ON THE $l$-INDEX BOUNDEDNESS OF ENTIRE RIDGE FUNCTIONS

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1. An entire function $\varphi \not\equiv \text{const}$ is called ridge provided $|\varphi(z)| \leq |\varphi(i \text{Im } z)|$ for all $z \in \mathbb{C}$. The concept of ridge function generalizes the concept of the characteristic functions of probability laws. In particular, entire characteristic functions form a proper subclass of ridge functions ([1, p. 42–51]). It is well known ([1, p. 45]) that zeros of an entire ridge function are symmetric with respect to the imaginary axis.

2. The aim of our note is to establish the $l$-index boundedness of some classes of ridge functions. We will begin with entire ridge functions of finite order. The following theorem is correct.

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Theorem 1. If an entire ridge function \( \varphi \) of finite order has only real zeros and its positive zeros \( a_k \) satisfy the condition \( a_k^2 - a_{k-1}^2 \nearrow +\infty \) as \( k \to \infty \), then \( \varphi \) is a function of bounded \( l \)-index with \( l(r) = r \) for \( r \geq r_0 > 0 \). The condition \( a_k^2 - a_{k-1}^2 \nearrow +\infty \) as \( k \to \infty \) can not be replaced by the condition \( a_k^2 - a_{k-1}^2 \to +\infty \) as \( k \to \infty \).

Proof. By a result of Gol’dberg and Ostrovskii ([3]) every entire ridge function \( \varphi \) of finite order with only real zeros has a form

\[
\varphi(z) = ce^{-\gamma z^2 + ibz} \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{a_k^2} \right),
\]

where \( c, \gamma, \beta \) and \( a_k \) are constant, \( \gamma \geq 0 \), \( \text{Im} \beta = 0 \), \( a_k > 0 \) and \( \sum_{k=1}^{\infty} \frac{1}{a_k^2} < +\infty \).

We put

\[
\varphi_1(z) = ce^{-\gamma z^2 + ibz}, \quad \varphi_2(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{a_k^2} \right)
\]

and we will show that each of these functions is of bounded \( l \)-index with \( l(r) = r \) for \( r \geq r_0 > 0 \).

Since the function \( \varphi_1 \) does not have zeros and

\[
\left| \frac{\varphi_1'(z)}{\varphi_1(z)} \right| = | -2\gamma z + i\beta | \leq 2\gamma r + |\beta| \leq \left( 2\gamma + \frac{|\beta|}{r_0} \right) r,
\]

this function is of bounded \( l \)-index with \( l(r) = r \) for \( r \geq r_0 > 0 \) by Lemma 1.

Now we consider

\[
\pi(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z}{b_k} \right), \quad b_k = a_k^2.
\]

It is clear that \( b_k - b_{k-1} \nearrow +\infty \) as \( k \to \infty \) and \( \sum_{k=1}^{\infty} \frac{1}{b_k} < +\infty \). G. Fricke ([4]) proved that the function \( \pi \) is of bounded index under such conditions, and \( b_k - b_{k-1} \nearrow +\infty \) as \( k \to \infty \) can not be replaced by the condition \( b_k - b_{k-1} \to +\infty \) as \( k \to \infty \).

In [5] it is proved that if the function \( l \in \Lambda \) satisfies \( l(r) = (1 + o(1))r^{n-1} \) as \( r \to +\infty \) for some \( n \geq 2 \), \( f \) is an entire function and \( Q(z) = c_n z^n + \cdots + c_1 z + c_0, c_n \neq 0 \), then the function \( g(z) = f(Q(z)) \) is of bounded \( l \)-index if and only if \( f \) is of bounded index. Therefore, since \( \varphi_2(z) = \pi(z^2) \) and \( \pi \) are functions of bounded index, \( \varphi_2 \) is of bounded \( l \)-index with \( l(r) = r \) for \( r \geq r_0 \).

Finally, since \( \varphi_1 \) and \( \varphi_2 \) are of bounded \( l \)-index with \( l(r) = r \) \( (r \geq r_0) \), by Theorem 2.3 from [2, p. 84] their product \( \varphi \) is also of bounded \( l \)-index with \( l(r) = r \) for \( r \geq r_0 \).

