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ASYMPTOTICS OF EIGENVALUES AND EIGENFUNCTIONS OF ENERGY-DEPENDENT STURM–LIOUVILLE EQUATIONS

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We study asymptotics of eigenvalues, eigenfunctions and norming constants of singular energy-dependent Sturm–Liouville equations with complex-valued potentials. The analysis essentially exploits the integral representation of solutions, which we derive using the connection between the problem under study and a Dirac system of a special form.

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Изучается асимптотика собственных значений, собственных функций и нормирующих множителей сингулярных уравнений Штурма–Лиувилля с комплексными потенциалами, зависящими от энергии. Анализ существенно использует интегральные представления решений, которые мы получаем, используя связь рассматриваемой задачи с системой Дирака специального вида.

1. Introduction. The main objective of the present paper is to study asymptotics of eigenvalues and eigenfunctions of Sturm–Liouville equations on (0,1) with energy-dependent potentials, viz.

$$-y'' + qy + 2\lambda py = \lambda^2 y. \tag{1}$$

Here $\lambda \in \mathbb{C}$ is the spectral parameter, p is a complex-valued function in $L_2(0,1)$ and q is a complex-valued distribution in the Sobolev space $W_2^{-1}(0,1)$, i.e. $q = r'$ with a complex-valued $r \in L_2(0,1)$. We consider equation (1) mostly under the Dirichlet boundary conditions

$$y(0) = y(1) = 0. \tag{2}$$

We restrict our attention to such boundary conditions just to concentrate on the ideas and to avoid unnecessary technicalities. Other separated boundary conditions can be treated analogously; in particular, in the last section we formulate some results for the case of mixed boundary conditions.

Energy-dependent Sturm–Liouville equations are of importance in classical and quantum mechanics. For instance, they are used for modelling mechanical systems vibrations in viscous media. The Klein–Gordon equations, which describe the motion of massless particles such as photons, can also be reduced to form (1). The corresponding evolution equations are used to model the interactions between colliding relativistic spinless particles. In such mechanical

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models the spectral parameter λ is related to the energy of the system, which explains the terminology "energy-dependent" used for spectral equation (1).

Asymptotic behaviour of eigenvalues, eigenfunctions, and other spectral characteristics for usual Sturm–Liouville operators are studied sufficiently well (see, e.g. [9, 11, 15]). The spectral problem for energy-dependent Sturm–Liouville equation (1) with $p \in W_2^1[0,\pi]$ and $q \in L_2[0,\pi]$ was considered by M. Gasymov and G. Guseinov in their short paper [3] of 1981. Some results on asymptotics are formulated there without proofs. Analogous problems under more general boundary conditions were also considered by I. Nabiev in [14]. The asymptotics of eigenvalues and eigenfunctions for Sturm–Liouville equations with singular potentials $q \in W_2^{-1}(0,1)$ were studied by A. Savchuk and A. Shkalikov in [18].

Our aim in this paper is to investigate the corresponding asymptotics for energy-dependent Sturm–Liouville equations (1) under minimal smoothness assumptions on the potentials p and q, including, e.g. the case where q contains Dirac delta-functions and/or Coulumb-like singularities. Our approach consists of the establishing a strong connection between spectral problem (1) and the spectral problem for a Dirac system of a special form. We then study the latter and, in particular, derive the integral representation for the solution of (1) which, in turn, is a basis for the subsequent asymptotic analysis.

The paper is organized as follows. In the next section, we rigorously set the spectral problem under study and give main definitions. The connection between spectral problem (1) and that for a special Dirac system is discussed in Section 3. In Section 4, we construct the transformation operator relating the solution of the obtained system with that of the Dirac system with zero potential. Next in Section 5, we derive the asymptotics of eigenvalues, eigenfunctions and the corresponding norming constants for problem (1), (2) and justify the factorization formula for its characteristic function. In the last section we formulate analogous results for spectral problem (1) under much more general boundary conditions. Appendix contains main definitions from the spectral theory for operator pencils. We also prove there that the algebraic multiplicity of λ as an eigenvalue of problem (1), (2) coincides with that of λ as an eigenvalue of the corresponding operator pencil.

Notations. Throughout the paper, we denote by $\mathcal{M}_2 = \mathcal{M}_2(\mathbb{C})$ the linear space of 2×2 matrices with complex entries endowed with the Euclidean operator norm. Next, p_0 will stand for $\int_0^1 p(s)ds$. The superscript t will signify the transposition of vectors and matrices, e.g. $(c_1, c_2)^{\mathrm{t}}$ is the column vector $\binom{c_1}{c_2}$.

2. Preliminaries. In this section, we recall main definitions and formulate the spectral problem under study more rigorously. To start with, introduce the differential expression

$$\ell(y) := -y'' + qy.$$

Since the potential q is a complex-valued distribution, we need to define the action of ℓ in detail. To do this, we use the regularization by quasi-derivative method due to Savchuk and Shkalikov [18, 19]. Take a function r from $L_2(0,1)$ such that $q = r'$ and for every absolutely continuous y introduce its quasi-derivative $y^{[1]} := y' - ry$. Then define $\ell(y)$ by

$$\ell(y) = -\bigl(y^{[1]}\bigr)' - ry^{[1]} - r^2 y$$

on the domain $\operatorname{dom} \ell = \{y \in AC[0,1] \mid y^{[1]} \in AC[0,1],\ \ell(y) \in L_2(0,1)\}$. A straightforward verification shows that so defined $\ell(y)$ coincides with $-y'' + qy$ in the distributional sense. Therefore we can recast equation (1) as

$$\ell(y) + 2\lambda py = \lambda^2 y. \tag{3}$$

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of problem (1), (2) if equation (3) possesses a nontrivial solution satisfying boundary conditions (2). This solution is then called an *eigenfunction* of problem (1), (2) corresponding to λ.

Assume that $y(x,z)$ is a solution of (3) with z instead of λ subject to the initial conditions $y(0)=0$, $y^{[1]}(0)=1$. This solution exists and is unique [18], so that λ is an eigenvalue of problem (1), (2) if and only if it is a zero of the *characteristic function* $\varphi(z) := y(1,z)$. The corresponding eigenfunction then coincides, up to a constant factor, with $y(\cdot,\lambda)$. The multiplicity of λ as a zero of $\varphi(z)$ is called an *algebraic multiplicity* of the eigenvalue λ of (1), (2).

