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TWO-RADII THEOREM FOR SOLUTIONS OF SOME MEAN VALUE EQUATIONS

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A description of solutions of some integral equations has been obtained. A two-radii theorem is obtained as well.

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В работе получено описание решений некоторых интегральных уравнений, а также теорема о двух радиусах.

1. Introduction. Characterization of solutions for differential equations in terms of various integral mean values has been studied by many authors (see [1]–[9] and references in these papers).

The classes of functions on subsets of the compact plane that satisfy the conditions of the next type is studied in this paper

$$\sum_{n=s}^{m-1} \frac{r^{2n+2}}{2(n-s)!(n+1)!}\left(\frac{\partial}{\partial z}\right)^{n-s}\left(\frac{\partial}{\partial \bar{z}}\right)^{n} f(z)=\frac{1}{2\pi}\iint\limits_{|\zeta-z|\le r} f(\zeta)(\zeta-z)^s d\xi d\eta, \tag{1}$$

where $m\in\mathbb{N}$ and $s\in\{0,\dots,m-1\}$ are fixed. Also r is fixed or belongs to the set of two elements.

We point out that this equation is satisfied for m-analytic functions (see [10]). A function from C^{2m-2-s} in some domain, that satisfies (1) with all possible z and r is of great interest.

The main results of the present paper are the following ones.

1) A description of all smooth solutions for (1) in a disk with radius $R>r$ with fixed r is obtained (see Theorem 1 below).
2) A two-radii theorem is obtained. It turns out that this theorem characterizes the class of solutions for the equation

$$\left(\frac{\partial}{\partial z}\right)^{m-s}\left(\frac{\partial}{\partial \bar{z}}\right)^{m} f=0 \tag{2}$$

in terms of equation (1) (see Theorem 2).

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Note that the case $s \geq m$ that corresponds to the zero integral mean value in the right hand side of (1), has been studied in papers of L. Zalcman and V. V. Volchkov (see [3], [11]–[12]). The first results that deal with the mean value theorem for polyanalytic functions, are contained in [13]–[14].

2. Main results. Let J_ν be the Bessel function of the first kind with index ν. For $\rho \geq 0$, $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$, let

$$\Phi_{\lambda,\eta,k}(\rho) = \left(\frac{d}{dz}\right)^{\eta} (J_k(z\rho))\,|_{z=\lambda}.$$

Let also

$$g_r(z) = \frac{J_{s+1}(rz)}{(zr)^{s+1}} - \sum_{n=s}^{m-1} \frac{(zr)^{2(n-s)}(-1)^{n-s}}{(n+1)!(n-s)!2^{2n-s+1}},$$

and $Z(g_r) = \{z \in \mathbb{C}\colon g_r(z) = 0\}$, $Z_r = Z(g_r) \setminus (\{z \in \mathbb{C}\colon \operatorname{Re} z > 0\} \cup \{z \in \mathbb{C}\colon \operatorname{Im} z \geq 0, \operatorname{Re} z = 0\})$. For $\lambda \in Z_r$ by the symbol n_λ we denote the multiplicity of zero λ of the entire function g_r.

Let $\mathbb{D}_R = \{z \in \mathbb{C}\colon |z| < R\}$. For any function $f \in C(\mathbb{D}_R)$ we assign the Fourier series

$$f(z) \sim \sum_{k=-\infty}^{\infty} f_k(\rho) e^{ik\varphi}, \tag{3}$$

where

$$f_k(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{it}) e^{-ikt} dt \tag{4}$$

and $0 \leq \rho < R$.

The next result gives a description for all solutions (1) in the class $C^\infty(\mathbb{D}_R)$ with fixed $r < R$.

Theorem 1. *Let $r > 0$, $m \in \mathbb{N}$ and $s \in \{0, \dots, m-1\}$ be fixed. Let also $R > r$ and a function f belong to $C^\infty(\mathbb{D}_R)$. Then the following statements are equivalent.*

1) With $|z| < R - r$ equality (1) holds.

2) For any $k \in \mathbb{Z}$ on $[0, R)$ the next equality holds

$$f_k(\rho) = \sum_{\substack{0 \leq p \leq s-1 \\ p+k \geq 0}} a_{k,p} \rho^{2p+k} + \sum_{p=0}^{m-s-1} b_{k,p} \rho^{2p+s+|k+s|} + \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) \tag{5}$$

where $a_{k,p} \in \mathbb{C}$, $b_{k,p} \in \mathbb{C}$, $c_{\lambda,\eta,k} \in \mathbb{C}$ and

$$c_{\lambda,\eta,k} = O(|\lambda|^{-\alpha}) \tag{6}$$

as $\lambda \to \infty$ for any fixed $\alpha > 0$.

