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BARI–MARKUS PROPERTY FOR DIRAC OPERATORS

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We prove the Bari–Markus property for spectral projectors of non-self-adjoint Dirac operators on $(0,1)$ with square-integrable matrix-valued potentials and some separated boundary conditions.

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Доказывается свойство Бари–Маркуса для спектральных проекторов несамосопряженного оператора Дирака на интервале $(0,1)$ с квадратично-интегрируемым матричным потенциалом и некоторыми разделенными краевыми условиями.

1. Introduction and main results. In the Hilbert space $\mathbb{H} := L_2((0,1),\mathbb{C}^{2r})$, we study the non-self-adjoint Dirac operator $T_Q := J\frac{\mathrm{d}}{\mathrm{d}x} + Q$ on the domain

$$D(T_Q) := \left\{(y_1,y_2)^\top \mid y_1,y_2 \in W_2^1((0,1),\mathbb{C}^r), \quad y_1(0)=y_2(0),\ y_1(1)=y_2(1)\right\}.$$

Here,

$$J := \frac{1}{\mathrm{i}}\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}, \quad Q := \begin{pmatrix} 0 & q_1 \\ q_2 & 0\end{pmatrix},$$

$I := I_r$ is the $r\times r$ identity matrix, $q_1, q_2 \in L_2((0,1),\mathcal{M}_r)$, $\mathcal{M}_r$ is the set of $r\times r$ matrices with complex entries and $W_2^1((0,1),\mathbb{C}^r)$ is the Sobolev space of $\mathbb{C}^r$-valued functions. All functions Q as above form the set $\mathfrak{Q}_2 := \{Q \in L_2((0,1),\mathcal{M}_{2r}) \mid JQ(x) = -Q(x)J \text{ a.e. on } (0,1)\}$ and will be called *potentials* of the operators T_Q.

The spectrum $\sigma(T_Q)$ of the operator T_Q consists of countably many isolated eigenvalues of finite algebraic multiplicities. We denote by $\lambda_j := \lambda_j(Q)$, $j\in\mathbb{Z}$, the pairwise distinct eigenvalues of the operator T_Q arranged by non-decreasing of their real — and then, if equal, imaginary — parts. For definiteness, we also assume that $\operatorname{Re}\lambda_0 \le 0 < \operatorname{Re}\lambda_1$. One can prove using the standard technique based on Rouche's theorem that the numbers λ_j, $j\in\mathbb{Z}$, satisfy the condition

$$\sup_{n\in\mathbb{Z}} \sum_{\lambda_j\in\Delta_n} 1 < \infty \tag{1}$$

and the asymptotics

$$\sum_{n\in\mathbb{Z}}\sum_{\lambda_j\in\Delta_n} |\lambda_j - \pi n|^2 < \infty, \tag{2}$$

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where $\Delta_n := \{\lambda \in \mathbb{C} \mid \pi n - \pi/2 < \operatorname{Re}\lambda \le \pi n + \pi/2\}$, $n \in \mathbb{Z}$. We then denote by P_{λ_j} the spectral projector of the operator T_Q corresponding to the eigenvalue λ_j (see [8, Chap.3]). We write

$$\mathcal{P}_n := \sum_{\lambda_j \in \Delta_n} P_{\lambda_j}, \quad n \in \mathbb{Z},$$

for the spectral projector of T_Q corresponding to the strip Δ_n.

In particular, in the free case $Q = 0$ one has $\sigma(T_0) = \{\pi n\}_{n \in \mathbb{Z}}$. We then write $\mathcal{P}_n^0$ for the spectral projector of the free operator T_0 corresponding to the strip Δ_n, $n \in \mathbb{Z}$.

The main result of this paper is the following theorem.

Theorem 1. *For every $Q \in \mathfrak{Q}_2$, we have*

$$\sum_{n \in \mathbb{Z}} \left\| \mathcal{P}_n - \mathcal{P}_n^0 \right\|^2 < \infty . \tag{3}$$

Inequality (3) is called the *Bari–Markus property* of spectral projectors of the operator T_Q.

