BOUNDEDNESS OF $L$-INDEX IN DIRECTION OF FUNCTIONS OF THE FORM $f(\langle z,m \rangle)$ AND EXISTENCE THEOREMS

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We obtain a criterion of boundedness of $L$-index in direction for functions $f(\langle z,m \rangle)$. Using this criterion we find sufficient conditions of boundedness $L$-index in direction for some class of entire functions with “plane” zeros. Moreover, we prove some existence theorems of an entire function $f(\langle z,m \rangle)$ of bounded $L$-index in direction for a given $L$ and of a positive continuous function $F$ for a given entire function $F(z)$ such that $F$ is of bounded $L$-index in direction.

1. Introduction. We introduced a class of entire functions of bounded $L$-index in direction as an object of study in [1]–[4]. There were investigated properties of these functions. As usually, the investigations have led to new open problems. For example, find conditions of boundedness of $L$-index in direction for a function $F(z) = f(\langle z,m \rangle)$ and some function $L$, where $\langle z,m \rangle = \sum_{j=1}^{n} z_{j}m_{j}$, $z, m \in \mathbb{C}^{n}$, and $f(t)$ is of bounded $l$-index. Especially, this problem is interesting for entire functions with “plane” zeros (definition see in [5]).

We need some standard notation. For $\eta > 0$, $z \in \mathbb{C}^{n}$, $b = (b_{1}, \ldots, b_{n}) \in \mathbb{C}^{n} \setminus \{0\}$ and a positive continuous function $L$ we define

$$
\lambda_{1}^{b}(z, \eta) = \inf \left\{ \inf \left\{ \frac{L(z + tb)}{L(z + t_{0}b)} : |t - t_{0}| \leq \frac{\eta}{L(z + t_{0}b)} \right\} : t_{0} \in \mathbb{C} \right\},
$$

$$
\lambda_{2}^{b}(z, \eta) = \sup \left\{ \sup \left\{ \frac{L(z + tb)}{L(z + t_{0}b)} : |t - t_{0}| \leq \frac{\eta}{L(z + t_{0}b)} \right\} : t_{0} \in \mathbb{C} \right\},
$$

$$
\lambda_{1}^{b} (\eta) = \inf \{ \lambda_{1}^{b}(z, \eta) : z \in \mathbb{C}^{n} \}, \quad \lambda_{2}^{b} (\eta) = \sup \{ \lambda_{2}^{b}(z, \eta) : z \in \mathbb{C}^{n} \}.
$$

By $Q_{b}^{c}$ we denote the class of functions $L$ which satisfy the condition for all $\eta \geq 0$, $0 < \lambda_{1}^{b}(\eta) \leq \lambda_{2}^{b}(\eta) < +\infty$.

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For a positive continuous function \( l(t), t \in \mathbb{C}, t_0 \in \mathbb{C} \) and \( \eta > 0 \) we set \( \lambda_1(t_0, \eta) \equiv \lambda_1^0(0, t_0, \eta) \) and \( \lambda_2(t_0, \eta) \equiv \lambda_2^0(0, t_0, \eta) \) in the case where \( z = 0, b = 1, n = 1, L \equiv l \), and also \( \lambda_1(\eta) = \inf \{ \lambda_1(t_0, \eta) : t_0 \in \mathbb{C} \}, \lambda_2(\eta) = \sup \{ \lambda_2(t_0, \eta) : t_0 \in \mathbb{C} \} \). As in [8], let \( Q \equiv Q_1^0 \) be the class of positive continuous functions \( l(t), t \in \mathbb{C} \), that satisfy the condition for all \( \eta > 0, 0 < \lambda_1(\eta) \leq \lambda_2(\eta) < +\infty \).

