $(r\to+\infty)$, *then the function $\pi(z)$ defined by (6) is a function of bounded L_2-index in the direction* $\mathbf{b}$.

Let $\widetilde{Q}$ be the class of nondecreasing functions $l(t)\in Q$. We obtain the next corollaries of Lemma 2 and Theorem 1 from [10].

Corollary 4. *Let* $L(z)=l(|\langle z,m\rangle|)$, $l\in Q$ *and (a^k) satisfy condition (8),* $l(|a^s|))=O(l(|a^{s+1}|))$ $s\to+\infty$, *for some $q_0>0$ and every $k\ge 1$*

$$|a^{k+1}|-|a^k|>\frac{2q_0}{L(|a^{k+1}|)},\ \sum_{k=1}^{s-1}\frac{1}{|a^s|-|a^k|}=O(L(|a^s|)),\sum_{k=s+2}^{\infty}\frac{1}{|a^k|-|a^s|}=O(L(|a^s|)),s\to\infty.$$

Then the function $\pi(z)$ of genus 0 defined by (6) is a function of bounded L-index in the direction $\mathbf{b}$.

Corollary 5. *If for some $\eta > 0$ and every $k \geq 1$ $(1+\eta)|a^k| \leq |a^{k+1}|$ and (a^k) satisfies condition (8) then there exists $L(z) = l(|\langle z, m\rangle|)$, $l \in \widetilde{Q}$ such, that $l(r) \sim \frac{n(r)}{r}$ $(r \to +\infty)$ and the function $\pi(z)$ of genus 0 defined by (6) is a function of bounded L-index in the direction $\mathbf{b}$.*

Theorem 1 from [11] gives one more corollary.

Corollary 6. *If $0 < |a^1| = d_1 \leq d_k = |a^k| - |a^{k-1}| \nearrow \infty$ $(2 \leq k \to \infty)$, (a^k) satisfies condition (8) then there exists $L(z) = l(|\langle z, m\rangle|)$, $l \in Q^n_{\mathbf{b}}$, such that $l(r) \to 0$ $r \to +\infty$, and the function $\pi(z)$ with genus 0 defined by (6) is a function of bounded L-index in the direction $\mathbf{b}$.*

Applying Lemma 9 from [1] to these Corollaries 1–6 and putting $L(\langle z, m\rangle) \equiv 1$, one can obtain corresponding sufficient conditions of boundedness index in the sense of Bordulyak-Sheremeta (see definition in [16]).

For the one-dimensional case, for some past time mathematicians were interested in the following two problems: the problem of the existence of an entire function of bounded l-index for a given l, and the problem of the existence of a function l for a given entire function f such that f is of bounded l-index (see [12]–[15]). It is clear that the same problems can be posed for the multidimensional case.

We note that the solution of the first problem in the one-dimensional case is given by a canonical product. The solution of the first problem in the multidimensional case also exists in the class of canonical product with "plane" zeros.

Theorem 4. *For every positive continuous function $L(z) = l(|\langle z, m\rangle|)$, where $m \in \mathbb{C}^n$ is a fixed vector, $l(t)\colon [0, +\infty) \to (0, +\infty)$ is a continuous function and $rl(r) \to +\infty (r \to +\infty)$ there exists an entire transcendental function F of bounded L-index in every direction $\mathbf{b}$.*

Proof. By Theorem 1 from [13] for every positive continuous function $l(|t|)$, $t \in \mathbb{C}$, such that $rl(r) \to +\infty$ $(r \to +\infty)$, there exists an entire function $f(t)$ of bounded l-index. We put $t = \langle z, m\rangle$ and by Theorem 3 we obtain that $F(z) = f(\langle z, m\rangle)$ is a function of bounded L-index in the direction $\mathbf{b}$. □

We consider the function $F(z^0 + t\mathbf{b})$ if $z^0 \in \mathbb{C}^n$ is fixed. If $F(z^0 + t\mathbf{b}) \not\equiv 0$, then we denote by $p_{\mathbf{b}}(z^0 + a^0_k\mathbf{b})$ the multiplicity of the zero a^0_k of the function $F(z^0 + t\mathbf{b})$. If $F(z^0 + t\mathbf{b}) \equiv 0$ for some $z^0 \in \mathbb{C}^n$, then we put $p_{\mathbf{b}}(z^0 + t\mathbf{b}) = \infty$.