As we shall see further, φ is an analytic nonconstant function, and so the set of its zeros is a discrete subset of $\mathbb{C}$. This shows that the set of eigenvalues of (1), (2) is discrete. Without loss of generality, we shall make the following standing assumption

(A) 0 is not a zero of the characteristic function $\varphi(z)$, i.e. it is not an eigenvalue of problem (1), (2).

In fact, (A) is achieved by shifting the spectral parameter λ if necessary; then for λ_0 such that $\varphi(\lambda_0) \neq 0$ problem (1), (2) with new spectral parameter $\mu := \lambda - \lambda_0$ and with p and q replaced with $p + \lambda_0$ and $q - 2\lambda_0 p - \lambda_0^2$ respectively, assumption (A) holds.

Spectral problem (1), (2) can be regarded as the spectral problem for the quadratic operator pencil T (see [13, 16]) defined by

$$T(\lambda)y := \lambda^2 y - 2\lambda p y + y'' - qy \tag{4}$$

on the λ-independent domain $\operatorname{dom} T := \{y \in \operatorname{dom} \ell \mid y(0) = y(1) = 0\}$. For the pencil T one can introduce the notions of the spectrum, the eigenvalues and corresponding eigenvectors, their geometric and algebraic multiplicities (see, e.g. [13] and Appendix). The spectral properties of T were discussed in [16]. In particular, it was proved therein that the spectrum of T consists only of eigenvalues, which can easily be shown to coincide with the eigenvalues of problem (1), (2) defined above. In Appendix, we show that the algebraic multiplicity of λ as an eigenvalue of (1), (2) coincides with that of λ as an eigenvalue of T.

3. Reduction to the Dirac system. In this section we reduce equation (1) to a λ-linear Dirac-type system of the first order. We shall further use the connection between (1) and this system to derive the asymptotics of interest.

The following observation plays an important role in the reduction procedure.

Lemma 1. *The equation $\ell(y) = 0$ possesses a complex-valued solution which does not vanish on $[0,1]$.*

Proof. Note firstly that for every complex a, b and every x_0 from $[0,1]$ the equation $\ell(y) = 0$ possesses a unique solution satisfying the conditions $y(x_0) = a$ and $y^{[1]}(x_0) = b$ (see, e.g. [18]).

Assume that y is a solution of $\ell(y) = 0$. We introduce the polar coordinates ρ and θ via $y(x) = \rho(x)\sin\theta(x)$ and $y^{[1]}(x) = \rho(x)\cos\theta(x)$. Clearly, the solution y vanishes at some point x_0 from $[0,1]$ if and only if $\theta(x_0) = \pi k$ for some $k \in \mathbb{Z}$. The function θ is called the Prüfer angle (see [4, 19]) and can be defined to be continuous; it then satisfies the equation

$$\theta'(x) = (\cos\theta(x) + r(x)\sin\theta(x))^2. \tag{5}$$

Equation (5) has the form $\theta' = f(x,\theta)$ with the right-hand side f that is not continuous in x. However, (5) is a so-called Caratheodory equation and subject to the initial condition $\theta(\xi) = 0$ with $\xi \in [0,1]$ it possesses a unique solution $\theta(x,\xi)$ (see e.g. [2, Theorem

1.1.2]); this solution depends continuously on ξ (see [2, Theorem 2.8.2]). Therefore the mapping $\xi \mapsto \theta(0,\xi)$ is continuous and its image I_0 is a compact subset of $\mathbb{C}$ containing 0 as a continuous image of the compactum $[0,1]$. Note further that for any $k \in \mathbb{Z}$ the solutions $\theta_k(x,\xi)$ of problem (5) with $\theta(\xi) = \pi k$ are equal to $\theta_k(x,\xi) = \theta(x,\xi)+\pi k$. The images I_k of the mappings $\xi \mapsto \theta_k(0,\xi)$ are compact and are the shifts of I_0 by πk along the real axis.

Now we take any complex number, say θ_0, outside the union of all the compacta I_k, $k \in \mathbb{Z}$, and consider problem (5) with $\theta(0) = \theta_0$. In view of the above arguments and uniqueness of the solution of the corresponding initial-value problem, the solution of (5) with $\theta(0) = \theta_0$ can be equal to πk, $k \in \mathbb{Z}$, at no point of the interval $[0,1]$. Consider the solution y_0 of $\ell(y) = 0$ subject to the initial conditions $y(0) = 1$, $y^{[1]}(0) = \cot\theta_0$. Then y_0 does not vanish on $[0,1]$, which is the assertion of the lemma. □

Denote by y_0 any solution of $\ell(y) = 0$ not vanishing on $[0,1]$ and set $v = y_0'/y_0$. Observe that $v \in L_2(0,1)$ and $q = v' + v^2$, i.e. q is a Miura potential (see [6]). Then the differential expression $-y'' + qy$ can be written in the factorized form, viz.

$$-y'' + qy = -\Big(\frac{d}{dx} + v\Big)\Big(\frac{d}{dx} - v\Big)y. \tag{6}$$

Remark 1. Observe that the function v satisfies the equality $v - r = y_0^{[1]}/y_0$, so that $v - r$ is a continuous function on $[0,1]$ and $(v-r)(x) = \cot\theta(x)$ for every $x \in [0,1]$, where θ is the Prüfer angle corresponding to y_0.

For $\lambda \neq 0$ consider the functions $u_2 := y$ and $u_1 := (y' - vy)/\lambda$ and recast spectral equation (1) as the following first order system for u_1 and u_2:

$$u_2' - vu_2 = \lambda u_1, \tag{7}$$
$$-u_1' - vu_1 + 2pu_2 = \lambda u_2. \tag{8}$$

Setting

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P := \begin{pmatrix} 0 & -v \\ -v & 2p \end{pmatrix}, \qquad \mathbf{u}(x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{9}$$

we see that the above system is the spectral problem $\ell(P)\mathbf{u} = \lambda\mathbf{u}$ for a Dirac differential expression $\ell(P)$ acting in $L_2(0,1) \times L_2(0,1)$ via

$$\ell(P)\mathbf{u} := J\frac{d\mathbf{u}}{dx} + P\mathbf{u} \tag{10}$$

on the domain $\operatorname{dom}\ell(P) = \{\mathbf{u} = (u_1,u_2)^{\mathrm{t}} \mid \mathbf{u} \in W_2^1(0,1) \times W_2^1(0,1)\}$.