An entire function of \( F(z), z \in \mathbb{C}^n \), is called (see [1]–[3]) a function of bounded \( L \)-index in direction \( b \), if there exists \( m_0 \in \mathbb{Z}_+ \) such that for every \( m \in \mathbb{Z}_+ \) and every \( z \in \mathbb{C}^n \) the following inequality is valid

\[
\frac{1}{m!L^m(z)} \left| \frac{\partial^m F(z)}{\partial b^m} \right| \leq \max \left\{ \frac{1}{k!L^k(z)} \left| \frac{\partial^k F(z)}{\partial b^k} \right| : 0 \leq k \leq m_0 \right\},
\]

(1)

where

\[
\frac{\partial^0 F(z)}{\partial b^0} = F(z), \quad \frac{\partial F(z)}{\partial b} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j = \langle \text{grad } F, b \rangle, \quad \frac{\partial^k F(z)}{\partial b^k} = \frac{\partial}{\partial b} \left( \frac{\partial^{k-1} F(z)}{\partial b^{k-1}} \right), \quad k \geq 2.
\]

Below we formulate assertions that indicate possible ways to construct a function \( L(z) \in Q_b^0 \), given a function \( l(t) \in Q \). Their proofs are based on the definitions of \( Q \) and \( Q_b^0 \). For \( l \in Q \) we denote \( l_1(z) = l(|z|), z \in \mathbb{C}^n \).

**Lemma 1.** If \( l \in Q \) then \( l_1 \in Q_b^0 \) for every \( b \in \mathbb{C}^n \).

**Proof.** Since \( l \in Q \) we have that for \( u \in \mathbb{C} \)

\[
0 < \inf_{u_0 \in \mathbb{C}} \lambda_1(u_0, \eta) \leq \inf \left\{ \frac{l(u)}{l(u_0)} : |u - u_0| \leq \frac{\eta}{l(u_0)} \right\} \leq 1 \\
\leq \sup \left\{ \frac{l(u)}{l(u_0)} : |u - u_0| \leq \frac{\eta}{l(u_0)} \right\} \leq \sup_{u_0 \in \mathbb{C}} \lambda_2(u_0, \eta) < +\infty.
\]

Using these inequalities we obtain

\[
\inf \left\{ \frac{l_1(z^0 + t b)}{l_1(z^0 + t_0 b)} : |t - t_0| \leq \frac{\eta}{l(|z^0 + t_0 b|)} \right\} = \inf \left\{ \frac{l(|z^0 + t b|)}{l(|z^0 + t_0 b|)} : |t - t_0| \leq \frac{\eta}{l(|z^0 + t_0 b|)} \right\} = \\
= \inf \left\{ \frac{l(|z^0 + t b|)}{l(|z^0 + t_0 b|)} : |z^0 + t b - (z^0 + t_0 b)| \leq \frac{|b||\eta|}{l(|z^0 + t_0 b|)} \right\} \geq \\
\geq \inf \left\{ \frac{l(|\tilde{z}|)}{l(|\tilde{z}_0|)} : ||\tilde{z}| - |\tilde{z}_0|| \leq \frac{|b||\eta|}{l(|\tilde{z}_0|)} \right\} \geq \inf \left\{ \frac{l(\tilde{t})}{l(\tilde{t}_0)} : |\tilde{t} - \tilde{t}_0| \leq \frac{|b||\eta|}{l(\tilde{t}_0)} \right\} \geq \lambda_1(|b\eta|) > 0,
\]

where \( \tilde{z} = z^0 + t b, \tilde{z}_0 = z^0 + t_0 b, \tilde{t} = |\tilde{z}|, \tilde{t}_0 = |\tilde{z}_0| \).

Using similar considerations we obtain

\[
\sup \left\{ \frac{l_1(z^0 + t_0 b)}{l_1(z^0 + t_0 b)} : |t - t_0| \leq \frac{\eta}{l(|z^0 + t_0 b|)} \right\} = \sup \left\{ \frac{l(|z^0 + t_0 b|)}{l(|z^0 + t_0 b|)} : |t - t_0| \leq \frac{\eta}{l(|z^0 + t_0 b|)} \right\} = \\
= \sup \left\{ \frac{l(|z^0 + t b|)}{l(|z^0 + t_0 b|)} : |z^0 + t b - (z^0 + t_0 b)| \leq \frac{|b||\eta|}{l(|z^0 + t_0 b|)} \right\} \leq \\
\leq \sup \left\{ \frac{l(|\tilde{z}|)}{l(|\tilde{z}_0|)} : ||\tilde{z}| - |\tilde{z}_0|| \leq \frac{|b||\eta|}{l(|\tilde{z}_0|)} \right\} \leq \sup \left\{ \frac{l(\tilde{t})}{l(\tilde{t}_0)} : |\tilde{t} - \tilde{t}_0| \leq \frac{|b||\eta|}{l(\tilde{t}_0)} \right\} \leq \lambda_2(|b\eta|) < +\infty.
\]

