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DAUGAVET CENTERS ARE SEPARABLY DETERMINED

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A linear bounded operator G acting from a Banach space X into a Banach space Y is a Daugavet center if every linear bounded rank-1 operator $T\colon X\to Y$ fulfills $\|G+T\| = \|G\|+\|T\|$. We prove that $G\colon X\to Y$ is a Daugavet center if and only if for every separable subspaces $X_1\subset X$ and $Y_1\subset Y$ there exist separable subspaces $X_2\subset X$ and $Y_2\subset Y$ such that $X_1\subset X_2$, $Y_1\subset Y_2$, $G(X_2)\subset Y_2$ and the restriction $G|_{X_2}\colon X_2\to Y_2$ of G is a Daugavet center. We apply this fact to study the set of G-narrow operators.

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Линейный ограниченный оператор G, действующий из банахова пространства X в банахово пространство Y, называется даугаветовым центром, если для любого линейного ограниченного одномерного оператора $T\colon X\to Y$ выполняется равенство $\|G+T\| = \|G\|+\|T\|$. В данной статье доказано, что оператор $G\colon X\to Y$ является даугаветовым центром тогда и только тогда, когда для любой пары сепарабельных подпространств $X_1\subset X$, $Y_1\subset Y$ существует пара сепарабельных подпространств $X_2\subset X$, $Y_2\subset Y$ такая, что $X_1\subset X_2$, $Y_1\subset Y_2$, $G(X_2)\subset Y_2$ и ограничение $G|_{X_2}\colon X_2\to Y_2$ оператора G является даугаветовым центром. Этот результат применён к изучению множества G-узких операторов.

The present paper is devoted to one of the ways to generalize the concept of the Daugavet property. A Banach space X is said to have *the Daugavet property* ([5]) if every linear bounded operator $T\colon X\to X$ of rank one satisfies the Daugavet equation

$$\|\mathrm{Id}+T\| = 1+\|T\|, \tag{1}$$

where Id denotes the identity operator in X.

Spaces with the Daugavet property are non-reflexive, do not have an unconditional basis, contain subspaces isomorphic to ℓ_1 ([5]). Among the examples of spaces with the Daugavet property there are $C(K)$ where K is perfect ([4]), $L_1(\mu)$ and $L_\infty(\mu)$ where μ has no atoms ([7]), certain functional algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ (see [8], [9]).

One of the ways of developing the study of spaces with the Daugavet property is to investigate the question how large the set of operators T satisfying (1) is. It was shown in [5] that if X has the Daugavet property then (1) holds true for every weakly compact operator $T\colon X\to X$; in [1], [6] the set of such operators T was extended further.

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The present paper concerns [6, Theorem 4.5] which claims that the Daugavet property is separably determined, i. e. that a Banach space X has the Daugavet property if and only if for every separable subspace $Y \subset X$ there exists a separable subspace $E \subset X$ which contains Y and has the Daugavet property.

Consider the following generalization of the concept of the Daugavet property, which we began to study in [3].

Definition 1. A linear bounded non-zero operator $G\colon X \to Y$ is said to be *a Daugavet center* if the norm identity

$$\|G + T\| = \|G\| + \|T\| \tag{2}$$

holds true for every linear bounded rank-1 operator $T\colon X \to Y$.

Banach spaces which have a Daugavet center acting from it, and Banach spaces which have a Daugavet center acting into it, are not obliged to have the Daugavet property, but for such spaces many facts known for spaces with the Daugavet property, hold true. In particular, if $G\colon X \to Y$ is a Daugavet center then X and Y are non-reflexive, do not have an unconditional basis, do not have the Radon-Nikodým property, contain isomorphic copies of ℓ_1 as well ([3]).

In the present paper we give for Daugavet centers an analog of the fact that the Daugavet property is separably determined, and apply this analog to study the set of operators T satisfying (2). Remark that our proof for the generalized result does not copy the particular case $G = \mathrm{Id}$ proof given in [6, Theorem 4.5]; in fact our proof is much easier to understand.

