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ON THE l-INDEX BOUNDEDNESS OF ENTIRE RIDGE FUNCTIONS

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The l-index boundedness of entire ridge functions of finite order with real zeros, entire characteristic functions of probability laws and finite Fourier-Stieltjes transforms was investigated.

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Исследована ограниченность l-индекса целых хребтовых функций конечного порядка с действительными нулями, целых характеристических функций вероятностных законов и финитных преобразований Фурье-Стилтьеса.

1. An entire function $\varphi \not\equiv \text{const}$ is called ridge provided $|\varphi(z)| \le |\varphi(i\text{Im}\, z)|$ for all $z \in \mathbb{C}$. The concept of ridge function generalizes the concept of the characteristic functions of probability laws. In particular, entire characteristic functions form a proper subclass of ridge functions ([1, p. 42–51]). It is well known ([1, p. 45]) that zeros of an entire ridge function are symmetric with respect to the imaginary axis.

Let Λ be a class of positive continuous functions on $[0, +\infty)$. For $l \in \Lambda$ an entire function f is called ([2, p. 5]) a function of bounded l-index provided there exists $N \in \mathbb{Z}_+$ such that

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \le k \le N\right\} \tag{1}$$

for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$. If (1) holds for $l(r) \equiv 1$, then f is called a function of bounded index. We need the following criterion of l-index boundedness.

Lemma 1. *Let $l \in \Lambda$ and $l(r + O(\frac{1}{l(r)})) = O(l(r))$ as $r \to +\infty$. Let $a_k \in \mathbb{C}$ be zeros of an entire function f, $n(r, z_0, \frac{1}{f}) = \sum_{|a_k - z_0| \le r} 1$ and $G_q(f) = \bigcup_k \{z : |z - a_k| \le \frac{q}{l(|a_k|)}\}$. Then f is of bounded l-index if and only if*

1) *for every $q > 0$ there exists $P = P(q) > 0$ such that $\frac{|f'(z)|}{|f(z)|} \le P(q)l(|z|)$ for all $z \in \mathbb{C} \setminus G_q(f)$;*
2) *for every $q > 0$ there exists $n^*(q) \in \mathbb{Z}_+$ such that $n\left(\frac{q}{l(|z_0|}, z_0, \frac{1}{f}\right) \le n^*(q)$ for all $z_0 \in \mathbb{C}$.*

2. The aim of our note is to establish the l-index boundedness of some classes of ridge functions. We will begin with entire ridge functions of finite order. The following theorem is correct.

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Theorem 1. *If an entire ridge function φ of finite order has only real zeros and its positive zeros a_k satisfy the condition $a_k^2 - a_{k-1}^2 \nearrow +\infty$ as $k \to \infty$, then φ is a function of bounded l-index with $l(r) = r$ for $r \geq r_0 > 0$. The condition $a_k^2 - a_{k-1}^2 \nearrow +\infty$ as $k \to \infty$ can not be replaced by the condition $a_k^2 - a_{k-1}^2 \to +\infty$ as $k \to \infty$.*

Proof. By a result of Gol'dberg and Ostrovskii ([3]) every entire ridge function φ of finite order with only real zeros has a form

$$\varphi(z) = ce^{-\gamma z^2 + i\beta z} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right), \tag{2}$$

where c, γ, β and a_k are constant, $\gamma \geq 0$, $\mathrm{Im}\beta = 0$, $a_k > 0$ and $\sum_{k=1}^{\infty} \frac{1}{a_k^2} < +\infty$.

We put

$$\varphi_1(z) = ce^{-\gamma z^2 + i\beta z}, \quad \varphi_2(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right)$$

and we will show that each of these functions is of bounded l-index with $l(r) = r$ for $r \geq r_0 > 0$.

Since the function φ_1 does not have zeros and

$$\frac{|\varphi_1'(z)|}{|\varphi_1(z)|} = |-2\gamma z + i\beta| \leq 2\gamma r + |\beta| \leq \left(2\gamma + \frac{|\beta|}{r_0}\right) r,$$

this function is of bounded l-index with $l(r) = r$ for $r \geq r_0 > 0$ by Lemma 1.

Now we consider

$$\pi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{b_k}\right), \quad b_k = a_k^2.$$

It is clear that $b_k - b_{k-1} \nearrow +\infty$ as $k \to \infty$ and $\sum_{k=1}^{\infty} \frac{1}{b_k} < +\infty$. G. Fricke ([4]) proved that the function π is of bounded index under such conditions, and $b_k - b_{k-1} \nearrow +\infty$ as $k \to \infty$ can not be replaced by the condition $b_k - b_{k-1} \to +\infty$ as $k \to \infty$.