Thus we proved that if \( l \in Q \) then for any \( b \in \mathbb{C}^n \) one has \( l_1 \in Q_b^0 \). \( \square \)
Lemma 2. If \( l(|t|) \in Q \) then for all \( m \in \mathbb{C}^n \) and every \( b \in \mathbb{C}^n \) we have \( l(|\langle z, m \rangle|) \in Q_b^n. \)

Proof. Since \( l(|t|) \in Q \) we have that for any \( q > 0 \)
\[
\sup \left\{ \frac{l(|t|)}{l(|t_0|)} : |t - t_0| \leq \frac{q}{l(t_0)} \right\} \leq \lambda_2(q) < +\infty.
\]
We substitute \( t = \langle z, m \rangle, t_0 = \langle z_0, m \rangle \) and obtain
\[
\sup \left\{ \frac{l(|\langle z, m \rangle|)}{l(|\langle z_0, m \rangle|)} : |\langle z, m \rangle - \langle z_0, m \rangle| \leq \frac{q}{l(|\langle z_0, m \rangle|)} \right\} \leq \lambda_2(q) < +\infty.
\]
Let \( z = \tilde{z} + tb, z^0 = \tilde{z} + t_0b. \) Then we have
\[
|\langle z, m \rangle - \langle z^0, m \rangle| = |\langle b, m \rangle| |t - t_0| \leq \frac{q}{l(|\langle z^0, m \rangle|)}.
\]
Hence
\[
\sup \left\{ \frac{l(|\langle \tilde{z} + tb, m \rangle|)}{l(|\langle \tilde{z} + t_0b, m \rangle|)} : |t - t_0| \leq \frac{q}{l(|\langle \tilde{z} + t_0b, m \rangle|)} \right\} \leq \lambda_2(q) < +\infty.
\]
We denote \( q^* = \frac{q}{l(|\langle b, m \rangle|)} \). Since the number \( q \) is arbitrary, we obtain that for every \( q^* > 0 \) the following inequality is valid
\[
\sup \left\{ \frac{l(|\langle \tilde{z} + tb, m \rangle|)}{l(|\langle \tilde{z} + t_0b, m \rangle|)} : |t - t_0| \leq \frac{q^*}{l(|\langle \tilde{z} + t_0b, m \rangle|)} \right\} \leq \lambda_2(q^*|\langle b, m \rangle|) < \infty. \tag{2}
\]
A similar inequality can be deduced for \( \inf \). Indeed, the condition \( l(t) \in Q \) implies the inequality
\[
\inf \left\{ \frac{l(|t|)}{l(|t_0|)} : |t - t_0| \leq \frac{q}{l(t_0)} \right\} \geq \lambda_1(q) > 0.
\]
As above we substitute \( t = \langle \tilde{z} + tb, m \rangle \) and \( t_0 = \langle \tilde{z} + t_0b, m \rangle \) and obtain
\[
\inf \left\{ \frac{l(|\langle \tilde{z} + tb, m \rangle|)}{l(|\langle \tilde{z} + t_0b, m \rangle|)} : |t - t_0| \leq \frac{q}{l(|\langle \tilde{z} + t_0b, m \rangle|)} \right\} \geq \lambda_1(q) > 0. \tag{3}
\]
Therefore from (2) and (3) we have that \( l(|\langle z, m \rangle|) \in Q_b^n. \)

We need an analogue of Hayman’s theorem for entire functions of bounded \( l \)-index.