Throughout this paper we use symbols X, Y and E to denote real Banach spaces. We denote $B(X)$ the unit ball of X and $S(X)$ the unit sphere. We use the symbol $L(X, Y)$ to denote the space of all linear bounded operators acting from X into Y.

We will use the following characterization of a Daugavet center which is a consequence of [3, Theorem 2.7].

Lemma 1. *A nonzero operator $G \in L(X, Y)$ is a Daugavet center if and only if for every $\varepsilon > 0$ and every $y \in Y$*

$$\overline{\mathrm{conv}}\{x \in B(X)\colon \|y - Gx\| > \|G\| + \|y\| - \varepsilon\} = B(X).$$

The following theorem is the key result of the present paper.

Theorem 1. *For any $G \in L(X, Y)$ the following assertions are equivalent:*

(i) G is a Daugavet center;

(ii) for every separable subspaces $X_1 \subset X$ and $Y_1 \subset Y$ there exist separable subspaces $X_2 \subset X$ and $Y_2 \subset Y$ such that $X_1 \subset X_2$, $Y_1 \subset Y_2$, $G(X_2) \subset Y_2$ and the restriction $G|_{X_2}\colon X_2 \to Y_2$ of G is a Daugavet center.

Proof. (i) $\Rightarrow$ (ii). In order to construct the desired subspaces X_2 and Y_2 we will select inductively two countable sets $\{x_{n,m}\}_{n,m\in\mathbb{N}} \subset X$ and $\{y_{n,m}\}_{n,m\in\mathbb{N}} \subset Y$ generating X_2 and Y_2 respectively. As the starting point we take a countable dense subset of $B(X_1)$ as $\{x_{1,m}\}_{m\in\mathbb{N}}$, and a countable dense subset of $\overline{\mathrm{lin}}\{Y_1 \cup \{Gx_{1,m}\}_{m\in\mathbb{N}}\}$ as $\{y_{1,m}\}_{m\in\mathbb{N}}$.

Assume that for a given positive integer k all the sequences $\{x_{n,m}\}_{m\in\mathbb{N}}$ and $\{y_{n,m}\}_{m\in\mathbb{N}}$ are constructed for every $n \in \{1, \ldots, k\}$. Now we construct the sequences $\{x_{k+1,m}\}_{m\in\mathbb{N}}$ and

$\{y_{k+1,m}\}_{m\in\mathbb{N}}$. Take a sequence of positive numbers $\{\varepsilon_i\}$ which converges to zero. Consider the Cartesian product

$$A_k := \left\{(i, x, y)\colon i \in \mathbb{N}, x \in \bigcup_{n=1}^{k}\{x_{n,m}\}_{m\in\mathbb{N}}, y \in \bigcup_{n=1}^{k}\{y_{n,m}\}_{m\in\mathbb{N}}\right\}.$$

Using Lemma 1 for every $\xi = (i, x, y) \in A_k$ we find a finite convex combination $\sum \lambda_{\xi,j}\hat{x}_{\xi,j}$ where $\hat{x}_{\xi,j} \in B(X)$, such that $\|x - \sum \lambda_{\xi,j}\hat{x}_{\xi,j}\| < \varepsilon_i$ and for every $\hat{x}_{\xi,j}$ the inequality $\|y - G\hat{x}_{\xi,j}\| > \|G\| + \|y\| - \varepsilon_i$ holds true.