Note that analogues of Theorem 1 for other equations related to ball mean values, were obtained by V. V. Volchkov for the first time (see [5]–[6] and the references in these papers).

Then let $Z(r_1, r_2) = Z_{r_1} \cap Z_{r_2}$.

We formulate now the local two-radii theorem for equation (1).

Theorem 2. *Let $r_1, r_2 > 0$, $m \in \mathbb{N}$ and $s \in \{0, \dots, m-1\}$ be fixed. Then:*

1) *if* $R > r_1 + r_2$, $Z(r_1, r_2) = \varnothing$, $f \in C^{2m-2-s}(\mathbb{D}_R)$ *and (1) holds with* $|z| < R - r$, *then* f *satisfies* $f \in C^\infty(\mathbb{D}_R)$ *and (2);*
2) *if* $\max\{r_1, r_2\} < R < r_1 + r_2$ *and* $Z(r_1, r_2) \neq \varnothing$, *then there is* $f \in C^\infty(\mathbb{D}_R)$, *that satisfies (1) with* $|z| < R - r$ *and does not satisfy (2).*

As regards other two-radii theorems see papers [1]–[9] and references in these papers.

3. Auxiliary statements. In this section we will obtain some auxiliary statements, that are necessary for the proof of main results.

First of all, we note that the function g_r is an even entire function of exponential type, that grows as a polynomial on the real axis (see, for example, [15], § 29). This together with the Hadamard theorem implies that the set Z_r is infinite.

Lemma 1. *Let* $\lambda \in Z_r$ *and* $|\lambda| > 4/r$. *Then*

$$|\mathrm{Im}\lambda| \leq c_1 \ln(1 + |\lambda|), \tag{7}$$

where the constant c_1 *does not depend on* λ. *Moreover, for all* λ *with sufficiently large absolute value*

$$|g_r'(\lambda)| > \frac{c_2}{|\lambda|}, \tag{8}$$

where c_2 *does not depend on* λ. *In addition, all zeros of* g_r *with sufficiently large absolute values are simple.*

Proof. By the condition $g_r(\lambda) = 0$ and the asymptotic expansion for $J_{s+1}(\lambda r)$ as $\lambda \to \infty$ (see [15], § 29) we have

$$\sqrt{\frac{2}{\pi\lambda r}}\left(\cos\Big(\lambda r - \frac{\pi s}{2} - \frac{3\pi}{4}\Big) - \frac{4(s^2+2s+1)-1}{8\lambda r}\sin\Big(\lambda r - \frac{\pi s}{2} - \frac{3\pi}{4}\Big)\right) +$$
$$+ O\left((\lambda r)^{-2} e^{|\mathrm{Im}(\lambda r)|}\right) = (\lambda r)^{s+1} \sum_{n=s}^{m-1} \frac{(\lambda r)^{2n-2s}(-1)^{n-s-1}}{(2n+2)(n-s)!n!2^{2n-s}}.$$

Hence, using $\lambda \in Z_r$, we obtain

$$\frac{e^{i(\lambda r - \frac{\pi s}{2} - \frac{\pi}{4})}}{2i} + O\left(\frac{e^{|\mathrm{Im}(\lambda r)|}}{\lambda r}\right) = \sqrt{\frac{\pi\lambda r}{2}} \sum_{n=s}^{m-1} \frac{(\lambda r)^{2n-s+1}(-1)^{n-s-1}}{(2n+2)(n-s)!n!2^{2n-s}}.$$

Denote by $p_1(\lambda r)$ the polynomial from the right hand side of the latter equation. Then

$$e^{i(\lambda r - \frac{\pi s}{2} - \frac{\pi}{4})} = 2ip_1(\lambda r) + O\left(\frac{2ie^{|\mathrm{Im}(\lambda r)|}}{\lambda r}\right).$$

Let us estimate

$$e^{|\mathrm{Im}(\lambda r)|} \leq |2ip_1(\lambda r)| + \frac{|2i|e^{|\mathrm{Im}(\lambda r)|}}{\lambda r} \leq |2ip_1(\lambda r)| + \frac{|i|e^{|\mathrm{Im}(\lambda r)|}}{2}.$$

Now one has $e^{|\mathrm{Im}(\lambda r)|} \leq 4|p_1(\lambda r)|$ and inequality (7) is proved. Inequality (8) can be proved in a similar way, by using [15], formula (6.3). □