Theorem 1 ([7]). An entire function \( f \) is of bounded \( l \)-index if and only if there exist numbers \( p \in \mathbb{Z}_+ \) and \( C > 0 \) such, that for every \( z \in \mathbb{C} \)
\[
\frac{|f^{(p+1)}(z)|}{l^{p+1}(z)} \leq C \max \left\{ \frac{|f^{(k)}(z)|}{l^k(z)} : 0 \leq k \leq p \right\}.
\]
This theorem was proved M. M. Sheremeta in [7].

In [1] we proved a proposition, which is a multidimensional analogue of Hayman’s theorem for functions of bounded \( L \)-index in direction.
Theorem 2 ([1]). Let \( L \in Q^n_b \). An entire function \( F(z) \), \( z \in \mathbb{C}^n \), is of bounded \( L \)-index in the direction \( \mathbf{b} \) if and only if there exist numbers \( p \in \mathbb{Z}_+ \) and \( c > 0 \) such that for every \( z \in \mathbb{C}^n \)
\[
\left| \frac{1}{L^{p+1}(z)} \frac{\partial^{p+1} F(z)}{\partial \mathbf{b}^{p+1}} \right| \leq C \max \left\{ \left| \frac{1}{L^k(z)} \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| : 0 \leq k \leq p \right\}. \tag{4}
\]

As a consequence, we obtain the following result.

Theorem 3. Let \( l(|t|) \in Q \). An entire function \( f(t) \), \( t \in \mathbb{C} \), is of bounded \( l \)-index if and only if the entire function \( f((z, m)) \) is of bounded \( L \)-index in the direction \( \mathbf{b} \in \mathbb{C}^n \), where \( L(z) = l(|(z, m)|) \), \( z \in \mathbb{C}^n \), \( m \in \mathbb{C}^n \), \( \langle \mathbf{b}, m \rangle \neq 0 \).

Proof. At first we calculate the directional derivative
\[
\frac{\partial^s f((z, m))}{\partial \mathbf{b}^s} = f^{(s)}((z, m))\langle \mathbf{b}, m \rangle^s \text{ for } s \geq 1. \tag{5}
\]

Since the function \( f(t) \) is of bounded \( l \)-index, by Theorem 1 there exist \( p \in \mathbb{Z}_+ \) and \( C^* > 0 \) such that for all \( t \in \mathbb{C} \)
\[
\frac{|f^{(p+1)}(t)|}{l^{p+1}(|t|)} \leq C^* \max \left\{ \frac{|f^{(k)}(z)|}{l^k(|t|)} : 0 \leq k \leq p \right\}.
\]

In other words, for \( t = (z, m) \) the following estimation holds
\[
\frac{1}{l^{p+1}(|(z, m)|)} \left| \frac{\partial^{p+1} f((z, m))}{\partial \mathbf{b}^{p+1}} \right| = \frac{|f^{(p+1)}((z, m))|}{l^{p+1}(|(z, m)|)} \cdot |\langle \mathbf{b}, m \rangle|^{p+1} \leq C^* |\langle \mathbf{b}, m \rangle|^{p+1} \max \left\{ \frac{|f^{(k)}((z, m))|}{l^k(|(z, m)|)} : 0 \leq k \leq p \right\} = C^* |\langle \mathbf{b}, m \rangle|^{p+1} \cdot \frac{1}{l^k(|(z, m)|)} \left| \frac{\partial^k f((z, m))}{\partial \mathbf{b}^k} \right| : 0 \leq k \leq p \leq C^* \max\{|\langle \mathbf{b}, m \rangle|^{p+1-k} : 0 \leq k \leq p\} \max \left\{ \frac{1}{l^k(|(z, m)|)} \left| \frac{\partial^k f((z, m))}{\partial \mathbf{b}^k} \right| : 0 \leq k \leq p \right\}.
\]

Hence there exist \( p \in \mathbb{Z}_+ \) and \( C = C^* \max\{|\langle \mathbf{b}, m \rangle|^{p+1-k} : 0 \leq k \leq p\} \), that for all \( z \in \mathbb{C}^n \) inequality (4) holds. Therefore by Theorem 2 the function \( f((z, m)) \) is of bounded \( L \)-index in the direction \( \mathbf{b} \) \( (L(z) = l(|(z, m)|) \in Q^*_b \) by Lemma 2).