It was shown in [17] that spectral problem (1), (2) is closely related to the spectral problem for the Dirac operator $\mathcal{D}(P)$ defined by the differential expression $\ell(P)$ on the domain

$$\operatorname{dom}\mathcal{D}(P) = \{\mathbf{u} = (u_1,u_2)^{\mathrm{t}} \mid \mathbf{u} \in \operatorname{dom}\ell(P),\ u_2(0) = u_2(1) = 0\}.$$

In particular, the nonzero spectra for both problems coincide counting with multiplicity. Moreover, $\mathbf{u} = (u_1,u_2)^{\mathrm{t}}$ is an eigenfunction of the operator $\mathcal{D}(P)$ corresponding to the eigenvalue $\lambda \neq 0$ if and only if $y = u_2$ is an eigenfunction of (1), (2) corresponding to λ and $u_1 = (u_2' - vu_2)/\lambda$.

By assumption (A), $\lambda = 0$ is not an eigenvalue of (1), (2); however, it is in the spectrum of $\mathcal{D}(P)$:

Lemma 2. *Under assumption (A), $\lambda = 0$ is an eigenvalue of $\mathcal{D}(P)$ of algebraic multiplicity one.*

Proof. A straightforward verification shows that $\lambda = 0$ is an eigenvalue of $\mathcal{D}(P)$, and every corresponding eigenfunction must be collinear to $\mathbf{u}_0 = (u_1, u_2)^{\mathrm{t}}$, where $u_1 = \exp\{-\int v\}$, $u_2 \equiv 0$. Therefore the eigenvalue $\lambda = 0$ is geometrically simple.

Suppose that the algebraic multiplicity of $\lambda = 0$ is greater than one. Then there exists a vector $\mathbf{w} = (w_1, w_2)^{\mathrm{t}}$ from the domain of $\mathcal{D}(P)$ associated with $\mathbf{u}_0$, i.e. satisfying the equality $\mathcal{D}(P)\mathbf{w} = \mathbf{u}_0$. Then $\left(\frac{d}{dx} - v\right) w_2 = u_1$ and thus $\ell w_2 = -\left(\frac{d}{dx} + v\right)\left(\frac{d}{dx} - v\right) w_2 = -\left(\frac{d}{dx} + v\right) u_1 = 0$. Moreover, $\mathbf{w} \in \operatorname{dom}\mathcal{D}(P)$ requires that $w_2(0) = w_2(1) = 0$ and thus either w_2 is an eigenfunction of (1), (2) for the eigenvalue $\lambda = 0$ or $\mathbf{w} \equiv 0$. The first possibility is ruled out by assumption (A), while the second one is impossible in view of the relation $w_2' - v w_2 = u_1$. The contradiction derived shows that no such $\mathbf{w}$ exists and finishes the proof. □

4. Transformation operator. In this section we construct the so-called transformation operator relating the solution of the system $\ell(P)\mathbf{u} = \lambda\mathbf{u}$ and that of $\ell(P_0)\mathbf{u} = \lambda\mathbf{u}$ with zero matrix potential P_0 (i.e. with matrix potential having all components zero).

Denote by $U(x, \lambda)$ a 2×2 matrix-valued function satisfying the equation

$$J\frac{dU}{dx} + PU = \lambda U \tag{11}$$

and the initial condition $U(0) = I$.

Theorem 1. *Let P in (11) be of the form (9) with p and v from $L_2(0,1)$. Then*

$$U(x, \lambda) = e^{a(x)J} + \int_0^x e^{-\lambda(x-2s)J} K(x,s)\,ds, \tag{12}$$

where $a(x) = a(x,\lambda) = \int_0^x p(s)ds - \lambda x$ and K is a matrix-valued function such that for every $x \in [0,1]$ the function $K(x,\cdot)$ is from $L_2((0,1),\mathcal{M}_2)$. Moreover, the mapping

$$x \mapsto K(x,\cdot) \in L_2((0,1),\mathcal{M}_2) \tag{13}$$

is continuous on $[0,1]$.

This theorem is very similar to the corresponding theorem of [1] and its proof requires only minor modifications.

Proof. Observe that system (11) can be rewritten as $J\frac{dU}{dx} + QU = (\lambda - p)U$ with

$$Q = \begin{pmatrix} -p & -v \\ -v & p \end{pmatrix} = pJ_1 - vJ_2, \quad J_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The variation of constants method shows that U satisfies the following integral equation

$$U(x) = e^{a(x)J} + \int_0^x e^{(a(x)-a(s))J} JQ(s)U(s)\,ds.$$

This equation can be solved by the method of successive approximation. Setting

$$U_0(x) = e^{a(x)J}, \quad U_n(x) = \int_0^x e^{(a(x)-a(s))J} JQ(s)U_{n-1}(s)\,ds, \tag{14}$$

we see that the solution of the above equation can formally be given by the sum $\sum_{n=0}^{\infty} U_n$. We shall prove below that

$$\sum_{n=0}^{\infty} \|U_n\|_\infty < \infty \tag{15}$$

(here $\|U_n\|_\infty := \sup_{x\in[0,1]} |U_n(x)|$, and $|U_n(x)|$ is the Euclidean norm of the matrix $U_n(x)$) whence the series $\sum_{n=0}^{\infty} U_n$ converges in the space $L_\infty([0,1],\mathcal{M}_2)$ to the solution of equation (11) which satisfies the initial condition $U(0) = I$.

Let us now prove (15). Set $\tilde{Q}(t) := \exp\{-2\int_0^t pJ\} JQ(t)$ and

$$\mathcal{Q}_n(t_1,\dots,t_n) := \tilde{Q}(t_n)\tilde{Q}(t_{n-1})\dots\tilde{Q}(t_1).$$

Observe firstly that the matrix Q anti-commutes with J and therefore $e^{-tJ}\tilde{Q}(s) = \tilde{Q}(s)e^{tJ}$. Using this, in relations (14), we obtain by induction that

$$U_n(x) = e^{\int_0^x p(s)ds J} \int\limits_{\Pi_n(x)} e^{-\lambda(x-2\xi_n(\mathrm{t}))J} \mathcal{Q}_n(t_1,\dots,t_n)dt_1\dots dt_n,$$

where

$$\Pi_n(x) = \{\mathrm{t} := (t_1,\dots,t_n) \in \mathbb{R}^n \mid 0 \le t_1 \le \dots \le t_n \le x\}, \quad \xi_n(\mathrm{t}) = \sum_{j=1}^{n} (-1)^{n-j} t_j.$$