In the scalar case $r = 1$, the Bari–Markus property for the operator T_Q, as well as for the operators with periodic and anti-periodic boundary conditions, was established in [1] to prove the unconditional convergence of spectral decompositions for such operators. Therein, P. Djakov and B. Mityagin used a technique based on Fourier representations of Dirac operators. This technique was further developed to prove the similar property for Dirac operators with regular boundary conditions in [3]. For Hill operators with singular potentials, the Bari–Markus property was established in [2].

A different and simpler technique based on some convenient representation of resolvents of the operators under consideration was used in [6] to establish the Bari–Markus property for Sturm–Liouville operators with matrix-valued potentials (see [6, Lemma 2.12]). Therein, this result was used to solve the inverse spectral problem for such operators. For the same purpose, the Bari–Markus property was established for self-adjoint Dirac operators with square-integrable matrix-valued potentials in [5].

In the present paper, we use the technique suggested in [6] to establish the Bari–Markus property for non-self-adjoint Dirac operators with square-integrable matrix-valued potentials. This result can be used to study the inverse spectral problems for non-self-adjoint Dirac operators on a finite intervals.

The paper is organized as follows. In the reminder of this sections, we introduce some notation that is used in this paper. In Sections 2 and 3, we provide some preliminary results and prove Theorem 1, respectively.

2. Notations. Throughout this paper, we identify $\mathcal{M}_r$ with the Banach algebra of linear operators in $\mathbb{C}^r$ endowed with the standard norm. If there is no ambiguity, we write simply $\|\cdot\|$ for norms of operators and matrices.

By $L_2((a,b),\mathcal{M}_r)$ we denote the Banach space of all strongly measurable functions $f\colon (a,b) \to \mathcal{M}_r$ for which the norm

$$\|f\|_{L_2} := \left(\int_a^b \|f(t)\|^2 \mathrm{d}t \right)^{1/2}$$

is finite. By $G_2(\mathcal{M}_r)$ we denote the set of all measurable functions $K\colon [0,1]^2 \to \mathcal{M}_r$ such that for all $x, t \in [0,1]$, the functions $K(x,\cdot)$ and $K(\cdot,t)$ belong to $L_2((0,1),\mathcal{M}_r)$ and, moreover, the mappings $[0,1] \ni x \mapsto K(x,\cdot) \in L_2((0,1),\mathcal{M}_r)$ and $[0,1] \ni t \mapsto K(\cdot,t) \in L_2((0,1),\mathcal{M}_r)$ are continuous. The symbol $G_2^+(\mathcal{M}_r)$ stands for the set of all functions $K \in G_2(\mathcal{M}_r)$ such that $K(x,t) = 0$ a.e. in the triangle $\Omega_- := \{(x,t) \mid 0 < x < t < 1\}$. The superscript $\top$ designates the transposition of vectors and matrices.

3. Preliminary results. In this section, we obtain some preliminary results and introduce some auxiliary objects that will be used in this paper.

For an arbitrary potential $Q \in \mathfrak{Q}_2$ and $\lambda \in \mathbb{C}$, we denote by $Y_Q(\cdot,\lambda) \in W_2^1((0,1),\mathcal{M}_{2r})$ the $2r \times 2r$ matrix-valued solution of the Cauchy problem

$$J\frac{\mathrm{d}}{\mathrm{d}x}Y + QY = \lambda Y, \quad Y(0,\lambda) = I_{2r}. \tag{4}$$

We set $\varphi_Q(\cdot,\lambda) := Y_Q(\cdot,\lambda)Ja^*$ and $\psi_Q(\cdot,\lambda) := Y_Q(\cdot,\lambda)a^*$, where $a := \frac{1}{\sqrt{2}}\begin{pmatrix} I, & -I\end{pmatrix}$, so that $\varphi_Q(\cdot,\lambda)$ and $\psi_Q(\cdot,\lambda)$ are the $2r \times r$ matrix-valued solutions of the Cauchy problems

$$J\frac{\mathrm{d}}{\mathrm{d}x}\varphi + Q\varphi = \lambda\varphi, \quad \varphi(0,\lambda) = Ja^*, \tag{5}$$

and $J\frac{\mathrm{d}}{\mathrm{d}x}\psi + Q\psi = \lambda\psi,\ \psi(0,\lambda) = a^*$, respectively. For an arbitrary $\lambda \in \mathbb{C}$, we introduce the operator $\Phi_Q(\lambda)\colon \mathbb{C}^r \to \mathbb{H}$ by the formula $[\Phi_Q(\lambda)c](x) := \varphi_Q(x,\lambda)c,\ x \in [0,1]$.