The proof of sufficiency is similar and uses (5).

This theorem is useful in the study of boundedness of \( L \)-index in direction for some infinite products.

Let \( \pi \) be an entire function in \( \mathbb{C}^n \) of genus \( p \) with “plane” zeros
\[
\pi(z) = \prod_{k=1}^{\infty} g((z, a^k|a^k|^2), p), \tag{6}
\]
\( p \neq 0 \) \( g(u, p) = (1 - u) \exp \left\{ u + \frac{u^2}{2} + \cdots + \frac{u^p}{p} \right\} \), \( p = 0 \) \( g(u, 0) = (1 - u) \),
where \( a^k \in \mathbb{C}^n \) is a sequence of genus \( p \), i.e.

\[
\sum_{k=1}^{\infty} \frac{1}{|a^k|^{p+1}} < +\infty, \quad \sum_{k=1}^{\infty} \frac{1}{|a^k|^p} = +\infty.
\]

(7)

We assume that the sequence \( (a^k) \) is ordered in such a way that \( |a^k| \leq |a^{k+1}| \) \( (k \geq 1) \). Moreover, we suppose that elements of sequence \( (a^k) \) are located on some ray

\[
a^k_j = m_j |a^k| \quad \text{for all} \quad k \geq 1,
\]

(8)

\( m = (m_1, m_2, \ldots, m_n) \). If condition (8) holds then \( \pi(z) \) is a function of \( \langle z, m \rangle \). For the class of such functions we obtained some conditions (see [4] and [1]) on the sequence \( a^k \), under which \( \pi(z) \) is a function of bounded \( L \)-index in direction.

We note that for these conditions the proof of sufficiency is similar to that for one-dimensional case ([9], [10]). But in view of Theorem 3 and Lemma 2 now we can apply the corresponding propositions for infinite products from [9], [10] to obtain sufficient conditions of boundedness \( L \)-index in direction for functions \( \pi(z) \). Thus the next corollaries of Propositions 2–4 from [9] are true. Let \( n(r) = \sum_{|a^k| < r} 1 \).

**Corollary 1.** If \( \frac{|a^k|^{p+1}}{k} \xrightarrow{\rho} \infty \quad (k \to \infty) \), \( (a^k) \) satisfies condition (8), \( L(z) = l(|\langle z, m \rangle|) \), \( l \in Q \), \( n(r) \ln n(r) = O(r \ell(l(r))) \) and

\[
r^{p-1} \sum_{l=1}^{n(r)} \frac{1}{|a^k|^p} + r^p \sum_{k=n(r)+1}^{\infty} \frac{1}{|a^k|^{p+1}} = O(l(r)), \quad r \to +\infty,
\]

then a function \( \pi(z) \) defined by (6) is a function of bounded \( L \)-index in the direction \( b \).

**Corollary 2.** Let \( \frac{|a^k|^{p+1}}{k} \xrightarrow{\rho} \infty \quad (k \to \infty) \), \( (a^k) \) satisfy condition (8), \( L_1(z) = l_1(|\langle z, m \rangle|) \), \( l_1 \in Q \) and \( l_1(r) \asymp r^p \sum_{k=1}^{n(r)} \frac{1}{|a^k|^p} \) \( (r_0 \leq r \to +\infty) \). If \( \frac{n(r) \ln n(r)}{r} = O(L_1(r)) \) \( (r \to +\infty) \), then the function \( \pi(z) \) defined by (6) is a function of bounded \( L_1 \)-index in the direction \( b \).