In [5] it is proved that if the function $l \in \Lambda$ satisfies $l(r) = (1 + o(1))r^{n-1}$ as $r \to +\infty$ for some $n \geq 2$, f is an entire function and $Q(z) = c_n z^n + \cdots + c_1 z + c_0$, $c_n \neq 0$, then the function $g(z) = f(Q(z))$ is of bounded l-index if and only if f is of bounded index. Therefore, since $\varphi_2(z) = \pi(z^2)$ and π are functions of bounded index, φ_2 is of bounded l-index with $l(r) = r$ $(r \geq r_0)$.

Finally, since φ_1 and φ_2 are of bounded l-index with $l(r) = r$ $(r \geq r_0)$, by Theorem 2.3 from [2, p. 84] their product φ is also of bounded l-index with $l(r) = r$ for $r \geq r_0$. □

3. Now we consider an entire function given by finite Fourier-Stieltjes transform

$$\varphi(z) = \int_{-a}^{a} e^{izt} dF(t), \tag{3}$$

where $0 < a < +\infty$ and F is a function of bounded variation. If F is non-decreasing (or non-increasing) on $[-a, a]$, then the function (3) is ridge. If F is non-decreasing on $[-a, a]$, $F(x) = 0$ for $x < -a$ and $F(x) = 1$ for $x \geq a$, then F is a probability law and (3) is its characteristic function.

We suppose that $F(-a)\neq F(-a+0)$ and $F(a)\neq F(a-0)$. Then ([6, p. 41–42]) all zeros of φ are contained in some strip $\{z\colon |\operatorname{Im} z|\le h\}$ and have the form

$$z_n=\frac{\pi n}{a}+O(1),\quad n\to\pm\infty. \tag{4}$$

Using this result it is easy to prove the following theorem.

Theorem 2. *If a function F is of bounded variation on $[-a,a]$, $F(-a)\neq F(-a+0)$ and $F(a)\neq F(a-0)$, then finite Fourier-Stieltjes transform (3) is a function of bounded index.*

Proof. Since the functions $w=z$ and $w=\sin z$ are of bounded l-index, by Theorem 2.3 from [2, p. 34] the function

$$\pi_1(z)=\frac{\sin z}{z}=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right),$$

and thus ([7]) the function $\pi_2(z)=\pi_1(az)$ are also of bounded index.

In the paper [8] it is proved that if zeros z_n of an entire function f of bounded index lie on the finite system of rays and $\psi_n=O(1)$ as $n\to\infty$, then the function f_ψ with zeros $z_n+\psi_n$ is also of bounded index. Since the entire function π_2 is of bounded index and has zeros $z_n=\frac{\pi n}{a}$, the function with zeros (4), that is the function (3), is of bounded index. □

We remark that in the case of characteristic functions Theorem 2 complements the following result of S. Shah ([9]): *if a function $p(t)\ge 0$ is absolutely continuous on $[-a,\,a]$, $p(-a)\neq 0$, $p(a)\neq 0$ and $\int_{-a}^{a}p(t)dt=1$, then the function $\varphi(z)=\int_{-a}^{a}e^{izt}p(t)dt$ is of bounded index.*

4. For a probability law F we put $W_F(x)=F(-x)+1-F(x)$, $x\ge 0$. Since the function F is non-decreasing on $(-\infty,+\infty)$ and continuous from the right, $F(-x)\to 0$ and $F(x)\to 1$ as $x\to+\infty$, we have $W_F(x)\searrow 0$ as $x\to+\infty$. The equality $W_F(x)=0$ for $x\ge x_0$ is a necessary condition for φ to be of bounded index, because if $W_F(x)\neq 0$ for all $x\ge 0$, then the characteristic function $\varphi(z)=\int_{-\infty}^{+\infty}e^{izt}dF(t)$ is of bounded index. Indeed, if φ is of bounded index, then by theorem of Hayman and Shah [10-11; 2, p. 59] φ is a function of exponential type σ. Hence it follows by Theorem 2.4.2 from [1, p. 53] that $W_F(x)>0$ for $0\le x<\sigma$ and $W_F(x)=0$ for $x>\sigma$, which is impossible.

We generalize now the above statement on the unboundedness of index to the statement on the unboundedness of l-index. For this purpose we denote by Ω a class of positive unbounded on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$. For $\Phi\in\Omega$ let ϕ be the function inverse to Φ', and let $\Psi(\sigma)=\sigma-\Phi(\sigma)/\Phi'(\sigma)$ be the function associated with Φ in the sense of Newton. Let P be an arbitrary function on $(0,+\infty)$ different from $+\infty$ (it can take a value $-\infty$, but $P\not\equiv-\infty$), and let $Q(r)=\sup\{P(t)+rt\colon t>0\}$ $(-\infty<r<+\infty)$ be the function, conjugated with P in the sense of Young.