We set $s_Q(\lambda) := a\varphi_Q(1,\lambda)$ and $c_Q(\lambda) := a\psi_Q(1,\lambda)$, $\lambda \in \mathbb{C}$. The function $m_Q(\lambda) := -s_Q(\lambda)^{-1}c_Q(\lambda)$ will be called the *Weyl–Titchmarsh function* of the operator T_Q. Note that in the free case $Q = 0$ one has $s_0(\lambda) = (\sin\lambda)I$, $c_0(\lambda) = (\cos\lambda)I$ and $m_0(\lambda) = -(\cot\lambda)I$.

The following proposition is a straightforward analogue of Lemma 2.1 in [5].

Proposition 1. *For an arbitrary potential $Q \in \mathfrak{Q}_2$ the following assertions are true:*

(i) there exists a unique function $K_Q \in G_2^+(\mathcal{M}_{2r})$ such that for every $x \in [0,1]$ and $\lambda \in \mathbb{C}$,

$$\varphi_Q(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x K_Q(x,s)\varphi_0(s,\lambda)\,\mathrm{d}s,$$

where $\varphi_0(\cdot,\lambda)$ is a solution of (5) in the free case $Q = 0$;

(ii) there exist unique functions $f_1 := f_{Q,1}$ and $f_2 := f_{Q,2}$ from $L_2((-1,1),\mathcal{M}_r)$ such that for every $\lambda \in \mathbb{C}$,

$$s_Q(\lambda) = (\sin\lambda)I + \frac{1}{\sqrt{2}}\int_{-1}^1 \mathrm{e}^{\mathrm{i}\lambda s}f_1(s)\,\mathrm{d}s, \quad c_Q(\lambda) = (\cos\lambda)I + \frac{1}{\sqrt{2}}\int_{-1}^1 \mathrm{e}^{\mathrm{i}\lambda s}f_2(s)\,\mathrm{d}s. \tag{6}$$

In particular, Proposition 1 implies the following corollary

Corollary 1. *For an arbitrary $Q \in \mathfrak{Q}_2$ and $\lambda \in \mathbb{C}$,*

$$\Phi_Q(\lambda) = (\mathcal{I} + \mathcal{K}_Q)\Phi_0(\lambda), \tag{7}$$

where $\mathcal{K}_Q$ is the integral operator with kernel K_Q and $\mathcal{I}$ is the identity operator in $\mathbb{H}$.

Using the first formula in (6) and repeating the proof of Theorem 3 in [7], one can also derive the following statement.

Corollary 2. *The set of zeros of the entire function* $\widetilde{s}_Q(\lambda) := \det s_Q(\lambda)$ *can be indexed (counting multiplicities) by numbers* $n \in \mathbb{Z}$ *so that the corresponding sequence* $(\xi_n)_{n\in\mathbb{Z}}$ *has the asymptotics* $\xi_{kr+j} = \pi k + \omega_{j,k},\ k \in \mathbb{Z},\ j \in \{0, \dots, r-1\}$, *where the sequences* $(\omega_{j,k})_{k\in\mathbb{Z}}$ *belong to* $\ell_2(\mathbb{Z})$.

Now let $\rho(T_Q)$ denote the resolvent set of the operator T_Q.