Setting $s = \xi_n(\mathrm{t})$, we can rewrite the equality for U_n as

$$U_n(x) = \int_0^x e^{-\lambda(x-2s)J} K_n(x,s)ds,$$

where $K_1(x,s) \equiv e^{\int_0^x pJ}\tilde{Q}(s)$ and for $n \ge 2$

$$K_n(x,s) = e^{\int_0^x pJ} \int\limits_{\Pi^*_{n-1}(x,s)} \mathcal{Q}_n(t_1,\dots,t_{n-1}, s+\xi_{n-1}(\mathrm{t}))dt_1\dots dt_{n-1}$$

with $0 \le s < x \le 1$ and

$$\Pi^*_{n-1}(x,s) = \{\mathrm{t} := (t_1,\dots,t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \le t_1 \le \dots \le t_{n-1} \le \xi_{n-1}(\mathrm{t}) + s \le x\}.$$

Let us estimate the L_2-norm of $K_n(x,\cdot)$

$$\|K_n(x,\cdot)\|_2^2 = \int\limits_0^1 |K_n(x,s)|^2 ds \le$$

$$\le \frac{1}{(n-1)!} \int\limits_0^1 ds \int\limits_{\Pi^*_{n-1}(x,s)} e^{2\int\limits_0^x |\mathrm{Im} p|} |\mathcal{Q}_n(t_1,\dots,t_{n-1}, s+\xi_{n-1}(\mathrm{t}))|^2 dt_1\dots dt_{n-1} \le$$

$$\le \frac{1}{(n-1)!} \int\limits_{\Pi_n(x)} e^{2\int\limits_0^x |\mathrm{Im} p|} |\mathcal{Q}_n(t_1,\dots,t_n)|^2 dt_1\dots dt_n =$$

$$= \frac{1}{((n-1)!)^2 n} e^{2\int\limits_0^x |\mathrm{Im} p|} \left(\int\limits_0^x |\tilde{Q}|^2 \right)^n \le \frac{e^{2\|p\|_1} \|\tilde{Q}\|_2^{2n}}{((n-1)!)^2},$$

where $\|K(\cdot)\|_2 := (\int_0^1 |K(s)|^2 ds)^{1/2}$ and $|K(x)|$ is the Euclidean norm of the matrix $K(x)$. Put $C := \max_{x\in[-1,1]}|e^{-\lambda xJ}|$. Then

$$|U_n(x)| \le C\int_0^x |K_n(x,s)|ds \le C\|K_n(x,\cdot)\|_2 \le C\frac{e^{\|p\|_1}\|\tilde{Q}\|_2^n}{(n-1)!},$$

which yields (15). The estimate of the norm $\|K_n(x,\cdot)\|_2$ implies also the convergence of the series $K(x,\cdot) := \sum_{n=1}^\infty K_n(x,\cdot)$ in $L_2((0,1),\mathcal{M}_2)$ with

$$\|K(x,\cdot)\|_2 \le \sum_{n=1}^{\infty}\frac{e^{\|p\|_1}\|\tilde{Q}\|_2^n}{(n-1)!} \le \|\tilde{Q}\|_2 \exp\{\|\tilde{Q}\|_2 + \|p\|_1\}.$$

Therefore the first statement of the theorem is proved.

To prove the continuity of (13) it is enough to verify the continuity of the mapping $x \mapsto \exp\{-\int_0^x pJ\}K(x,\cdot) =: \tilde{K}(x,\cdot)$. Take $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$. Set $\tilde{K}_n(x,s) := \exp\{-\int_0^x pJ\}K_n(x,\cdot)$; then

$$\tilde{K}_n(x_2,s) - \tilde{K}_n(x_1,s) = \int\limits_{\Pi^{**}_{n-1}(x_1,x_2,s)} \mathcal{Q}_n(t_1,\dots,t_{n-1}, s+\xi_{n-1}(\mathrm{t}))dt_1\dots dt_{n-1},$$

where

$$\Pi^{**}_{n-1}(x_1,x_2,s) := \{\mathrm{t} := (t_1,\dots,t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \le t_1 \le \dots \le t_{n-1} \le \xi_{n-1}(\mathrm{t}) + s,$$
$$x_1 \le \xi_{n-1}(\mathrm{t}) + s \le x_2\}.$$

Therefore,

$$\int\limits_0^1 |\tilde{K}_n(x_2,s) - \tilde{K}_n(x_1,s)|^2 ds \le$$

$$\le \frac{1}{(n-1)!}\int\limits_0^1 \int\limits_{\Pi^{**}_{n-1}(x_1,x_2,s)} |\mathcal{Q}_n(t_1,\dots,t_{n-1},s+\xi_{n-1}(\mathrm{t})|^2 dt_1\dots dt_{n-1}ds \le$$

$$\le \frac{1}{(n-1)!}\int\limits_{x_1}^{x_2} dt_n \int\limits_{\Pi_{n-1}(x_2)} |\mathcal{Q}_n(t_1,\dots,t_n)|^2 dt_1\dots dt_{n-1} \le \frac{1}{((n-1)!)^2}\|\tilde{Q}\|_2^{2(n-1)} \cdot \int\limits_{x_1}^{x_2} |\tilde{Q}(t)|^2 dt.$$

This yields the estimate of the norms

$$\|\tilde{K}_n(x_2,\cdot) - \tilde{K}_n(x_1,\cdot)\|_2 \le \frac{1}{(n-1)!}\|\tilde{Q}\|_2^{n-1}\left[\int_{x_1}^{x_2} |\tilde{Q}(t)|^2 dt\right]^{1/2}$$

and so

$$\|\tilde{K}(x_1,\cdot) - \tilde{K}(x_2,\cdot)\| \le C\left[\int_{x_1}^{x_2} |\tilde{Q}(t)|^2 dt\right]^{1/2} \exp\{\|\tilde{Q}\|\}$$

with some constant C depending on p. This shows that the mapping $x \mapsto \tilde{K}(x,\cdot)$ is continuous from $[0,1]$ to $L_2((0,1),\mathcal{M}_2)$. □

Observe that the 2×2 matrix $U_0 = e^{-\lambda xJ}$ is a solution of the system $\ell(P_0)U = \lambda U$ with zero potential P_0. Therefore, in view of the later theorem, the solution U of problem (11) can be obtained from U_0 by means of the transformation operator $\mathscr{T} = \mathscr{R} + \mathscr{K}$, where $\mathscr{R}$ is the operator of multiplication by $\exp\{J\int_0^x p\}$ and $\mathscr{K}$ is the integral operator acting as follows $\mathscr{K} f(x) = \int_0^x f(x-2s)K(x,s)ds$. This transformation operator also performs similarity of the corresponding differential expressions, namely, $\ell(P)\mathscr{T} = \mathscr{T}\ell(P_0)$.

