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DIAGONAL REDUCTION OF MATRICES OVER FINITE STABLE RANGE RINGS

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The aim of this review is to present the results of the participants of the scientific seminar "Problems of elementary divisor rings" concerning the Bezout rings of finite stable range.

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Представлены, в основном, результаты участников научного семинара "Проблемы колец элементарных делителей", которые относятся к кольцам Безу конечного стабильного ранга.

This paper is a report of B. V. Zabavsky at the 9-th International Algebraic Conference in Ukraine. All definitions and known facts presented in monographs [25], [27].

The aim of this review is to present the results of the participants of the scientific seminar "Problems of elementary divisor rings" concerning the Bezout rings of finite stable range and investigate the conditions under which these rings are elementary divisor rings. These rings were introduced by I. Kaplansky.

Definition 1 ([1])**.** A ring R is called an *elementary divisor ring*, if for an arbitrary matrix A of order $n\times m$ over, there exist invertible matrices $P\in GL_n(R)$ and $Q\in GL_m(R)$ such that 1) $PAQ=D$ is diagonal matrix, $D=(d_i)$; 2) $Rd_{i+1}R\subseteq Rd_i\cap d_iR$ for each i.

Definition 2 ([2])**.** If for any $a,b\in R$ there exists an invertible matrix $P\in GL_2(R)$ such that $(a,b)P=(d,0)$ for some $d\in R$ then R is a *right Hermite ring.* If for any $a,b\in R$ there exists an invertible matrix $P\in GL_2(R)$ such that $P\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}d\\0\end{pmatrix}$ for some element $d\in R$, then R is a *left Hermite ring.* If a ring is both left and right Hermite, then it is called an *Hermite ring.*

Definition 3. A *right (left) Bezout ring* is a ring in which every finitely generated right (left) ideal is principal, and if both R is a Bezout ring.

Theorem 1 ([2])**.** *Every right (left) Hermite ring is a right (left) Bezout ring.*

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Theorem 2 ([1]). *Let R be an elementary divisor ring. Then every finitely presented R-module can be represented as direct sum of cyclic modules.*

In the case of commutative rings the converse statement is proved.

Theorem 3 ([3]). *A commutative ring R is an elementary divisor ring if and only if every finitely presented R-modules is a direct sum of cyclic modules.*

Gillman and Henriksen constructed an example of a commutative Bezout ring which is not an elementary divisor ring ([26]).

Open Problems:

1. *Is a commutative Bezout domain an elementary divisor ring([4], [5])?*
2. *Under which conditions is a Bezout ring a Hermite ring([4])?*
3. *Is any commutative Bezout ring with compact space of minimal prime ideals a Hermite ring([13])?*

Let R be a simple ring, then for every nonzero element $a \in R$ we obtain that $RaR = R$.

Definition 4. If for every nonzero element there are elements $u_1, u_2, v_1, v_2 \in R$ such that $u_1av_1 + u_2av_2 = 1$ then the ring R is called a 2-*simple ring.*

Theorem 4 ([7]). *A simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.*

Theorem 5 ([8], [9]). *A distributive ring is an elementary divisor ring if it is a duo-ring.*

Definition 5 ([10]). A row $(a_1, \dots, a_n) \in R^n$ is called a *unimodular row* if $a_1R + a_2R + \dots + a_nR = R$. It is said that n is a *stable range* of a ring R if for any unimodular row $(a_1, \dots, a_n, a_{n+1}) \in R^{n+1}$ there are elements $x_1, \dots, x_n \in R$ such that the row $(a_1 + a_{n+1}x_1, a_2 + a_{n+1}x_2, \dots, a_n + a_{n+1}x_n) \in R^n$ is unimodular.

Theorem 6 ([11]). *A commutative Bezout ring is a Hermite ring if and only if its stable range is 2.*

Definition 6. $\min R$ is defined to be the space of minimal prime ideals of a commutative ring R.

Let $x \in R$. If we let $D(x) = \{P \in \min R \mid x \notin P\}$ then the set of all $D(x)$ is a base of the Zarisky topology an $\min R$. If we say that $\min R$ is compact then we mean that it is compact in this topology.