Let symbol D_k denote the set of all $\hat{x}_{\xi,j}$ that we have chosen this way for all $\xi \in A_k$. Note that D_k is a countable subset of $B(X)$. Let us extend D_k to a countable dense subset of the unit ball of the subspace $\overline{\operatorname{lin}}\{D_k\}$ and take the set obtained this way as $\{x_{k+1,m}\}_{m\in\mathbb{N}}$. As $\{y_{k+1,m}\}_{m\in\mathbb{N}}$ we take a countable dense subset of the subspace

$$\overline{\operatorname{lin}}\left\{Y_1 \cup \bigcup_{n=1}^{k+1}\{Gx_{n,m}\}_{m\in\mathbb{N}}\right\}.$$

Consider $X_2 := \overline{\operatorname{lin}}\,\{\{x_{n,m}\}_{n,m\in\mathbb{N}}\}$ and $Y_2 := \overline{\{y_{n,m}\}_{n,m\in\mathbb{N}}}$, and show that these two subspaces satisfy our requirements. X_2 and Y_2 are clearly separable subspaces of X and Y, respectively. It is easy to see that $X_1 \subset X_2$, $Y_1 \subset Y_2$ and $G(X_2) \subset Y_2$.

Let us prove that $G|_{X_2}\colon X_2 \to Y_2$ is a Daugavet center. By Lemma 1 it is sufficient to show that for every $\varepsilon > 0$ and every $y \in Y_2$

$$B(X_2) \subset \overline{\operatorname{conv}}\{z \in B(X_2)\colon \|y - Gz\| > \|G\| + \|y\| - \varepsilon\}.$$

Take any $x \in B(X_2)$ and $\delta > 0$. By our construction, x can be approximated by a finite linear combination $u \in B(X_2)$ of some vectors $x_{i,j}$. This u belongs to the unit ball of $\overline{\operatorname{lin}}\{D_n\}$ for some n, so in turn, u can be approximated by an element that belongs to $\{x_{n+1,m}\}_{m\in\mathbb{N}}$. Putting all these together we deduce that there is $\hat{x} \in \{x_{n,m}\}_{n,m\in\mathbb{N}}$ with $\|x - \hat{x}\| < \delta/2$. By the construction of Y_2, there is $\hat{y} \in \{y_{n,m}\}_{n,m\in\mathbb{N}}$ with $\|y - \hat{y}\| < \varepsilon/3$. Let $k \in \mathbb{N}$ be such that

$$\hat{x} \in \bigcup_{n=1}^{k}\{x_{n,m}\}_{m\in\mathbb{N}} \text{ and } \hat{y} \in \bigcup_{n=1}^{k}\{y_{n,m}\}_{m\in\mathbb{N}}.$$

Then there exists a convex combination $\sum \lambda_j\hat{x}_j$ with $\hat{x}_j \in \{x_{k+1,m}\}_{m\in\mathbb{N}}$ such that $\|\hat{x} - \sum \lambda_j\hat{x}_j\| < \delta/2$ and for every $\hat{x}_j$ the inequality $\|\hat{y} - G\hat{x}_j\| > \|G\| + \|\hat{y}\| - \varepsilon/3$ holds true. Then

$$\|y - G\hat{x}_j\| > \|\hat{y} - G\hat{x}_j\| - \varepsilon/3 > \|G\| + \|\hat{y}\| - 2\varepsilon/3 > \|G\| + \|y\| - \varepsilon$$

and $\|x - \sum \lambda_j\hat{x}_j\| < \delta$. Hence

$$x \in \overline{\operatorname{conv}}\{z \in B(X_2)\colon \|y - Gz\| > \|G\| + \|y\| - \varepsilon\}.$$

(ii) $\Rightarrow$ *(i)*. Let $\varepsilon > 0$, $T \in L(X, Y)$ be a rank-1 operator, $x \in B(X)$ be such that $\|Tx\| > \|T\| - \varepsilon$ and $z \in B(X)$ be such that $\|Gz\| > \|G\| - \varepsilon$. For $X_1 := \operatorname{lin}\{x, z\}$ and $Y_1 := T(X)$ let us pick separable subspaces X_2 and Y_2 as in *(ii)*. Consider the restriction $T|_{X_2}\colon X_2 \to Y_2$. Since $x, z \in X_2$, we have

$$\|T|_{X_2}\| \geq \|T|_{X_2}x\| = \|Tx\| \geq \|T\| - \varepsilon$$

and by the analogous argument $\|G|_{X_2}\| \geq \|G\| - \varepsilon$. By *(ii)* $G|_{X_2} \colon X_2 \to Y_2$ is a Daugavet center, hence

$$\|G + T\| \geq \|G|_{X_2} + T|_{X_2}\| = \|G|_{X_2}\| + \|T|_{X_2}\| > \|G\| + \|T\| - 2\varepsilon.$$

Since the above inequality holds true for an arbitrary $\varepsilon > 0$, G is a Daugavet center. □

Let us show an application of this result. In [2] we introduced the following concept.