Lemma 1. *For an arbitrary* $Q \in \mathfrak{Q}_2$ *we have* $\rho(T_Q) = \{\lambda \in \mathbb{C} \mid \ker s_Q(\lambda) = \{0\}\}$ *and for each* $\lambda \in \rho(T_Q)$,

$$(T_Q - \lambda\mathcal{I})^{-1} = \Phi_Q(\lambda) m_Q(\lambda) \Phi_{Q^*}(\overline{\lambda})^* + \mathcal{T}_Q(\lambda), \tag{8}$$

where $\mathcal{T}_Q$ *is an entire operator-valued function. The spectrum of the operator* T_Q *consists of countably many isolated eigenvalues of finite algebraic multiplicities.*

Proof. A direct verification shows that $\frac{\mathrm{d}}{\mathrm{d}x}\big(JY_{Q^*}(x,\overline{\lambda})^* JY_Q(x,\lambda)\big) = 0$. Therefore, taking into account (4), we find that $-JY_{Q^*}(x,\overline{\lambda})^* JY_Q(x,\lambda) = I_{2r}$ for every $x \in [0,1]$ and thus

$$Y_Q(x,\lambda) J Y_{Q^*}(x,\overline{\lambda})^* = J, \quad x \in [0,1].$$

Since $J = Ja^*a + a^*aJ$, the latter can be rewritten as

$$\varphi_Q(x,\lambda)\psi_{Q^*}(x,\overline{\lambda})^* - \psi_Q(x,\lambda)\varphi_{Q^*}(x,\overline{\lambda})^* = J, \quad x \in [0,1]. \tag{9}$$

Using (9), one can verify that for an arbitrary $f \in \mathbb{H}$ and $\lambda \in \mathbb{C}$, the function

$$g(x,\lambda) = [\mathcal{T}_Q(\lambda)f](x) := \psi_Q(x,\lambda)\int_0^x \varphi_{Q^*}(t,\overline{\lambda})^* f(t)\,\mathrm{d}t + \varphi_Q(x,\lambda)\int_x^1 \psi_{Q^*}(t,\overline{\lambda})^* f(t)\,\mathrm{d}t$$

solves the Cauchy problem

$$Jy' + Qy = \lambda y + f, \quad y_1(0) = y_2(0). \tag{10}$$

Since for every $c \in \mathbb{C}^r$, the function $h(\cdot,\lambda) := \varphi_Q(\cdot,\lambda)c$ solves (10) with $f = 0$, it then follows that a generic solution of (10) takes the form $y = \varphi_Q(\cdot,\lambda)c + \mathcal{T}_Q(\lambda)f$, $c \in \mathbb{C}^r$. If $\lambda \in \mathbb{C}$ is such that the $r \times r$ matrix $s_Q(\lambda) := a\varphi_Q(1,\lambda)$ is non-singular, then setting

$$c = -s_Q(\lambda)^{-1} c_Q(\lambda) \int_0^1 \varphi_{Q^*}(t,\overline{\lambda})^* f(t)\,\mathrm{d}t$$

we obtain that $ay(1) = 0$, i.e. $y_1(1) = y_2(1)$. Therefore, every $\lambda \in \mathbb{C}$ such that $\ker s_Q(\lambda) = \{0\}$ is a resolvent point of the operator T_Q and for such a λ one has

$$(T_Q - \lambda\mathcal{I})^{-1} = \Phi_Q(\lambda) m_Q(\lambda) \Phi_{Q^*}(\overline{\lambda})^* + \mathcal{T}_Q(\lambda).$$

To complete the proof, it remains to observe that the function $y = \varphi_Q(\cdot,\lambda)c$ is a non-zero solution of the problem

$$Jy' + Qy = \lambda y, \quad y_1(0) = y_2(0), \quad y_1(1) = y_2(1)$$

if and only if $c \in \ker s_Q(\lambda) \setminus \{0\}$. Since the values of the resolvent of the operator T_Q are compact operators, it follows that all spectral projectors P_{λ_j}, $j \in \mathbb{Z}$, are finite dimensional. In particular, it then follows (see, e.g., [4, Theorem 2.2]) that all eigenvalues of the operator T_Q are of finite algebraic multiplicities. □

From Lemma 1 we obtain that eigenvalues of the operator T_Q are zeros of the entire function $\widetilde{s}_Q(\lambda) := \det s_Q(\lambda)$. In view of Corollary 2 we then obtain the following consequence.