Theorem 7 ([12], [25]). *A stable range of a commutative Bezout ring with compact space of minimal prime ideals is 2 and then it is a Hermite ring.*

Definition 7. A ring R is called a *Kazimirskii ring* if for any nonzero elements $a, b \in R$ there are elements $s, t \in R$ and a unit $u \in U(R)$ such that $as + ubt = 1$.

Theorem 8 ([7]). *A simple Bezout domain of stable range 1 is an elementary divisor ring if and only if it is a Kazimirskii ring.*

Theorem 9 ([14]). *Let R be an Ore domain which is a Kazimirskii domain. Then for any full matrix A of order n there exist invertible matrices P, $Q \in GL_n(R)$ such that*

$$PAQ = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & a & b \\ 0 & 0 & \dots & 0 & c & d \end{pmatrix}$$

Among all the commutative rings that are elementary divisor rings we have a special class of adequate rings.

Definition 8 ([15]). A commutative ring R is called an *adequate ring*, if for every pair of elements $a, b \in R$ $(a \neq 0)$ there exists a pair of element $a_1, d \in R$ such that 1) $a = a_1 d$; 2) $a_1 R + bR = R$; 3) for every noninvertible factor d' of d we have $d'R + bR \neq R$.

Theorem 10 ([16]). *A stable range of adequate Bezout ring is 2.*

Theorem 11 ([16]). *Any adequate ring is an elementary divisor ring if and only if it is a Bezout ring.*

Theorem 12 ([17], [19]). *A commutative Bezout ring is an adequate ring if and only if for any nonzero element $a \in R$ the factor-ring R/aR is a semi-regular ring.*

Definition 9 ([18]). A ring R is *left morphic* if for every $a \in R$ we have $R/Ra \cong l(a)$, where $l(a)$ denotes the left annihilator *of* a.

Theorem 13 (Zabavsky B., Pihura O., 2014). *Let R be a commutative Bezout domain and $a \in R \setminus \{0\}$. Then the factor-ring R/aR is a morphic ring.*

A ring is called *clean* if every element is the sum of a unit and an idempotent.

Question ([18]). *Is every left and right morphic ring clean?*

The answer is “not”.

Definition 10. A ring R is said to be a *left Kasch* ring if $r(I) \neq 0$ for every left ideal I of R where $r(I)$ denotes the right annihilator of I.

Theorem 14. *The following assertions are equivalent for a commutative Bezout domain. 1) R/aR is a Kasch ring for every $a \in R \setminus \{0\}$. 2) Every maximal ideal R is principal.*

Definition 11. An element a of a ring R is called an *adequate element* if for any $b \in R$ there are $r, s \in R$ such that $a = rs$, where $rR + bR = R$ and for any noninvertible divisor s' of s we have $s'R + bR \neq R$.

Definition 12. A ring R is a ring of *idempotent stable range 1* if the condition $aR + bR = R$ implies that there exists an idempotent $e \in R$ such that $a + be$ is an invertible element of the ring R.

Theorem 15 ([19]). *Let R be a commutative Bezout ring in which 0 is an adequate element. Then R is a ring of idempotent stable range 1.*

Theorem 16 ([19]). *Let R be a commutative Bezout domain and a be an adequate element. Then R/aR is a ring in which 0 is an adequate element (= exchange ring = clean ring).*

Theorem 17 ([19]). *A commutative Bezout ring R of stable range 2 is an elementary divisor ring if and only if for each ideal $I \neq 0$ every finitely generated projective R/I-module is a direct sum of principal ideals generated by idempotents.*

Definition 13. A ring R is said to be an *ID-ring* if every idempotent matrix over R is diagonalizable under the similarity transformation.

Theorem 18 ([19]). *A commutative Bezout domain is an elementary divisor ring if and only if for each ideal I the ring R/I is an ID-ring.*

Definition 14. A ring R is called an *f-ring (F-ring)* if every pure ideal of R is generated by idempotents (idempotent).

Theorem 19 ([19]). *Let R be a commutative Bezout ring of stable range 2, and for each ideal I the factor-ring R/I be an f (F)-ring. Then R is an elementary divisor ring.*

Gilman introduced the notion of a sharp domain via property (#). We say that a commutative domain R has property (#) if for any two distinct subsets M and N of $\operatorname{mspec} R$ we have $\bigcap_{P\in N} R_P \neq \bigcap_{P\in M} R_P$. A domain R is called a *sharp domain* if each overring of R has property (#).