5. Asymptotics. In this section, we derive the asymptotics of eigenvalues and eigenfunctions and the corresponding norming constants of problem (1) under the Dirichlet boundary conditions (2). We also obtain a factorization of the characteristic function for the problem under study.

5.1. Asymptotics of the eigenvalues. Consider the vector $\mathbf{u}(x,\lambda) = U(x,\lambda)(1,0)^{\mathrm{t}} = (u_1(x,\lambda), u_2(x,\lambda))^{\mathrm{t}}$. In view of (12), the second component $u_2(x,\lambda)$ of the vector $\mathbf{u}(x,\lambda)$ is given by

$$u_2(x,\lambda) = -\sin a(x) + \int_0^x k_{11}(x,s)\sin(\lambda(x-2s))ds + \int_0^x k_{21}(x,s)\cos(\lambda(x-2s))ds.$$

Observe that the function $u_2(x,\lambda)$ solves equation (1) and satisfies the initial condition $u_2(0,\lambda) = 0$. However, $u_2^{[1]}(0,\lambda) = \lambda(u_1(0,\lambda) + cu_2(0,\lambda)) = \lambda$, where $c = (v-r)(0)$ (see Remark 1). Therefore $\varphi(\lambda) = u_2(1,\lambda)/\lambda$ is the characteristic function for spectral problem (1), (2).

Further observe that $u_2(1,\lambda)$ can be written as

$$u_2(1,\lambda) = \sin(\lambda - p_0) + \int_0^1 f(s)e^{i\lambda(1-2s)}ds \tag{16}$$

with $f(s) = \frac{1}{2}\left[k_{21}(1,s) + k_{21}(1,1-s) - ik_{11}(1,s) + ik_{11}(1,1-s)\right]$ and $p_0 := \int_0^1 p(s)ds$. Let us use the substitution $z := \lambda - p_0$, and consider the function $\delta(z) = \sin z + \int_0^1 \tilde f(s)e^{iz(1-2s)}ds$, where $\tilde f(s) = f(s)e^{ip_0(1-2s)}$. Clearly, $\delta(z) = u_2(1, z+p_0)$. By Theorem 4 of [10], the zeros of $\delta(z)$ can be labeled according to their multiplicities as z_n, $n \in \mathbb{Z}$, so that $z_n = \pi n + \tilde\lambda_n$, where the sequence $(\tilde\lambda_n)_{n\in\mathbb{Z}}$ belongs to $\ell_2(\mathbb{Z})$. Hence the zeros of the function $u_2(1,\lambda)$ can be labeled according to their multiplicities as λ_n, $n \in \mathbb{Z}$, so that $\lambda_n = \pi n + p_0 + \tilde\lambda_n$. In view of this asymptotics, all but finitely many zeros of $u_2(1,\lambda)$ are simple.

Next note that $u_2(1,\lambda)$ is the characteristic function of the operator $\mathcal{D}(P)$ defined in Section 3, whence $\lambda = 0$ is a zero of $u_2(1,\lambda)$ of order 1 (see Lemma 2). However, under assumption (A) it is not a zero of the characteristic function $\varphi(\lambda)$. Therefore the set of all eigenvalues of problem (1), (2) coincides with the set of zeros of the function $u_2(1,\lambda)$ different from 0. Thus the following theorem holds true.

Theorem 2. *The eigenvalues of problem* (1), (2) *can be labeled according to their multiplicities as* λ_n *with* $n \in \mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$ *so that*

$$\lambda_n = \pi n + p_0 + \tilde\lambda_n \tag{17}$$

with an ℓ_2*-sequence* $(\tilde\lambda_n)$.

Next we construct a factorization of the characteristic function $\varphi(\lambda)$, which allows to determine $\varphi(\lambda)$ via the eigenvalues of (1), (2). We use this factorization to derive the formula determining the norming constants of (1), (2) via two spectra of equation (1) under two types of boundary conditions (see [16]).

Theorem 3. *Suppose that* λ_n, $n \in \mathbb{Z}^*$, *are the eigenvalues of spectral problem* (1), (2). *Then the characteristic function* $\varphi(\lambda)$ *can be factorized in the following way*

$$\varphi(\lambda) = \begin{cases} \text{V.p.} \prod\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\lambda_n - \lambda}{\pi n}, & \textit{if } p_0 \neq \pi l,\ l \in \mathbb{Z}; \\ (-1)^l\ \text{V.p.} \prod\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\lambda_n - \lambda}{\pi n}, & \textit{if } p_0 = \pi l,\ l \in \mathbb{Z}. \end{cases}$$

Proof. Suppose firstly that $p_0 \neq \pi l$, $l \in \mathbb{Z}$. Observe that the function $u_2(1,\lambda)$ is given by (16) and so it is of exponential type 1. Recall also that, by Lemma 2, $\lambda = 0$ is a zero of $u_2(1,\lambda)$ of order 1. Therefore by Hadamard factorization theorem (see e.g. [20]) we have[1]

$$u_2(1,\lambda) = \lambda e^{A\lambda+B} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) e^{\frac{\lambda}{\lambda_n}},$$

where A and B are some constants and λ_n are the zeros of the function $u_2(1,\lambda)$. In view of asymptotic distribution (17), the series $\sum \frac{\lambda}{\lambda_n}$ converges in the principal value sense. Indeed,

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{-n}}\right) = \sum_{n=1}^{\infty} \frac{\pi^2 n^2}{\lambda_n \lambda_{-n}} \cdot \frac{\lambda_n + \lambda_{-n}}{\pi^2 n^2},$$

and we note that the series $\sum_{n=1}^{\infty} \frac{\lambda_n + \lambda_{-n}}{\pi^2 n^2}$ is absolutely convergent and the sequence $\left(\frac{\pi^2 n^2}{\lambda_n \lambda_{-n}}\right)$ is bounded. Therefore we can write $u_2(1,\lambda) = \lambda e^{A'\lambda+B} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)$ with some constant A'.