Definition 2. Let $G \in S(L(X, Y))$ be a Daugavet center, E be a Banach space. An operator $T \in L(X, E)$ is called a *G-narrow operator* if for every $\varepsilon > 0$, every $x \in S(X)$, every $y \in S(Y)$ and every $x^* \in S(X^*)$ there exists $z \in B(X)$ satisfying the inequalities

$$\|T(x - z)\| + |x^*(x - z)| < \varepsilon, \ \|y + Gz\| > 2 - \varepsilon.$$

The concept of a G-narrow operator is a generalization to Daugavet centers of the concept of a narrow operator, which was introduced in [6] for the case $G = \mathrm{Id}$. Note that if we consider the definition of a G-narrow operator for a general $G \in L(X, Y)$ we obtain that for a non-zero $G \in L(X, Y)$ there exists a G-narrow operator if and only if G is a Daugavet center ([2]), and so for convenience we require G to be a Daugavet center in the above definition. It is not difficult to derive by Definition 2 that for a Daugavet center $G \in S(L(X, Y))$ every its G-narrow operator $T \in L(X, Y)$ satisfies $\|G + T\| = \|G\| + \|T\|$. We showed in [2] that for every Daugavet center G the set of all G-narrow operators is rather large.

Proposition 1. *For a Daugavet center $G \in S(L(X, Y))$ all weakly compact operators, all operators not fixing a copy of ℓ_1, and all strong Radon-Nikodým operators acting from X into some Banach space E, are G-narrow.*

Unfortunately, as we have discovered recently, in the proof of Proposition 1 we used an argument from [1] that requires implicitly separability of E, so in fact only the case of separable E is proved in [2]. Now we are ready to fix that gap. For this purpose we need the following fact.

Proposition 2. *Let $G \in S(L(X, Y))$ be a Daugavet center, $T \in L(X, E)$. Suppose that for every pair of separable subspaces $X_1 \subset X$ and $Y_1 \subset Y$ such that the restriction $G|_{X_1} \colon X_1 \to Y_1$ is a Daugavet center, the restriction of T to X_1 is a $G|_{X_1}$-narrow operator. Then T is a G-narrow operator.*

Proof. Take any $\varepsilon > 0$, $x \in S(X)$, $y \in S(Y)$ and $x^* \in S(X^*)$. Using Theorem 1, pick separable subspaces X_2 and Y_2 for $X_1 := \mathrm{lin}\{x\}$ and $Y_1 := \mathrm{lin}\{y\}$. Since $G|_{X_2} \colon X_2 \to Y_2$ is a Daugavet center and the restriction of T on X_2 is a $G|_{X_2}$-narrow operator, there is $z \in B(X_2) \subset B(X)$ such that $\|T(x - z)\| + |x^*(x - z)| < \varepsilon$ and $\|y + Gz\| > 2 - \varepsilon$. □

Proof of Proposition 1. As we have mentioned already, the case of separable E is proved correctly in [2], so it remains to consider the non-separable case. Let $T \in L(X, E)$ be an operator which belongs to one of the sets of operators listed in the statement of this proposition. Let $X_1 \subset X$ and $Y_1 \subset Y$ be such separable subspaces that the restriction $G|_{X_1} \colon X_1 \to Y_1$ is a Daugavet center. Remark that the restriction $T|_{X_1}$ has a separable image and belongs to the same set of operators to which T belongs, hence by the "separable" version of our proposition $T|_{X_1}$ is a $G|_{X_1}$-narrow operator. Then by Proposition 2 T is G-narrow. □

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