**Corollary 3.** Let \( \frac{|a^k|^2}{k} \xrightarrow{\rho} \infty \quad (k \to \infty) \), \( (a^k) \) satisfy condition (8), \( L_2(z) = l_2(|\langle z, m \rangle|) \), \( l_2 \in Q \) and \( l_2(r) \asymp r^p \sum_{k=n(r)+1}^{\infty} \frac{1}{|a^k|^p} \) \( (r_0 \leq r \to +\infty) \). If \( \frac{n(r) \ln n(r)}{r} = O(l_2(r)) \) \( (r \to +\infty) \), then the function \( \pi(z) \) defined by (6) is a function of bounded \( L_2 \)-index in the direction \( b \).

Let \( \widetilde{Q} \) be the class of nondecreasing functions \( l(t) \in Q \). We obtain the next corollaries of Lemma 2 and Theorem 1 from [10].

**Corollary 4.** Let \( L(z) = l(|\langle z, m \rangle|) \), \( l \in Q \) and \( (a^k) \) satisfy condition (8), \( l(|a^s|) = O(l(|a^{s+1}|)) \) \( s \to +\infty \), for some \( q_0 > 0 \) and every \( k \geq 1 \)

\[
|a^{k+1}| - |a^k| \geq \frac{2q_0}{L(|a^{k+1}|)}, \quad \sum_{k=1}^{s-1} \frac{1}{|a^s| - |a^k|} = O(L(|a^s|)), \quad \sum_{k=s+2}^{\infty} \frac{1}{|a^k| - |a^s|} = O(L(|a^s|)), \quad s \to \infty.
\]

Then the function \( \pi(z) \) of genus \( 0 \) defined by (6) is a function of bounded \( L \)-index in the direction \( b \).
Corollary 5. If for some $\eta > 0$ and every $k \geq 1 \ (1 + \eta)|a^k| \leq |a^{k+1}|$ and $(a^k)$ satisfies condition (8) then there exists $L(z) = l(|\langle z, m \rangle|)$, $l \in Q$, such that $l(r) \sim \frac{n(r)}{r^p} (r \to +\infty)$ and the function $\pi(z)$ of genus 0 defined by (6) is a function of bounded $L$-index in the direction $b$.

Theorem 1 from [11] gives one more corollary.

Corollary 6. If $0 < |a^1| = d_1 \leq d_k = |a^k| - |a^{k-1}| \nearrow \infty \ (2 \leq k \to \infty)$, $(a^k)$ satisfies condition (8) then there exists $L(z) = l(|\langle z, m \rangle|)$, $l \in Q^b$, such that $l(r) \to 0 \ r \to +\infty$, and the function $\pi(z)$ with genus 0 defined by (6) is a function of bounded $L$-index in the direction $b$.

Applying Lemma 9 from [1] to these Corollaries 1–6 and putting $L(|\langle z, m \rangle|) \equiv 1$, one can obtain corresponding sufficient conditions of boundedness index in the sense of Bordulyak-Shcheremeta (see definition in [16]).

For the one-dimensional case, for some past time mathematicians were interested in the following two problems: the problem of the existence of an entire function of bounded $l$-index for a given $l$, and the problem of the existence of a function $l$ for a given entire function $f$ such that $f$ is of bounded $l$-index (see [12]–[15]). It is clear that the same problems can be posed for the multidimensional case.

We note that the solution of the first problem in the one-dimensional case is given by a canonical product. The solution of the first problem in the multidimensional case also exists in the class of canonical product with “plane” zeros.

Theorem 4. For every positive continuous function $L(z) = l(|\langle z, m \rangle|)$, where $m \in \mathbb{C}^n$ is a fixed vector, $l(t) : [0, +\infty) \to (0, +\infty)$ is a continuous function and $rl(r) \to +\infty (r \to +\infty)$ there exists an entire transcendental function $F$ of bounded $L$-index in every direction $b$.

Proof. By Theorem 1 from [13] for every positive continuous function $l(|t|)$, $t \in \mathbb{C}$, such that $rl(r) \to +\infty (r \to +\infty)$, there exists an entire function $f(t)$ of bounded $l$-index. We put $t = \langle z, m \rangle$ and by Theorem 3 we obtain that $F(z) = f(\langle z, m \rangle)$ is a function of bounded $L$-index in the direction $b$.