Theorem 20 ([28]). *A sharp Bezout domain is an elementary divisor ring.*

Theorem 21 ([28]). *A sharp Bezout domain is an adequate domain if and only if each nonzero prime ideal is contained in a unique maximal ideal.*

Let us define the following types of column (row) operations with a matrix A over a ring R

1) column (row) switching;
2) column (row) right (left) multiplication by an invertible element;
3) column (row) addition of one column (row) right (left) multiplied by some element to another column (row).

Definition 15. A matrix A over a ring R is said *to admit an elementary diagonal reduction* if it can be reduced to the canonical diagonal form using only elementary column and row operations

$$\begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \varepsilon_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $R\varepsilon_{i+1}R \subset \varepsilon_i R \cap R\varepsilon_i$.

If every matrix over a ring R admits an elementary diagonal reduction then R is called a *ring with elementary matrix reduction*.

Definition 16. Let $a, b \in R$, $a \neq 0$, and n be an arbitrary positive integer. An n-stage right division chain starting from the pair (a, b) is a sequence of equations $b = aq_1 + r_1,\ a = r_1 q_2 + r_2,\ \ldots,\ r_{n-2} = r_{n-1} q_n + r_n$.

Let R be a domain and there be a function $N\colon R \to \mathbb{Z}$ such that 1) $N(0) = 0$; 2) $N(a) > 0$ if $a \neq 0$. Then we say that N is a norm on R.

A domain R is called a right *ω-Euclidean ring* relatively to a norm N if for every pair of elements $a, b \in R$, $a \neq 0$, one can find $k \in \mathbb{N}$ and a right divisibility chain of lenght k such that $N(r_k) < N(b)$.

Definition 17. A ring R is called a *GE_n-ring* if $GE_n(R) = GL_n(R)$.

Proposition 1 ([20])**.** *A commutative domain is an ω-Euclidean domain if and only if it is both a GE_2-ring and a Bezout domain.*

Proposition 2. *A ring with an elementary matrix reduction is an ω-Euclidean ring.*

Definition 18. A commutative domain R with a norm N is called a 2-*Euclidean ring* with respect to the norm N if for any $a \in R$ and $b \in R \setminus \{0\}$ al least one of the following assertions is true

1) there exist $q, r \in R$ such that $a = bq + r$ where $r = 0$ or $N(r) < N(b)$;
2) there exist $q_1, r_1, q_2, r_2 \in R$ such $b = aq_1 + r_1,\ a = r_1 q_2 + r_2$, where $r_2 = 0$ or $N(r_2) < N(b)$.

Theorem 22 ([21])**.** *Any 2-Euclidean commutative ring is a ring with an elementary matrix reduction.*

Theorem 23 ([22])**.** *If R is a right ω-Euclidean Bezout ring then R is also a left ω-Euclidean ring.*

Theorem 24 ([25])**.** *Let R be an elementary divisor ring. Then for any $n \times m$ matrix A ($n > 2$, $m > 2$) there exists invertible matrices $P \in GE_n(R)$ and $Q \in GE_m(R)$ such that*

$$PAQ = \begin{pmatrix} \varepsilon_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \varepsilon_2 & \ldots & 0 & 0 & \ldots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \ldots & \varepsilon_r & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \multicolumn{5}{c}{\dotfill} & A_0 & \\ 0 & 0 & \ldots & 0 & 0 & & \end{pmatrix}$$

where $R\varepsilon_{i+1}R \subset \varepsilon_i R \cap R\varepsilon_i$ and A_0 is either $2 \times k$ or $k \times 2$ matrix for some $k \in \mathbb{N}$.

Theorem 25 ([25])**.** *Let R be an elementary divisor ring. Then for any $n \times m$ matrix A with $m - n = 2$ there are invertible matrices $P \in GL_n(R)$ and $Q \in GE_m(R)$ such that*

$$PAQ = \begin{pmatrix} \varepsilon_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \varepsilon_2 & \ldots & 0 & 0 & \ldots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \ldots & \varepsilon_r & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}$$

where $R\varepsilon_{i+1}R \subset \varepsilon_i R \cap R\varepsilon_i$.