Lemma 2. *Let $\lambda \in \mathbb{C}$, $f(z) = e^{i\lambda(x\cos\alpha + y\sin\alpha)}$, $r > 0$. Then for $z \in \mathbb{C}$ we have*

$$\iint\limits_{|\zeta - z| \leq r} f(\zeta)(\zeta - z)^s d\xi d\eta - \sum_{n=s}^{m-1} \frac{2\pi r^{2n+2}}{2(n-s)!(n+1)!} \left(\frac{\partial}{\partial z}\right)^{n-s} \left(\frac{\partial}{\partial \bar{z}}\right)^n f(z) =$$
$$= 2\pi g_r(\lambda) e^{i\alpha s} i^{s+2} \frac{r^{s+1}}{\lambda} e^{i\lambda(x\cos\alpha + y\sin\alpha)}.$$

Proof. We substitute the function $e^{i\lambda(x\cos\alpha+y\sin\alpha)}$ to the right hand side of equation (1).

First, we have

$$\iint\limits_{|w| \leq r} f(w+z) w^s du dv = \iint\limits_{|w| \leq r} e^{i\lambda((x+u)\cos\alpha + (y+v)\sin\alpha)} w^s du dv =$$
$$= e^{i\lambda(x\cos\alpha+y\sin\alpha)} \int_{-\pi}^{\pi} \int_0^r \left(\rho e^{i\varphi}\right)^s e^{i\lambda\rho\cos(\varphi-\alpha)} \rho d\varphi d\rho.$$

Now we make the substitution $t = \varphi - \alpha$. Then

$$e^{i\lambda(x\cos\alpha+y\sin\alpha)} e^{i\alpha s} \int_{-\pi}^{\pi} \int_0^r \rho^{s+1} e^{its} e^{i\lambda\rho\cos t} dt d\rho =$$
$$= e^{i\lambda(x\cos\alpha+y\sin\alpha)} e^{i\alpha s} \int_0^r \rho^{s+1}(-1) \int_{-\pi}^{\pi} e^{-i(t+\frac{\pi}{2})s} e^{i\frac{\pi}{2}s} e^{i\lambda\rho\sin(\frac{\pi}{2}+t)} d\left(\frac{\pi}{2}+t\right) d\rho.$$

Continuing the consideration, we obtain

$$e^{i\lambda(x\cos\alpha+y\sin\alpha)} e^{i\alpha s} \int_{-\pi}^{\pi} \int_0^r \rho^{s+1} e^{its} e^{i\lambda\rho\cos t} dt d\rho =$$
$$= e^{i\lambda(x\cos\alpha+y\sin\alpha)} e^{i\alpha s} i^s 2\pi(-1) \int_0^r \rho^{s+1} J_s(\lambda\rho) d\rho.$$

Now properties of the Bessel function $J_s(z)$ imply

$$e^{i\lambda(x\cos\alpha+y\sin\alpha)} e^{i\alpha s} i^s (-2\pi) \frac{1}{\lambda^{s+2}} \int_0^r (\lambda\rho)^{s+1} J_s(\lambda\rho) d(\lambda\rho) =$$
$$= e^{i\lambda(x\cos\alpha+y\sin\alpha)} e^{i\alpha s} i^s \frac{(-2\pi)}{\lambda} r^{s+1} J_{s+1}(\lambda).$$

Then we substitute the function $e^{i\lambda(x\cos\alpha+y\sin\alpha)}$ to the left hand side of equation (1).

$$2\pi \sum_{n=s}^{m-1} \frac{r^{2n+2}}{(2n+2)(n-s)!n!} \left(\frac{\partial}{\partial z}\right)^{n-s} \left(\frac{\partial}{\partial \bar{z}}\right)^n \left(e^{i\lambda(x\cos\alpha+y\sin\alpha)}\right) =$$
$$= 2\pi \sum_{n=s}^{m-1} \frac{r^{2n+2}}{(2n+2)(n-s)!n!} \frac{i^{2n-s}}{2^{2n-s}} \lambda^{2n-s} e^{i\alpha s} e^{i\lambda(x\cos\alpha+y\sin\alpha)}.$$

It is clear that the difference of the obtained expressions for the right and left hand sides has the form $2\pi g_r(\lambda) e^{i\alpha s} i^{s+2} \frac{r^{s+1}}{\lambda} e^{i\lambda(x\cos\alpha+y\sin\alpha)}$. □