We consider the function $F(z^0 + tb)$ if $z^0 \in \mathbb{C}^n$ is fixed. If $F(z^0 + tb) \equiv 0$, then we denote by $p_b(z^0 + a^0_k b)$ the multiplicity of the zero $a^0_k$ of the function $F(z^0 + tb)$. If $F(z^0 + tb) \equiv 0$ for some $z^0 \in \mathbb{C}^n$, then we put $p_b(z^0 + tb) = \infty$.

Theorem 5. In order that for an entire function $F$ there exist a positive continuous function $L(z)$ such that $F(z)$ is a function of bounded $L$-index in the direction $b$ it is necessary and sufficient that $\exists p \in \mathbb{Z}_+ \ \forall z^0 \in \mathbb{C}^n$ such that $F(z^0 + tb) \not\equiv 0$, and $\forall k \ p_b(z^0 + a^0_k b) \leq p$.

Proof. Necessity. To simplify the notation we consider everywhere in the proof $p^0_k = p_b(z^0 + a^0_k b)$. Necessity follows from the definition of bounded $L$-index in direction. Indeed, assume on the contrary that $\forall p \in \mathbb{Z}_+ \ \exists z^0 \ \exists k \ p^0_k > p$. This means that

$$\frac{\partial^0 k F(z^0 + a^0_k b)}{\partial b^{p^0_k}} \neq 0 \quad \text{and} \quad \frac{\partial^j F(z^0 + a^0_k b)}{\partial b^j} = 0$$

for all $j \in \{1, \ldots, p^0_k - 1\}$. Therefore $L$-index in the direction $b$ at the point $z^0 + a^0_k b$ is not less than $p^0_k > p$

$$N_b(F, L, z^0 + a^0_k b) > p.$$
If \( p \to +\infty \), then we obtain that \( N_b(F, L, z^0 + a_k^0 b) \to +\infty \). But this contradicts the boundedness of \( L \)-index in the direction of the function \( F \).

**Sufficiency.** If for some \( z^0 \in \mathbb{C}^n \), \( F(z^0 + t b) \equiv 0 \), then inequality (1) is obvious.

Let \( p \) be the smallest integer such that \( \forall z^0 \in \mathbb{C}^n \) \( F(z^0 + t b) \neq 0 \), and \( \forall k p_k(z^0) \leq p \). For any point \( z \in \mathbb{C}^n \) we define unambiguously the choice of \( z^0, t_0 \), \( t_0 \in \mathbb{C} \) such that \( z = z^0 + t_0 b \). We choose a point \( z^0 \) on a hyperplane \( \langle z, m \rangle = 1 \), where \( \langle b, m \rangle = 1 \) (actually it is sufficient that \( \langle b, m \rangle \neq 0 \), i. e. the hyperplane is not parallel to \( b \)). Therefore \( t_0 = (z, m) - 1, z^0 \equiv z - (\langle z, m \rangle - 1) b \). We put \( K_R = \{ t \in \mathbb{C} : \max \{ 0, R - 1 \} \leq |t| \leq R + 1 \} \) for all \( R \geq 0 \) and

\[
m_1(z^0, R) = \min_{a_k^0 \in K_R} \left\{ \frac{1}{p_k^0!} \left| \frac{\partial^{p_k^0} F(z^0 + a_k^0 b)}{\partial b_k^{p_k^0}} \right| \right\}.
\]

Since \( F \) is an entire function, there exists \( \varepsilon = \varepsilon(z^0, R) > 0 \) such that

\[
\frac{1}{p_k^0!} \left| \frac{\partial^{p_k^0} F(z^0 + t b)}{\partial b_k^{p_k^0}} \right| \geq \frac{m_1(z^0, R)}{2}
\]

for all \( k \) and all \( t \in K_R \cap \{ t \in \mathbb{C} : |t - a_k^0| < \varepsilon(R, z^0) \} \). We denote \( G_\varepsilon = \bigcup_{a_k^0 \in K_R} \{ t \in \mathbb{C} : |t - a_k^0| < \varepsilon \} \), \( m_2(z^0, R) = \min \{ |F(z^0 + t b)| : |t| \leq R + 1, t \notin G_\varepsilon \} \),

\[
Q(R, z^0) = \min \left\{ \frac{m_1(R, z^0)}{2}, m_2(R, z^0) \right\}.
\]