In [12] it is proved that if $\Phi\in\Omega$, then $Q(r)\le\Phi(r)$ for all $r\ge r_0$ if and only if $P(x)\le -x\Psi(\phi(x))$ for all $x\ge x_0$. Therefore, if we put $\mu(r,\varphi)=\sup\{W_F(x)e^{rx}\colon x\ge 0\}$ for $r\ge 0$ and $P(x)=W_F(x)$, then $Q(r)=\mu(r,\varphi)$ for $r\ge 0$ and the following lemma is correct.

Lemma 2. *Let $\Phi\in\Omega$. In order that $\ln\mu(r,\varphi)\le\Phi(r)$ for all $r\ge r_0$, it is necessary and sufficient that $\ln W_F(x)\le -x\Psi(\phi(x))$ for all $x\ge x_0$.*

Using Lemma 2 we prove the following theorem.

Theorem 3. *If $\Phi \in \Omega$ and*

$$\varlimsup_{x\to+\infty} \frac{1}{x}\Phi'\left(\Psi^{-1}\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)\right) = 0$$

then the function φ is of unbounded l-index with $l(r) = \Phi'(r)$ for all $r > 0$ large enough.

Proof. We suppose that φ is of bounded l-index. Then φ is of bounded l-M-index, that is ([2, p. 74]) there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $r \geq 0$

$$\frac{M(r,\varphi^{(n)})}{n!l^n(r)} \leq \max\left\{\frac{M(r,\varphi^{(k)})}{k!l^k(r)} : 0 \leq k \leq N\right\},$$

where $M(r,\varphi) = \max\{|\varphi(z)| : |z| = r\}$.

Theorem 4.4 from [2, p. 83] implies that if the function l is positive and continuously differentiable on $[0,+\infty)$,

$$\varlimsup_{r\to+\infty} \frac{(-l'(r))^+}{l^2(r)} < +\infty \quad \text{and} \quad \int_0^{+\infty} l(t)dt = +\infty,$$

and if φ is an entire function of bounded l-M-index, then $\ln M(r,\varphi) = O(L(r))$ as $r \to +\infty$, where $L(r) = \int_0^r l(t)dt$. Since $\Phi \in \Omega$ is positive and continuously differentiable on $[0,+\infty)$, the function $l(r) = \Phi'(r)$ for r large enough satisfies the condition of Theorem 4.4 from [2, p. 83] and, thus, $\ln M(r,\varphi) = O(\Phi(r))$ as $r \to +\infty$. Taking into account [1, p.54] $\mu(r,\varphi) \leq 2M(r,\varphi)$ we have $\ln \mu(r,\varphi) \leq K\Phi(r)$ with $K = \text{const} > 0$ for all $r \geq r_0$. Using Lemma 2 with $K\Phi(r)$ instead of $\Phi(r)$, we obtain the inequality $\ln W_F(x) \leq -x\Psi(\varphi(x/K))$ for $x \geq x_0$, i.e.

$$\frac{1}{x}\Phi'\left(\Psi^{-1}\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)\right) \geq \frac{1}{K} > 0,$$

which is impossible. □

Theorem 3 an for entire characteristic function of finite order implies the following corollary.

Corollary 1. *If $\varrho \in (1,+\infty)$ and*

$$\varliminf_{x\to+\infty} \frac{1}{x^{\varrho/(\varrho-1)}} \ln\frac{1}{W_F(x)} = 0, \tag{5}$$

then φ is a function of unbounded l-index with $l(r) = r^{\varrho-1} (r \geq r_0)$.

Indeed, if we choose $\Phi(r) = r^\varrho$ for $r \geq r_0$, then we have $\Phi'(r) = \varrho r^{\varrho-1}$ $(r \geq r_0)$ and $\Psi^{-1}(x) = \frac{\varrho x}{\varrho-1}$ $(x \geq x_0)$. Therefore,

$$\frac{1}{x}\Phi'\left(\Psi^{-1}\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)\right) = \left(\frac{\varrho r}{\varrho-1}\right)^{\varrho-1} \frac{1}{x^\varrho}\ln^{\varrho-1}\frac{1}{W_F(x)}$$

and the condition of Theorem 3 holds provided (5). Since $\ell(r) = \Phi'(r) = \varrho r^{\varrho-1}$ $(r \geq r_0)$, the proof of the corollary is complete.

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