Corollary 1. *Let $\lambda \in Z_r$, $\eta \in \{0, ..., n_\lambda - 1\}$, $\alpha \in R^1$. Then the function $\left(\frac{\partial}{\partial z}\right)^\eta e^{i\lambda(x\cos\alpha + y\sin\alpha)}$ satisfies (1) for all $z \in \mathbb{C}$. The same statement is true for the function $\Phi_{\lambda,\eta,k}(\rho)e^{ik\varphi}$ with any $k \in \mathbb{Z}$.*

Proof. The corollary follows from Lemma 2 and [5, formula (1.5.29)]. □

Lemma 3. *Let $m \in \mathbb{N}$ and $s \in \{0, \dots, m-1\}$ be fixed. Then $f \in C^{2m-s}(\mathbb{D}_R)$ satisfies (2) if and only if for all $k \in \mathbb{Z}$ and $\rho \in [0, R)$ the following equality is true*

$$f_k(\rho) = \sum_{\substack{0 \le p \le s-1 \\ p+k \ge 0}} a_{k,p}\rho^{2p+k} + \sum_{p=0}^{m-s-1} b_{k,p}\rho^{2p+s+|k+s|}, \tag{9}$$

where $a_{k,p} \in \mathbb{C}$ and $b_{k,p} \in \mathbb{C}$.

Proof. In the case where $b_{k,p} = 0$ and the equality $\left(\frac{\partial}{\partial \bar{z}}\right)^m f = 0$ is considered instead of (2) a similar statement was proved in [10]. In our case the proof is similar. □

Lemma 4. *Let $m \in \mathbb{N}$ and $s \in \{0, \dots, m-1\}$ be fixed. Assume that a function $f \in C^\infty(\mathbb{D}_R)$ satisfies (1) with fixed $r < R$ and all $z \in \mathbb{D}_{R-r}$. Let $f = 0$ in $\mathbb{D}_r$. Then $f \equiv 0$.*

Proof. The statement of Lemma 4 is a special case Theorem 1 from [16]. □

4. Proof of Theorem 1. *Sufficiency.* First, let $f \in C^\infty(\mathbb{D}_R)$ and equality (5) hold on $[0, R)$ for any $k \in \mathbb{Z}$ with the coefficients, that satisfy (6). From Lema 2 and Corollary 1 we see, that the function $f_k(\rho)e^{ik\varphi}$ satisfies (1) with $|z| < R-r$. Because of the arbitrariness of $k \in \mathbb{Z}$ this together with (3), (4) implies (see, for example, [5, Section 1.5.2] that the function f also satisfies (1) with $|z| < R - r$. Hence, implication 2) ⇒ 1) is proved.

Now we prove the reverse statement.

Let $\mathcal{E}'_\natural(\mathbb{C})$ denote the space of radial compactly supported distributions on $\mathbb{C}$. Let $f \in C^\infty(\mathbb{D}_R)$ and assume that equality (1) holds for $|z| < R - r$. From [5, statement 1.5.6] the functions $F_k(z) = f_k(\rho)e^{ik\varphi}$ satisfy (1) for $|z| < R - r$. Using the Paley-Wiener theorem for the spherical transform (see [5, Section 3.2.1 and Theorem 1.6.5]), we define the distribution $T \in \mathcal{E}'_\natural(\mathbb{C})$ with support in $\overline{\mathbb{D}}_r$ by the following formula $\widetilde{T}(z) = g_r(z), z \in \mathbb{C}$. A calculation shows that equality (1) holds for the function F_k with $|z| < R - r$. This is equivalent to the following convolution equation

$$F_k * \left(\frac{\partial}{\partial z}\right)^{m-s} \left(\frac{\partial}{\partial \bar{z}}\right)^m T = 0 \tag{10}$$

in $\mathbb{D}_{R-r}$.

We solve this equation (10) using Lemmas 1–4. Then we have (see [5, Section 3.2.4]) statement 2). Hence, the theorem is proved.

5. Proof of Theorem 2. Let $R > r_1 + r_2$, $Z(r_1, r_2) = \varnothing$, $f \in C^{2m-2-s}(\mathbb{D}_R)$ and assume that equality (1) holds for $|z| < R - r$. Let us prove that f satisfies (2) in $\mathbb{D}_R$.

Without loss of generality we may assume that $f \in C^\infty(\mathbb{D}_R)$ (the general case can be reduced to this one by the standard smoothing, see [5, Section 1.3.3]).