Corollary 3. *For an arbitrary potential $Q \in \mathfrak{Q}_2$, eigenvalues of the operator T_Q satisfy condition (1) and asymptotics (2).*

Now we can introduce the spectral projectors of the operator T_Q as explained in the previous section. Formulas (7) and (8) will serve as an efficient tool to prove Theorem 1.

4. Proof of Theorem 1. We start with the following auxiliary lemma.

Lemma 2. *For an arbitrary $\lambda \in \mathbb{C}$, let an operator $A(\lambda)\colon L_2((-1,1),\mathcal{M}_r) \to \mathcal{M}_r$ act by the formula*

$$A(\lambda)f := \frac{1}{\sqrt{2}}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\lambda t} f(t)\,\mathrm{d}t.$$

Then for an arbitrary $f \in L_2((-1,1),\mathcal{M}_r)$ and $\lambda \in \mathbb{T}_0 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$,

$$\sum_{n\in\mathbb{Z}} \|A(\pi n + \lambda)f\|^2 \le 9r\|f\|^2_{L_2}. \tag{11}$$

Proof. Let $f \in L_2((-1,1),\mathcal{M}_r)$, $\lambda \in \mathbb{T}_0$ and $\|S\|_2$ denote the Hilbert–Schmidt norm of a matrix $S \in \mathcal{M}_r$. Since $\left\{\frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi n t}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L_2(-1,1)$, it follows that

$$\sum_{n\in\mathbb{Z}} \|A(\pi n)f\|^2 \le \sum_{n\in\mathbb{Z}} \|A(\pi n)f\|_2^2 = \int_{-1}^{1} \|f(x)\|_2^2\,\mathrm{d}x \le r\int_{-1}^{1} \|f(x)\|^2\,\mathrm{d}x.$$

Taking into account that $A(\pi n + \lambda)f = A(\pi n)f_1$ with $f_1(t) := \mathrm{e}^{\mathrm{i}\lambda t}f(t)$ and that $\|f_1\|_{L_2} < 3\|f\|_{L_2}$, we then arrive at (11). □

Remark 1. In the notation of the above lemma, formulas (6) can be rewritten as

$$s_Q(\lambda) = (\sin\lambda)I + A(\lambda)f_1, \quad c_Q(\lambda) = (\cos\lambda)I + A(\lambda)f_2. \tag{12}$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Using formula (8) and the asymptotics (2) of eigenvalues of the operator T_Q, we obtain that there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{Z}$ with $|n| > N$,

$$\mathcal{P}_n := -\frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}_n} \Phi_Q(\lambda)m_Q(\lambda)\Phi_{Q^*}(\overline{\lambda})^*\mathrm{d}\lambda, \quad \mathcal{P}_n^0 := -\frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}_n} \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^*\mathrm{d}\lambda,$$

where $\mathbb{T}_n := \{\lambda \in \mathbb{C} \mid |\lambda - \pi n| = 1\}$. Therefore, for each $n \in \mathbb{Z}$ such that $|n| > N$,

$$\|\mathcal{P}_n - \mathcal{P}_n^0\| = \left\|-\frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}_n} \left(\Phi_Q(\lambda)m_Q(\lambda)\Phi_{Q^*}(\overline{\lambda})^* - \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^*\right)\mathrm{d}\lambda\right\| \le \|\alpha_n\| + \|\beta_n\|,$$

where

$$\alpha_n := -\frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}_n} \Phi_Q(\lambda)(m_Q(\lambda) - m_0(\lambda))\Phi_{Q^*}(\overline{\lambda})^*\mathrm{d}\lambda \tag{13}$$

and

$$\beta_n := -\frac{1}{2\pi\mathrm{i}}\oint_{\mathbb{T}_n} \left(\Phi_Q(\lambda)m_0(\lambda)\Phi_{Q^*}(\overline{\lambda})^* - \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^*\right)\mathrm{d}\lambda.$$

The theorem will be proved once we show that $\sum_{|n|>N} \|\alpha_n\|^2 < \infty$ and $\sum_{|n|>N} \|\beta_n\|^2 < \infty$.