3. Now we consider an entire function given by finite Fourier-Stieltjes transform

\[
\varphi(z) = \int_{-a}^{a} e^{izt} dF(t),
\]

where \( 0 < a < +\infty \) and \( F \) is a function of bounded variation. If \( F \) is non-decreasing (or non-increasing) on \([-a, a]\), then the function (3) is ridge. If \( F \) is non-decreasing on \([-a, a]\), \( F(x) = 0 \) for \( x < -a \) and \( F(x) = 1 \) for \( x \geq a \), then \( F \) is a probability law and (3) is its characteristic function.
We suppose that \( F(-a) \neq F(-a+0) \) and \( F(a) \neq F(a-0) \). Then ([6, p. 41–42]) all zeros of \( \varphi \) are contained in some strip \( \{ z: |\text{Im} \ z| \leq h \} \) and have the form

\[
z_n = \frac{\pi n}{a} + O(1), \quad n \to \pm \infty.
\] (4)

Using this result it is easy to prove the following theorem.

**Theorem 2.** If a function \( F \) is of bounded variation on \([-a, a]\), \( F(-a) \neq F(-a+0) \) and \( F(a) \neq F(a-0) \), then finite Fourier-Stieltjes transform (3) is a function of bounded index.

**Proof.** Since the functions \( w = z \) and \( w = \sin z \) are of bounded \( l \)-index, by Theorem 2.3 from [2, p. 34] the function

\[
\pi_1(z) = \frac{\sin z}{z} = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\pi^2 n^2} \right),
\]

and thus ([7]) the function \( \pi_2(z) = \pi_1(az) \) are also of bounded index.

In the paper [8] it is proved that if zeros \( z_n \) of an entire function \( f \) of bounded index lie on the finite system of rays and \( \psi_n = O(1) \) as \( n \to \infty \), then the function \( f_\psi \) with zeros \( z_n + \psi_n \) is also of bounded index. Since the entire function \( \pi_2 \) is of bounded index and has zeros \( z_n = \frac{\pi n}{a} \), the function with zeros (4), that is the function (3), is of bounded index. \( \square \)

We remark that in the case of characteristic functions Theorem 2 complements the following result of S. Shah ([9]): if a function \( p(t) \geq 0 \) is absolutely continuous on \([-a, a]\), \( p(-a) \neq 0, p(a) \neq 0 \) and \( \int_{-a}^{a} p(t) dt = 1 \), then the function \( \varphi(z) = \int_{-a}^{a} e^{itz} p(t) dt \) is of bounded index.

4. For a probability law \( F \) we put \( W_F(x) = F(-x) + 1 - F(x), x \geq 0 \). Since the function \( F \) is non-decreasing on \((-\infty, +\infty)\) and continuous from the right, \( F(-x) \to 0 \) and \( F(x) \to 1 \) as \( x \to +\infty \), we have \( W_F(x) \to 0 \) as \( x \to +\infty \). The equality \( W_F(x) = 0 \) for \( x \geq x_0 \) is a necessary condition for \( \varphi \) to be of bounded index, because if \( W_F(x) \neq 0 \) for all \( x \geq 0 \), then the characteristic function \( \varphi(z) = \int_{-\infty}^{+\infty} e^{itz} dF(t) \) is of bounded index. Indeed, if \( \varphi \) is of bounded index, then by theorem of Hayman and Shah [10; 11; 2, p. 59] \( \varphi \) is a function of exponential type \( \sigma \). Hence it follows by Theorem 2.4.2 from [1, p. 53] that \( W_F(x) > 0 \) for \( 0 \leq x < \sigma \) and \( W_F(x) = 0 \) for \( x > \sigma \), which is impossible.

We generalize now the above statement on the unboundedness of index to the statement on the unboundedness of \( l \)-index. For this purpose we denote by \( \Omega \) a class of positive unbounded on \((-\infty, +\infty)\) functions \( \Phi \) such that the derivative \( \Phi' \) is positive continuously differentiable and increasing to \(+\infty\) on \((-\infty, +\infty)\). For \( \Phi \in \Omega \) let \( \phi \) be the function inverse to \( \Phi' \), and let \( \Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma) \) be the function associated with \( \Phi \) in the sense of Newton. Let \( P \) be an arbitrary function on \((0, +\infty)\) different from \(+\infty\) (it can take a value \(-\infty\), but \( P \neq -\infty \)), and let \( Q(r) = \sup \{ P(t) + rt: t > 0 \} \) \((-\infty < r < +\infty)\) be the function, conjugated with \( P \) in the sense of Young.