Theorem 26 ([25]). *Let R be a commutative adequate ring. Then for any nonsigular matrix of order n we can find unimodular matrices $P \in GE_n(R)$ and $Q \in GL_m(R)$ such that $PAQ = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, where $R\varepsilon_{i+1}R \subset \varepsilon_i R \cap R\varepsilon_i$.*

Definition 19. A ring R is said to be *n-fold stable range 1* if for $a_1R + b_1R = R$, $a_2R + b_2R = R, \dots, a_nR + b_nR = R$, there exists some $y \in R$ such that $(a_1 + b_1y)R = R$, $(a_2 + b_2y)R = R, \ \dots, \ (a_n + b_ny)R = R$.

The class of n-fold stable range 1 rings is very large. It includes the ring of all totally real algebraic integers in $\mathbb{C}$ and the rings of continuous complex valued functions on a 1-dimensional complex space.

Theorem 27. *Let R be a ring. Then the following assertions are equivalent:*

1) *R is an n-fold stable range 1 ring;*
2) *$Ra_1+Rb_1 = R, \ \dots, \ Ra_n+Rb_n = R$ implies that $R(a_1+rb_1) = R, \ \dots, \ R(a_n+rb_n) = R$ for some $z \in R$.*

Theorem 28. *Let R be an adequate domain and $a \in R\setminus\{0\}$. Then the following statements are equivalent: 1) R/aR is an n-fold stable range 1 ring; 2) $|R/M| > n$ for all maximal ideals M containing a.*

One can easily prove the following facts.

1) R is a 1-fold stable range 1 ring if and only if R has stable range 1;
2) if R is a 2-fold stable range 1 ring than R has unit 1-stable range;
3) if R is a 3-fold stable range 1 it satisfies the Goodearl-Menal conditions (for any $x, y \in R$); there exists some $u \in U(R)$ such that $x - u, y - u^{-1} \in U(R)$ — conditions Goodearl-Menal).

Definition 20. A commutative ring R is said to be of *n-fold stable range 2* if for any $a_1R + b_1R + c_1R = R, \ a_2R + b_2R + c_2R = R, \ \dots, \ a_nR + b_nR + c_nR = R$ there exists $y \in R$ such that $(a_1 + b_1y)R + c_1R = R, \ (a_2 + b_2y)R + c_2R = R, \ \dots, \ (a_n + b_ny)R + c_nR = R$.

We know that $\mathbb{C}[x]$ is an n-fold stable range 2 ring.

Theorem 29 ([23]). *Let R be an n-fold stable range 1 Bezout ring. Then for any n-matrices $A_1, \dots, A_n \in M_2(R)$ there exist $P \in E_2(R)$ and $Q_1, Q_2, \dots, Q_n \in GL_2(R)$ such that $PA_1Q_1 = \begin{pmatrix} \varepsilon_1^1 & 0 \\ * & \varepsilon_2^1 \end{pmatrix}, \dots, \ PA_nQ_n = \begin{pmatrix} \varepsilon_1^n & 0 \\ * & \varepsilon_2^n \end{pmatrix}$, where $\varepsilon_1^i, \varepsilon_2^i$ are elementary divisors of matrices A_i.*

Theorem 30 ([24]). *Let R be an n-fold stable range 2 Bezout ring. Then for any n-matrices $A_1, \dots, A_n \in M_2(R)$ there exists $P \in GE_2(R)$ and $Q_1, Q_2, \dots, Q_n \in GL_2(R)$ such that $PA_1Q_1 = \begin{pmatrix} \varepsilon_1^1 & 0 \\ * & \varepsilon_2^1 \end{pmatrix}, \dots, \ PA_nQ_n = \begin{pmatrix} \varepsilon_1^n & 0 \\ * & \varepsilon_2^n \end{pmatrix}$, where $\varepsilon_1^i, \varepsilon_2^i$ are elementary divisors of matrices A_i.*

Definition 21. A commutative ring R is said to be of *neat range 1* if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that for the element $a + bt = c$ the ring R/cR is a clean ring.