Let us prove the claim for (α_n) first. Taking into account (12), observe that

$$m_Q(\lambda) - m_0(\lambda) = s_Q(\lambda)^{-1}\left[(\cot\lambda)A(\lambda)f_1 - A(\lambda)f_2\right], \tag{14}$$

where $A(\lambda)$ is defined in Lemma 2. Note that by virtue of the Riemann–Lebesgue lemma, without loss of generality we may assume that

$$\sup_{|n|>N}\sup_{\lambda\in\mathbb{T}_n} \|A(\lambda)f_1\| \le \frac{1}{4}.$$

Since for every $\lambda \in \mathbb{T}_n$ one has $|\sin\lambda| \ge 1/2$, in view of the first formula in (12) it then holds

$$\|s_Q(\lambda)^{-1}\| \le |\sin\lambda|^{-1}(1 - |\sin\lambda|^{-1}\|A(\lambda)f_1\|)^{-1} \le 4, \quad \lambda\in\mathbb{T}_n, \quad |n|>N.$$

Since $|\cot\lambda| \le \sqrt{3}$ as $\lambda \in \mathbb{T}_n$, from (14) we then obtain that

$$\|m_Q(\lambda) - m_0(\lambda)\|^2 \le 64(\|A(\lambda)f_1\|^2 + \|A(\lambda)f_2\|^2), \quad \lambda\in\mathbb{T}_n, \quad |n|>N. \tag{15}$$

Next, taking into account (7), observe that for an arbitrary $Q \in \mathfrak{Q}_2$ and $\lambda \in \mathbb{T}_n$ one has

$$\|\Phi_Q(\lambda)\| \le \|\mathcal{I} + \mathcal{K}_Q\|\|\Phi_0(\lambda)\| \le 2\|\mathcal{I} + \mathcal{K}_Q\|. \tag{16}$$

By virtue of the Cauchy–Bunyakovsky inequality we then obtain from (13), (15) and (16) that for every $n \in \mathbb{Z}$ such that $|n| > N$,

$$\|\alpha_n\|^2 \le C\int_0^{2\pi} \left(\|A(\pi n + \mathrm{e}^{\mathrm{i}t})f_1\|^2 + \|A(\pi n + \mathrm{e}^{\mathrm{i}t})f_2\|^2\right)\mathrm{d}t$$

with some $C > 0$. In view of Lemma 2 we then obtain that $\sum_{|n|>N} \|\alpha_n\|^2 < \infty$.

Thus it remains only to prove that $\sum_{|n|>N} \|\beta_n\|^2 < \infty$. For this purpose, take into account (7) and observe that

$$\begin{gathered}\Phi_Q(\lambda)m_0(\lambda)\Phi_{Q^*}(\overline{\lambda})^* - \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^* = \\ = \mathcal{K}_Q\Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^* + \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^*\mathcal{K}_{Q^*}^* + \mathcal{K}_Q\Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^*\mathcal{K}_{Q^*}^*.\end{gathered}$$

Therefore, $\beta_n = \mathcal{K}_Q\mathcal{P}_n^0 + [\mathcal{K}_{Q^*}\mathcal{P}_n^0]^* + \mathcal{K}_Q\mathcal{P}_n^0\mathcal{K}_{Q^*}^*$ and thus the claim will be proved once we show that for an arbitrary $Q \in \mathfrak{Q}_2$,

$$\sum_{|n|>N} \|\mathcal{K}_Q\mathcal{P}_n^0\|^2 < \infty. \tag{17}$$

To this end, note that the operator $\mathcal{K}_Q$ belongs to the Hilbert–Schmidt class $\mathcal{B}_2$ and that the sequence $(\mathcal{P}_n^0)_{n\in\mathbb{Z}}$ consists of pairwise orthogonal projectors. Therefore, we get

$$\sum_{n\in\mathbb{Z}} \|\mathcal{K}_Q\mathcal{P}_n^0\|^2 \le \sum_{n\in\mathbb{Z}} \|\mathcal{K}_Q\mathcal{P}_n^0\|_{\mathcal{B}_2}^2 \le \|\mathcal{K}_Q\|_{\mathcal{B}_2}^2.$$

Hence (17) follows and the proof is complete. □

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