Theorem 5. *In order that for an entire function F there exist a positive continuous function $L(z)$ such that $F(z)$ is a function of bounded L-index in the direction $\mathbf{b}$ it is necessary and sufficient that $\exists p \in \mathbb{Z}_+$ $\forall z^0 \in \mathbb{C}^n$ such, that $F(z^0 + t\mathbf{b}) \not\equiv 0$, and $\forall k$ $p_{\mathbf{b}}(z^0 + a^0_k\mathbf{b}) \leq p$.*

Proof. Necessity. To simplify the notation we consider everywhere in the proof $p^0_k \equiv p_{\mathbf{b}}(z^0 + a^k_0\mathbf{b})$. Necessity follows from the definition of bounded L-index in direction. Indeed, assume on the contrary that $\forall p \in \mathbb{Z}_+$ $\exists z^0$ $\exists k$ $p^0_k > p$. This means that

$$\frac{\partial^{p^0_k} F(z^0 + a^0_k\mathbf{b})}{\partial \mathbf{b}^{p^0_k}} \neq 0 \ \text{ and } \ \frac{\partial^j F(z^0 + a^0_k\mathbf{b})}{\partial \mathbf{b}^j} = 0$$

for all $j \in \{1, \ldots, p^0_k - 1\}$. Therefore L-index in the direction b at the point $z^0 + a^0_k\mathbf{b}$ is not less than $p^0_k > p$

$$N_{\mathbf{b}}(F, L, z^0 + a^0_k\mathbf{b}) > p.$$

If $p\to+\infty$, then we obtain that $N_{\mathbf{b}}(F,L,z^0+a_k^0\mathbf{b})\to+\infty$. But this contradicts the boundedness of L-index in the direction of the function F.
Sufficiency. If for some $z^0\in\mathbb{C}^n$, $F(z^0+t\mathbf{b})\equiv 0$, then inequality (1) is obvious.

Let p be the smallest integer such that $\forall z^0\in\mathbb{C}^n$ $F(z^0+t\mathbf{b})\not\equiv 0$, and $\forall k$ $p_k(z^0)\le p$. For any point $z\in\mathbb{C}^n$ we define unambiguously the choice of $z^0\in\mathbb{C}^n$ and $t_0\in\mathbb{C}$ such that $z=z^0+t_0\mathbf{b}$. We choose a point z^0 on a hyperplane $\langle z,m\rangle=1$, where $\langle\mathbf{b},m\rangle=1$ (actually it is sufficient that $\langle\mathbf{b},m\rangle\ne 0$, i. e. the hyperplane is not parallel to $\mathbf{b}$). Therefore $t_0=\langle z,m\rangle-1$, $z^0=z-(\langle z,m\rangle-1)\mathbf{b}$. We put $K_R=\{t\in\mathbb{C}\colon \max\{0,R-1\}\le|t|\le R+1\}$ for all $R\ge 0$ and

$$m_1(z^0,R)=\min_{a_k^0\in K_R}\left\{\frac{1}{p_k^0!}\left|\frac{\partial^{p_k^0}F(z^0+a_k^0\mathbf{b})}{\partial\mathbf{b}^{p_k^0}}\right|\right\}.$$

Since F is an entire function, there exists $\varepsilon=\varepsilon(z^0,R)>0$ such that

$$\frac{1}{p_k^0!}\left|\frac{\partial^{p_k^0}F(z^0+t\mathbf{b})}{\partial\mathbf{b}^{p_k^0}}\right|\ge\frac{m_1(z^0,R)}{2}$$

for all k and all $t\in K_R\cap\{t\in\mathbb{C}\colon |t-a_k^0|<\varepsilon(R,z^0)\}$. We denote $G_\varepsilon^0=\bigcup_{a_k^0\in K_R}\{t\in\mathbb{C}\colon |t-a_k^0|<\varepsilon\}$, $m_2(z^0,R)=\min\{|F(z^0+t\mathbf{b})|\colon |t|\le R+1, t\notin G_\varepsilon^0\}$,