In [12] it is proved that if \( \Phi \in \Omega \), then \( Q(r) \leq \Phi(r) \) for all \( r \geq r_0 \) if and only if \( P(x) \leq -x\Psi(\phi(x)) \) for all \( x \geq x_0 \). Therefore, if we put \( \mu(r, \varphi) = \sup \{ W_F(x) e^{rx}: x \geq 0 \} \) for \( r \geq 0 \) and \( P(x) = W_F(x) \), then \( Q(r) = \mu(r, \varphi) \) for \( r \geq 0 \) and the following lemma is correct.

**Lemma 2.** Let \( \Phi \in \Omega \). In order that \( \ln \mu(r, \varphi) \leq \Phi(r) \) for all \( r \geq r_0 \), it is necessary and sufficient that \( \ln W_F(x) \leq -x\Psi(\phi(x)) \) for all \( x \geq x_0 \).
Using Lemma 2 we prove the following theorem.

**Theorem 3.** If $\Phi \in \Omega$ and

$$\lim_{x \to +\infty} \frac{1}{x} \Phi' \left( \Psi^{-1} \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right) \right) = 0$$

then the function $\varphi$ is of unbounded $l$-index with $l(r) = \Phi'(r)$ for all $r > 0$ large enough.

**Proof.** We suppose that $\varphi$ is of bounded $l$-index. Then $\varphi$ is of bounded $l$-$M$-index, that is ([2, p. 74]) there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $r \geq 0$

$$M(r, \varphi^{(n)}) \leq \max \left\{ M(r, \varphi^{(k)}) : 0 \leq k \leq N \right\},$$

where $M(r, \varphi) = \max\{ |\varphi(z)| : |z| = r \}$.

Theorem 4.4 from [2, p. 83] implies that if the function $l$ is positive and continuously differentiable on $[0, +\infty)$,

$$\lim_{r \to +\infty} \frac{(-l'(r))}{P^2(r)} < +\infty \quad \text{and} \quad \int_0^{+\infty} l(t) dt = +\infty,$$

and if $\varphi$ is an entire function of bounded $l$-$M$-index, then $\ln M(r, \varphi) = O(L(r))$ as $r \to +\infty$, where $L(r) = \int_0^r l(t) dt$. Since $\Phi \in \Omega$ is positive and continuously differentiable on $[0, +\infty)$, the function $l(r) = \Phi'(r)$ for $r$ large enough satisfies the condition of Theorem 4.4 from [2, p. 83] and, thus, $\ln M(r, \varphi) = O(\Phi(r))$ as $r \to +\infty$. Taking into account [1, p.54] $\mu(r, \varphi) \leq 2M(r, \varphi)$ we have $\ln \mu(r, \varphi) \leq K\Phi(r)$ with $K = \text{const} > 0$ for all $r \geq r_0$. Using Lemma 2 with $K\Phi(r)$ instead of $\Phi(r)$, we obtain the inequality $\ln W_F(x) \leq -x\Psi(\varphi(x/K))$ for $x \geq x_0$, i.e.

$$\frac{1}{x} \Phi' \left( \Psi^{-1} \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right) \right) \geq \frac{1}{K} > 0,$$

which is impossible. \qed

Theorem 3 an for entire characteristic function of finite order implies the following corollary.

**Corollary 1.** If $\varrho \in (1, +\infty)$ and

$$\lim_{x \to +\infty} \frac{1}{x^{\varrho/(\varrho-1)}} \ln \frac{1}{W_F(x)} = 0,$$  \hspace{1cm} (5)

then $\varphi$ is a function of unbounded $l$-index with $l(r) = \varrho^{r-1}(r \geq r_0)$.

Indeed, if we choose $\Phi(r) = r^\varrho$ for $r \geq r_0$, then we have $\Phi'(r) = \varrho r^{\varrho-1}$ ($r \geq r_0$) and $\Psi^{-1}(x) = \frac{x}{\varrho-1}$ ($x \geq x_0$). Therefore,

$$\frac{1}{x} \Phi' \left( \Psi^{-1} \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right) \right) = \left( \frac{\varrho r}{\varrho - 1} \right)^{\varrho-1} \frac{1}{x^\varrho} \ln^{\varrho-1} \frac{1}{W_F(x)}$$

and the condition of Theorem 3 holds provided (5). Since $\ell(r) = \Phi'(r) = \varrho r^{\varrho-1}$ ($r \geq r_0$), the proof of the corollary is complete.
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