To find the values A' and B, consider the ratio $\frac{u_2(1,\lambda)}{\sin(\lambda - p_0)}$ and find its limits along the ray $\lambda = re^{i\theta}$, $\theta \neq 0, \pi$. In view of (16) and a refined version of the Riemann–Lebesgue lemma [12, Lemma 1.3.1], we have

$$\frac{u_2(1, re^{i\theta})}{\sin(re^{i\theta} - p_0)} = 1 + o(1), \quad r \to \infty. \tag{18}$$

Recall (see e.g. [20]) that the function $\sin(\lambda - p_0)$ can be factorized as follows

$$\sin(\lambda - p_0) = (\lambda - p_0) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\nu_n - \lambda}{\pi n},$$

where ν_n are the zeros of $\sin(\lambda - p_0)$, i.e. $\nu_n = \pi n + p_0$. Therefore we have

$$\frac{u_2(1,\lambda)}{\sin(\lambda - p_0)} = \frac{\lambda e^{A'\lambda+B}}{\lambda - p_0} \cdot \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\pi n}{\lambda_n} \cdot \frac{\lambda_n - \lambda}{\nu_n - \lambda}. \tag{19}$$

Let us show that $A' = 0$. If A' were not 0, then one would choose the direction θ such that $\operatorname{Re} A're^{i\theta}$ tends to infinity as $r \to \infty$. Next note that by Lemma 3 given below the product $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\pi n}{\lambda_n}$ is convergent and by Lemma 4 the product $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}}$ converges to 1 as $r \to \infty$ and $\theta \neq 0, \pi$. These arguments together with (18) and (19) give a contradiction. Thus $A' = 0$ and $e^B \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\pi n}{\lambda_n} = 1$, yielding that $\varphi(\lambda) = \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - \lambda}{\pi n}$.

If $p_0 = \pi l$ for some $l \in \mathbb{Z}$, then $\sin(\lambda - p_0) = (-1)^l \lambda \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\pi n - \lambda}{\pi n}$, and so

$$\frac{u_2(1,\lambda)}{\sin(\lambda - p_0)} = (-1)^l e^{A'\lambda+B} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\pi n}{\lambda_n} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - \lambda}{\pi n - \lambda}.$$

[1]Here and hereafter all infinite products and sums are understood in the principal value sense and the symbol V.p. will be omitted.

Combining this, (18) and the arguments analogous to those used in the case of $p_0 \neq \pi l$, we obtain that $A' = 0$ and $e^B = (-1)^l \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n}{\pi n}$. Thus for $p_0 = \pi l$

$$\varphi(\lambda) = (-1)^l \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - \lambda}{\pi n}. \qquad \square$$

Lemma 3. *The product* $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\lambda_n}{\pi n}$ *is convergent.*

Proof. We start with proving the following inequality for $z \in \mathbb{C}$

$$|(1+z)e^{-z}| \leq e^{|z|^2}. \tag{20}$$

Indeed, $(1+z)e^{-z} = \sum_{k=0}^{\infty} c_k z^k$ with $c_0 = 1$ and $c_k = (-1)^{k+1}(k-1)/k!$. For $|z| < 1$ we have $|(1+z)e^{-z}| \leq 1 + \sum_{k=1}^{\infty}(|c_{2k}|+|c_{2k+1}|)|z|^{2k}$ while for $|z| \geq 1$ $|(1+z)e^{-z}| \leq 1+\sum_{k=1}^{\infty}(|c_{2k-1}|+|c_{2k}|)|z|^{2k}$. It remains to observe that $|c_{2k}|+|c_{2k+1}| \leq 1/k!$ and that $|c_{2k}|+|c_{2k-1}| \leq 1/k!$ for all $k \geq 1$.

Now consider a sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers from $\ell_2(\mathbb{Z})$ such that $1 + a_n \neq 0$ and the series $\sum a_n$ converges. Applying (20) we see that the product $\prod(1 + a_n)$ converges and moreover

$$\left|\prod(1+a_n)\right| \leq \left|\exp\sum a_n\right| \cdot \exp\Big\{\sum |a_n|^2\Big\}.$$

Next we show that this result is applicable to $\prod_{n\in\mathbb{Z}^*} \frac{\lambda_n}{\pi n}$. Set $1+a_n := \frac{\lambda_n}{\pi n}$, i.e. $a_n = \frac{p_0+\tilde\lambda_n}{\pi n}$. Since $\frac{1}{\pi n}$ belongs to ℓ_2 and a constant $C < \infty$ exists such that $|p_0+\tilde\lambda_n| < C$ for every $n \in \mathbb{Z}^*$, the sequence (a_n) belongs to ℓ_2. Note that V.p.$\sum_{n\in\mathbb{Z}^*} \frac{p_0}{\pi n} = 0$ and $\sum_{n\in\mathbb{Z}^*} \frac{\tilde\lambda_n}{\pi n}$ converges as the sequences $(\tilde\lambda_n)$ and $\frac{1}{\pi n}$ are from ℓ_2. This implies the convergence of the series $\sum_{n\in\mathbb{Z}^*} a_n$, thus finishing the proof. $\square$

Lemma 4. *The product* $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}}$ *converges to* 1 *as* $r \to \infty$ *with* $\theta \neq 0, \pi$.

Proof. Consider the series

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \ln \frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \ln\left(1 + \frac{\tilde\lambda_n}{\nu_n - re^{i\theta}}\right). \tag{21}$$

One can find N sufficiently large such that $|\tilde\lambda_n| < 1/2$ if $|n| > N$. Also, for all $r > R_\theta$ with $R_\theta = (1+|p_0|)/\sin\theta$ we have $|\nu_n - re^{i\theta}| > 1$. Since the constant $C < \infty$ exists such that $|\ln(1+z)| \leq C|z|$ if $|z| \leq 1/2$, for such n and r

$$\left|\ln\left(1+\frac{\tilde\lambda_n}{\nu_n - re^{i\theta}}\right)\right| \leq C\left|\frac{\tilde\lambda_n}{\nu_n - re^{i\theta}}\right|.$$

Next observe that $|\nu_n - re^{i\theta}| \geq |\pi n - re^{i\theta}| - |p_0| \geq |\pi n \sin\theta| - |p_0|$. Since $|p_0| < \frac{1}{2}\pi N \sin\theta$ for sufficiently large N, for all n with $|n| > N$ we have $|\nu_n - re^{i\theta}| > \frac{|\sin\theta|}{2}|\pi n|$. Therefore,

$$\left|\frac{\tilde{\lambda}_n}{\nu_n - re^{i\theta}}\right| < \frac{2}{\pi|\sin\theta|}\frac{|\tilde{\lambda}_n|}{|n|}.$$

Since the sequences $(\tilde{\lambda}_n)$ and $(1/n)$ belong to ℓ_2, the series $\sum\limits_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{|\tilde{\lambda}_n|}{|n|}$ is convergent and so the series (21) is convergent uniformly in $r > R_\theta$ for a fixed θ, $\theta \neq 0, \pi$. Therefore,

$$\lim_{r\to\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\ln\frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\lim_{r\to\infty}\ln\frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}} = 0,$$

which means that the product $\prod\limits_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\lambda_n - re^{i\theta}}{\nu_n - re^{i\theta}}$ converges to 1 as $r\to\infty$. □