By Theorem 1, for any $k \in \mathbb{Z}$ and $\rho \in [0, R)$ the next equality holds

$$f_k(\rho)e^{ik\varphi} = \sum_{\substack{0 \le p \le s-1 \\ p+k \ge 0}} a_{k,p}\rho^{2p+k}e^{ik\varphi} + \sum_{p=0}^{m-s-1} b_{k,p}\rho^{2p+s+|k+s|}e^{ik\varphi} + \sum_{\lambda \in Z_{r_1}} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k}\Phi_{\lambda,\eta,k}(\rho)e^{ik\varphi}, \quad (11)$$

where $a_{k,p} \in \mathbb{C}$, $b_{k,p} \in \mathbb{C}$ and the constants $c_{\lambda,\eta,k}$ satisfy (6).

Condition (6) implies that the series in (10) converges in the space $C^\infty(\mathbb{D}_R)$ (see [5, Lemma 3.2.7]).

Let

$$F_k(z) = \left(\frac{\partial}{\partial z}\right)^{m-s} \left(\frac{\partial}{\partial \bar{z}}\right)^{m} \left(f_k(\rho)e^{ik\varphi}\right) = \sum_{\lambda \in Z_{r_1}} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k} \left(\frac{\partial}{\partial z}\right)^{m-s} \left(\frac{\partial}{\partial \bar{z}}\right)^{m} \Phi_{\lambda,\eta,k}(\rho)e^{ik\varphi}. \quad (12)$$

In view of (11) we see that $F_k * T_1 = 0$ in $\mathbb{D}_{R-r_1}$, where the distribution $T_1 \in \mathcal{E}'_\natural(\mathbb{C})$ with support in $\overline{\mathbb{D}}_{r_1}$ is determined by the equality $\widetilde{T}_1(z) = g_{r_1}(z)$ (see [5, Theorem 1.6.5]).

Similarly, using Theorem 1 for $r = r_2$, we conclude that $F_k * T_2 = 0$ in $\mathbb{D}_{R-r_2}$, where $T_2 \in \mathcal{E}'_\natural(\mathbb{C})$ with support in $\overline{\mathbb{D}}_{r_2}$ is determined by the equality $\widetilde{T}_2(z) = g_{r_2}(z)$.

If $Z(r_1, r_2) = \varnothing$ then from [5, Theorem 3.4.1] we conclude that $F_k = 0$.

Then it follows from (11) that the function $f_k(\rho)e^{ik\varphi}$ satisfies (2) for all $k \in \mathbb{Z}$. This means that (see [5, proof of Lemma 2.1.4]) f satisfies (2). Thus the first statement of Theorem 2 is proved.

We now establish the second statement.

If there is $\lambda \in Z(r_1, r_2)$ then the function $f(z) = \Phi_{\lambda,0,0}(|z|)$ does not satisfy (2). In addition, it satisfies (1) for all $z \in \mathbb{C}$ and $r = r_1, r_2$ (see Corollary 1). Then we henceforth assume that $Z(r_1, r_2) = \varnothing$.

Suppose that $T_1, T_2 \in \mathcal{E}'_\natural(\mathbb{C})$ are defined as above. If $R < r_1 + r_2$, in view of [5, Theorem 3.4.9] we conclude, that there is a nonzero radial function $f \in C^\infty(\mathbb{D}_R)$. It satisfies the conditions $f * T_1 = 0$ in $\mathbb{D}_{R-r_1}$ and $f * T_2 = 0$ in $\mathbb{D}_{R-r_2}$.

Applying [5, Theorem 3.2.3] we infer that for $r = r_1, r_2$ the following equality holds

$$f(z) = \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta}(r)\Phi_{\lambda,\eta,0}(|z|),$$

where $z \in \mathbb{D}_R$ and the constants $c_{\lambda,\eta}(r)$ satisfy (5). Moreover, these constants are not all equal to zero.

From this equality and Corollary 1 one deduces that f satisfies (1) for $|z| < R - r$, $r = r_1, r_2$. Suppose now that f satisfies (2).

Then

$$f(z) = \sum_{0 \le p \le s-1} a_p|z|^{2p} + \sum_{p=0}^{m-s-1} b_p|z|^{2p+2s}$$

in $\mathbb{D}_R$ and the convolutions $f * T_1$ and $f * T_2$ are polynomials. This means that $f * T_1 = f * T_2 = 0$ in $\mathbb{C}$.

Since $Z(r_1, r_2) = \varnothing$, from [5, Theorem 3.4.1] we infer that $f = 0$. This is impossible because of the definition of f.

Therefore, the function f satisfies all the requirements of the second statement of Theorem 2.

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