We take \( R = |t_0| \). Then at least one of the numbers \( |F(z^0 + t_0 b)|, \left| \frac{\partial F(z^0 + t_0 b)}{\partial b} \right|, \ldots, \left| \frac{\partial^j F(z^0 + t_0 b)}{\partial b^j} \right| \) is not less than \( Q(R, z^0) \) (respectively, \( \left| \frac{\partial^j F(z^0 + t_0 b)}{\partial b^j} \right| \) for \( t_0 \in G_\varepsilon \) and \( |F(z^0 + t_0 b)| \) for \( t \notin G_\varepsilon \)). Hence

\[
\max \left\{ \frac{1}{j!} \left| \frac{\partial^j F(z^0 + t_0 b)}{\partial b^j} \right| : 0 \leq j \leq p \right\} \geq Q(R, z^0).
\]

On the other hand, for \( |t_0| = R \) and \( j \geq p + 1 \) Cauchy’s inequality is valid

\[
\frac{1}{j!} \left| \frac{\partial^j F(z^0 + t_0 b)}{\partial b^j} \right| = \left| \frac{1}{2\pi i} \int_{|\tau - t_0| = 1} \frac{F(z^0 + \tau b)}{(\tau - t_0)^{j + 1}} d\tau \right| \leq \max \{ |F(z^0 + \tau b)| : |\tau| \leq R + 1 \}.
\]

We choose a positive continuous function \( L(z) \) such that

\[
L(z_0 + t_0 b) \geq \max \left\{ \frac{\max \{ |F(z^0 + t b)| : |\tau| \leq |R + 1| \}}{Q(R, z^0)}, 1 \right\}.
\]

From (9) and (10) with \( |t_0| = R \) and \( j \geq p + 1 \) we obtain

\[
\max \left\{ \frac{1}{k!L(z_0 + t b)} \left| \frac{\partial^k F(z_0 + t_0 b)}{\partial b^k} \right| : 0 \leq k \leq p \right\} \leq \frac{L^{-j}(z_0 + t b)}{Q(R, z^0)L^{-p}(z_0 + t b)} \times
\]

\[
\times \max \{ |F(z^0 + t b)| : |\tau| \leq R + 1 \} \leq L^{p+1-j}(z_0 + t b) \leq 1.
\]

Since \( z = z^0 + t b \), we have

\[
\frac{1}{j!L(z)} \left| \frac{\partial^j F(z)}{\partial b^j} \right| \leq \max \left\{ \frac{1}{k!L(z)} \left| \frac{\partial^k F(z)}{b^k} \right| : 0 \leq k \leq p \right\}.
\]

But \( z \) is arbitrary. So \( F \) is a function of bounded \( L \)-index in the direction \( b \). \( \square \)
Let $\gamma_F(z)$ be a multiplicity of the zero point of function $F$

$$\gamma_F(z) = \min_{a_k \neq 0} \|k\|$$

for $F(z) = \sum_{\|k\|=0}^{\infty} a_k (z - z_0)^k$, $\|k\| = k_1 + \ldots + k_n$, $k \in \mathbb{Z}_+^n$, $z \in \mathbb{C}^n$. If $F(z_0) = 0$ and for all $j \in \{1, \ldots, p\}$ \(\frac{\partial F(z_0)}{\partial b_j} = 0\) and \(\frac{\partial^{p+1} F(z_0)}{\partial b_j} \neq 0\), then the point $z_0$ is called zero of multiplicity $p$ in the direction $b$, and we denote this multiplicity by $p_b(z)$. It is clear that $\gamma_F(z) \leq p_b(F)$. Using the proved theorem we obtain the following corollary.

**Corollary 7.** If $F$ is an entire function of bounded index in the direction $b$ (i.e. $L(z) = 1$), then the multiplicities of the zero points of function $F$ are uniformly bounded.

**REFERENCES**


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