5.2. Asymptotics of eigenfunctions and norming constants. Let us now consider the vectors $\mathbf{u}_n := \mathbf{u}(x,\lambda_n)$. Put $\mathbf{u}_{n,0} := (\cos\lambda_n x, \sin\lambda_n x)^{\mathrm{t}}$. Then, in view of Theorem 1, we have

$$\mathbf{u}_n = \mathscr{R}\mathbf{u}_{n,0} + \mathscr{L}\mathbf{u}_{n,0},$$

where the operator $\mathscr{R}$ was defined at the end of Section 4 and $\mathscr{L}\mathbf{u}(x) = \int_0^x L(x,s)\mathbf{u}(x-2s)ds$ with

$$L(x,s) = \begin{pmatrix} k_{11}(x,s) & -k_{21}(x,s)\\ k_{21}(x,s) & k_{11}(x,s)\end{pmatrix}$$

and k_{ij} being the corresponding entries of K. This yields the following

Theorem 4. *The eigenfunctions y_n of problem* (1), (2) *corresponding to the eigenvalues λ_n satisfy the asymptotics*

$$y_n(x) = \sin\Big(\lambda_n x - \int_0^x p\Big) + \tilde{y}_n(x),$$

where $\tilde{y}_n(x) = (0,1)\mathscr{L}\mathbf{u}_{n,0}$ and the sequence $(\|\tilde{y}_n\|)$ is from ℓ_2.

Proof. It only remains to prove the statement on $\|\tilde{y}_n\|$, and to this end we shall show that the sequence of norms $(\|\mathscr{L}\mathbf{u}_{n,0}\|)$ belongs to ℓ_2. Put $\mathbf{v}_n := (\cos(\pi n + p_0)x, \sin(\pi n + p_0)x)^{\mathrm{t}}$ and $\mathbf{v}_{n,0} := (\cos\pi n x, \sin\pi n x)^{\mathrm{t}}$. Then the vectors $\mathbf{v}_{n,0}$ form an orthonormal basis in $L_2(0,1)\times L_2(0,1)$ and $\mathbf{v}_n = \mathscr{P}\mathbf{v}_{n,0}$, where $\mathscr{P}$ is an operator of multiplication by $\exp\{p_0 x J\}$. Next observe that

$$\|\mathbf{u}_{n,0} - \mathbf{v}_n\| = \big\|(e^{\tilde{\lambda}_n xJ} - I)\mathbf{v}_n\big\| \le \bigg\|\int_0^x \frac{d}{dt}e^{\tilde{\lambda}_n tJ}dt\bigg\|\|\mathbf{v}_n\| \le C|\tilde{\lambda}_n|,$$

with $C := \max_{n\in\mathbb{Z}^*} e^{|\tilde{\lambda}_n|}\|\mathscr{P}\|$ so that

$$\sum_{n\in\mathbb{Z}^*}\|\mathbf{u}_{n,0} - \mathbf{v}_n\|^2 < \infty. \tag{22}$$

Now we write the inequality

$$\|\mathscr{L}\mathbf{u}_{n,0}\| \le \|\mathscr{L}(\mathbf{u}_{n,0} - \mathbf{v}_n)\| + \|\mathscr{L}\mathbf{v}_n\| \tag{23}$$

and note that the sequence $(\|\mathscr{L}(\mathbf{u}_{n,0} - \mathbf{v}_n)\|)$ belongs to ℓ_2 in view of (22). Finally, we observe that the operator $\mathscr{L}$ is of the Hilbert–Schmidt class $\mathfrak{S}_2$; then the same is true for $\mathscr{L}\mathscr{P}$, and therefore

$$\sum\|\mathscr{L}\mathbf{v}_n\|^2 = \sum\|\mathscr{L}\mathscr{P}\mathbf{v}_{n,0}\|^2 = \|\mathscr{L}\mathscr{P}\|_{\mathfrak{S}_2},$$

where $\|\cdot\|_{\mathfrak{S}_2}$ is the Hilbert–Schmidt norm of the operator $\mathscr{L}\mathscr{P}$(see e.g. [7, V.2.4.]). These observations and (23) complete the proof. □

Observe that the following estimate holds

$$|\|\mathbf{u}_n\| - \|\mathscr{R}\mathbf{v}_n\|| \le \|\mathbf{u}_n - \mathscr{R}\mathbf{v}_n\| \le \|\mathbf{u}_n - \mathscr{R}\mathbf{u}_{n,0}\| + \|\mathscr{R}\mathbf{u}_{n,0} - \mathscr{R}\mathbf{v}_n\| = \\ = \|\mathscr{L}\mathbf{u}_{n,0}\| + \|\mathscr{R}\|\|\mathbf{u}_{n,0} - \mathbf{v}_n\|.$$

Similarly as it was done in the proof of the above theorem, one can prove that the sequence of $|\|\mathbf{u}_n\| - \|\mathscr{R}\mathbf{v}_n\||$ belongs to ℓ_2. Assume now that p is real-valued. In this case the operators $\mathscr{P}$ and $\mathscr{R}$ are unitary, whence $\mathbf{v}_n$ form an orthonormal basis and $\|\mathscr{R}\mathbf{v}_n\| = 1$. As a consequence, $\|\mathbf{u}_n\| = 1 + \tilde{\alpha}_n$, where $(\tilde{\alpha}_n)$ belongs to ℓ_2.

If additionally q is real-valued then all but finitely many eigenvalues of (1), (2) are real and simple, and the norming constants of (1), (2) corresponding to such eigenvalues coincide with the norms of eigenvectors of the operator $\mathcal{D}(P)$ (see [5]). Therefore the following holds true.

Theorem 5. *If p and q are real-valued, the norming constants α_n of problem* (1), (2) *corresponding to the eigenvalues λ_n satisfy the asymptotics $\alpha_n = 1 + \tilde{\alpha}_n$, where $(\tilde{\alpha}_n) \in \ell_2$.*

6. Results for the mixed boundary conditions. In this section, we consider equation (1) under the boundary conditions, which we call the mixed ones; namely,

$$y(0) = y^{[1]}(1) + hy(1) = 0 \tag{24}$$

with some complex h. As the proofs are analogous to those applied in the case of the Dirichlet boundary conditions, we shall only reformulate the results.