Theorem 31. *Let R be a commutative Bezout domain and let $c \in R \setminus \{0\}$. Then $\overline{R} = R/cR$ is a clean ring if and only if for any elements $a, b \in R$ such $aR + bR + cR = R$ there exist elements $r, s \in R$ such that $c = rs$ and $rR + aR = R$, $sR + bR = R$, $rR + sR = R$.*

Proof. Let $\overline{R}$ be a clean ring. By [27], $\overline{R}$ is an exchange ring. Let $\bar{a} = a + cR$, $\bar{b} = b + cR$. Then there exists an idempotent $\bar{e} \in \overline{R}$ such that $\bar{e} \in \bar{a}\overline{R}$ and $\bar{1} - \bar{e} \in \bar{b}\overline{R}$. Since $\bar{e} \in \bar{a}\overline{R}$, one has that $e - ap = cs$ for some elements $p, s \in R$. Similarly $1 - e - b\alpha = c\beta$ for some elements $\alpha, \beta \in R$. By the substitution $e = cs + ap$ and $1 - e - b\alpha = c\beta$ we get $apR + cR + bR = R$.

Since $\bar{e} = \bar{e}^2$, we obtain $e(1 - e) = ct$ for some element $t \in R$. Let $eR + cR = dR$. Since stable range R equals 2 and by [11] we have $e = de_0$, $c = dc_0$, where $e_0R + c_0R = R$, $e_0(1 - e) = c_0t$ and then $e + c_0j = 1$ for some element $j \in R$. Taking $r = c_0$, $s = d$. We obtain the decomposition $c = rs$ where $rR + eR = R$ and $sR \subset eR$. Since $e = ap + cs$, we have $rR + apR = R$, $sR \subset apR$. Since $aR + bR + cR = R$ and $pR + bR + cR = R$, we get $apR + bR + cR = R$. If $sR \subset apR$ then $sR + bR = R$. Obviously, $rR + sR = R$ and $rR + aR = R$. Let $aR + bR + cR = R$ and $c = rs$, where $rR + sR = R$, $rR + aR = R$ and $sR + bR = R$. Let $\bar{r} = r + cR$, $\bar{s} = s + cR$. Since $rR + sR = R$, one has $ru + sv = 1$ and $\bar{r}^2\bar{u} = \bar{r}$, $\bar{s}^2\bar{v} = \bar{s}$. Let $\bar{s}\bar{v} = \bar{e}$, obviously $\bar{e}^2 = \bar{e}$ and $\bar{1} - \bar{e} = \bar{r}\bar{u}$. Since $rR + aR = R$, we obtain $\bar{a}\bar{\beta}\bar{e} = \bar{e}$ for some element $\bar{\beta} \in \overline{R}$. Similarly $\bar{b}\bar{x}(\bar{1} - \bar{e}) = \bar{1} - \bar{e}$ for some element $\bar{x} \in \overline{R}$. We prove that if $\bar{a}\overline{R} + \bar{b}\overline{R} = \overline{R}$ then there exists an idempotent $\bar{e}$ such that $\bar{e} \in \bar{a}\overline{R}$ and $\bar{1} - \bar{e} \in \bar{b}\overline{R}$. We have proved that $\overline{R}$ is a clean ring. □

Theorem 32 ([29], [30])**.** *A commutative Hermite ring is an elementary divisor ring if and only if for any elements $a, b, c \in R$ such that $aR + bR + cR = R$ there exist elements $t, r, s \in R$ such that $b + ct = rs$ and $rR + aR = R$, $sR + cR = R$ and $rR + sR = R$.*

Theorem 33. *A commutative Bezout domain is an elementary divisor ring if and only if R is a ring of neat range 1.*

Definition 22. A commutative domain R is said to be a *Dirichlet ring* if for any elements $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that the element $a + bt$ is an atom of the ring R.

Obvious examples of Dirichlet ring are $\mathbb{Z}$, $R = \{z_0 + a_1x + \dots + a_nx^n + \dots \mid z_0 \in \mathbb{Z}, a_1 \in \mathbb{Q}\}$, $P[x]$, where P is a finite field. A Dirichlet Bezout ring is a ring of neat range 1.

As an obvious consequence we have the following result.

Theorem 34. *A Dirichlet Bezout ring is an elementary divisor ring.*

Open question. *Let R be a commutative Bezout domain in which every maximal ideal is principal. Under what conditions the ring R is a Dirichlet ring?*

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