$$Q(R,z^0)=\min\left\{\frac{m_1(R,z^0)}{2},m_2(R,z^0)\right\}.$$

We take $R=|t_0|$. Then at least one of the numbers $|F(z^0+t_0\mathbf{b})|$, $\left|\frac{\partial F(z^0+t_0\mathbf{b})}{\partial\mathbf{b}}\right|$, …, $\frac{1}{p!}\left|\frac{\partial^p F(z^0+t_0\mathbf{b})}{\partial\mathbf{b}^p}\right|$ is not less than $Q(R,z^0)$ (respectively, $\frac{1}{p_k^0!}\left|\frac{\partial^{p_k^0}F(z^0+t_0)\mathbf{b})}{\partial\mathbf{b}^{p_k^0}}\right|$ for $t_0\in G_\varepsilon^0$ and $|F(z^0+t_0\mathbf{b})|$ for $t\notin G_\varepsilon$). Hence

$$\max\left\{\frac{1}{j!}\left|\frac{\partial^j F(z^0+t_0\mathbf{b})}{\partial\mathbf{b}^j}\right|\colon 0\le j\le p\right\}\ge Q(R,z^0). \tag{9}$$

On the other hand, for $|t_0|=R$ and $j\ge p+1$ Cauchy's inequality is valid

$$\frac{1}{j!}\left|\frac{\partial^j F(z^0+t_0\mathbf{b})}{\partial\mathbf{b}^j}\right|=\left|\frac{1}{2\pi i}\int_{|\tau-t_0|=1}\frac{F(z^0+\tau\mathbf{b})}{(\tau-t_0)^{j+1}}d\tau\right|\le\max\{|F(z^0+\tau\mathbf{b})|\colon |\tau|\le R+1\}. \tag{10}$$

We choose a positive continuous function $L(z)$ such that

$$L(z^0+t_0\mathbf{b})\ge\max\left\{\frac{\max\{|F(z^0+t\mathbf{b})|\colon |\tau|R+1\}}{Q(R,z^0)},1\right\}.$$

From (9) and (10) with $|t_0|=R$ and $j\ge p+1$ we obtain

$$\frac{\frac{1}{j!L^j(z^0+t_0\mathbf{b})}\cdot\left|\frac{\partial^j F(z^0+t_0\mathbf{b})}{\partial\mathbf{b}^j}\right|}{\max\left\{\frac{1}{k!L^k(z^0+t_0\mathbf{b})}\left|\frac{\partial^k F(z^0+t_0\mathbf{b})}{\partial\mathbf{b}^k}\right|\colon 0\le k\le p\right\}}\le\frac{L^{-j}(z^0+t\mathbf{b})}{Q(R,z^0)L^{-p}(z^0+t\mathbf{b})}\times$$
$$\times\max\{|F(z^0+t\mathbf{b})|\colon |\tau|\le R+1\}\le L^{p+1-j}(z^0+t\mathbf{b})\le 1.$$

Since $z=z^0+t\mathbf{b}$, we have

$$\frac{1}{j!L^j(z)}\left|\frac{\partial^j F(z)}{\partial\mathbf{b}^j}\right|\le\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\mathbf{b}^k}\right|\colon 0\le k\le p\right\}.$$

But z is arbitrary. So F is a function of bounded L-index in the direction $\mathbf{b}$. □

Let $\gamma_F(z)$ be a multiplicity of the zero point of function F

$$\gamma_F(z) = \min_{a_k \neq 0} \|k\|$$

for $F(z) = \sum_{\|k\|=0}^{\infty} a_k (z - z_0)^k$, $\|k\| = k_1 + \ldots + k_n$, $k \in \mathbb{Z}_+^n$, $z \in \mathbb{C}^n$. If $F(z^0) = 0$ and for all $j \in \{1, \ldots, p\}$ $\frac{\partial^j F(z^0)}{\partial \mathbf{b}^j} = 0$ and $\frac{\partial^{p+1} F(z^0)}{\partial \mathbf{b}} \neq 0$, then the point z^0 is called zero of multiplicity p in the direction $\mathbf{b}$, and we denote this multiplicity by $p_{\mathbf{b}}(z)$. It is clear that $\gamma_F(z) \leq p_{\mathbf{b}}(F)$. Using the proved theorem we obtain the following corollary.

Corollary 7. *If F is an entire function of bounded index in the direction $\mathbf{b}$ (i. e. $L(z) = 1$), then the multiplicities of the zero points of function F are uniformly bounded.*

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