Without loss of generality, we assume that $\mu = 0$ is not an eigenvalue of problem (1), (24). Consider the function

$$\psi(\mu) := u_1(1, \mu) + (h_1 + h)u_2(1, \mu)/\mu,$$

where u_1 and u_2 are the solutions of system (7), (8) and $h_1 := (v - r)(1)$. The function $y(\cdot, \mu) := u_2(\cdot, \mu)$ solves the equation $\ell(y) + 2\mu py = \mu^2 y$ and satisfies the relation

$$y^{[1]} + hy = (y' - vy) + (v - r)y + hy = \mu u_1 + (v - r + h)u_2;$$

in particular, $y(0, \mu) = 0$ and $y^{[1]}(1, \mu) + hy(1, \mu) = \mu\psi(\mu)$. Therefore ψ is the characteristic function for problem (1), (24), i.e. the zeros of $\psi(\mu)$ are the eigenvalues of the mentioned problem.

Using the asymptotics form of u_1 and u_2, we have

$$\psi(\mu) = \cos a(1, \mu) - \frac{(h_1 + h)}{\mu} \sin a(1, \mu) - \\ - \int_0^1 k_{21}(1, s) \sin(\mu(1 - 2s))ds + \int_0^1 k_{11}(1, s) \cos(\mu(1 - 2s))ds + \\ + \frac{h_1 + h}{\mu} \left(\int_0^1 k_{11}(1, s) \sin(\mu(1 - 2s))ds + \int_0^1 k_{21}(1, s) \cos(\mu(1 - 2s))ds \right).$$

Taking into account Theorem 4 from [10], we obtain

Theorem 6. *The eigenvalues of* (1), (24) *can be labeled according to their multiplicities as μ_n, $n \in \mathbb{Z}$, so that they satisfy the following asymptotics*

$$\mu_n = \pi\left(n + \frac{1}{2}\right) + p_0 + \tilde{\mu}_n, \tag{25}$$

with ℓ_2-sequence $(\tilde{\mu}_n)$. In particular, all eigenvalues μ_n with large enough $|n|$ are simple.

Analogously to the case of the Dirichlet boundary conditions, we prove the following

Theorem 7. *Let μ_n, $n \in \mathbb{Z}$, be the eigenvalues of* (1), (24)*. Then the characteristic function $\psi(\mu)$ can be factorized in the following way*

$$\psi(\mu) = \begin{cases} -\mathrm{V.p.} \prod\limits_{n=-\infty}^{\infty} \dfrac{\mu_n - \mu}{\pi\,(n+1/2)}, & \text{if } p_0 \neq \dfrac{\pi}{2} + \pi l,\ l \in \mathbb{Z}; \\ (-1)^{l+1}(\mu_0 - \mu)\mathrm{V.p.} \prod\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\mu_n - \mu}{\pi n}, & \text{if } p_0 = \dfrac{\pi}{2} + \pi l,\ l \in \mathbb{Z}. \end{cases}$$

Theorem 8. *The eigenfunctions y_n of* (1), (24) *corresponding to the eigenvalues μ_n satisfy the asymptotics*

$$y_n(x) = \cos\Big(\lambda_n x - \int_0^x p\Big) + \tilde{y}_n(x),$$

where the sequence $(\|\tilde{y}_n(x)\|)$ is from ℓ_2.

Theorem 9. *If p and q are real-valued, the norming constants β_n of* (1), (24) *corresponding to the eigenvalues μ_n satisfy the asymptotics $\beta_n = 1 + \tilde{\beta}_n$, with an ℓ_2-sequence $(\tilde{\beta}_n)$.*

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7. Appendix: algebraic multiplicities of the eigenvalues. In this appendix we recall the main notions of the spectral theory for the operator pencils. We also show that the algebraic multiplicity of λ as an eigenvalue of the operator pencil T of (4) coincides with the corresponding multiplicity of λ as an eigenvalue of problem (1), (2).

An *operator pencil* T is an operator-valued function defined on $\mathbb{C}$. The *spectrum* of an operator pencil T is the set $\sigma(T)$ of all $\lambda \in \mathbb{C}$ such that $T(\lambda)$ is not boundedly invertible, i.e. $\sigma(T) = \{\lambda \in \mathbb{C} \mid 0 \in \sigma(T(\lambda))\}$. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if $T(\lambda)y = 0$ for some non-zero function y from $\operatorname{dom} T$, which is then the corresponding *eigenfunction.*

Vectors $y_1, ..., y_{m-1}$ from $\operatorname{dom} T$ are said to be associated with an eigenvector y_0 corresponding to an eigenvalue λ if

$$\sum_{k=0}^{j} \frac{1}{k!} T^{(k)}(\lambda) y_{j-k} = 0, \quad j \in \{1, ..., m-1\}. \tag{26}$$

Here $T^{(k)}$ denotes the k-th derivative of T with respect to λ. The number m is called the length of the chain $y_0, \dots, y_{m-1}$ of an eigen- and associated vectors. The maximal length of a chain starting with an eigenvector y_0 is called the *algebraic multiplicity* of an eigenvector y_0.

For an eigenvalue λ of T the dimension of the null-space of $T(\lambda)$ is called the *geometric multiplicity* of λ. The eigenvalue is said to be *geometrically simple* if its geometric multiplicity equals to one.

All the eigenvalues of the pencil T of (4) are geometrically simple (see [16]), and then the *algebraic multiplicity* of an eigenvalue is the algebraic multiplicity of the corresponding eigenvector. (If the eigenvalue λ is not geometrically simple, its algebraic multiplicity is the number of vectors in the corresponding canonical system, see [8, 13]). An eigenvalue is said to be *algebraically simple* (or just *simple*) if its algebraic multiplicity is one.

In the next proposition we show that the order of λ as a zero of the characteristic function φ coincides with the algebraic multiplicity of λ as an eigenvalue of the operator pencil T defined by (4).

Proposition 1. *Suppose λ is an eigenvalue of spectral problem* (1), (2). *Then λ is a zero of the characteristic function φ of order m if and only if λ is an eigenvalue of the operator pencil T given by* (4) *of algebraic multiplicity m.*

Proof. Suppose that $y(x,z)$ is the solution of (1) subject to the initial conditions $y(0,z)=0$, $y^{[1]}(0,z)=1$ and that λ is a zero of $\varphi(z)=y(1,z)$ of order m. Then $y(x,\lambda)$ is an eigenfunction of (1), (2) corresponding to λ. Clearly, $y(x,\lambda)$ is also an eigenfunction of the operator pencil T corresponding to the eigenvalue λ. Consider the chain of the vectors y_j with $j\in\{0,1,\dots\}$, such that $y_0=y(x,\lambda)$ and $y_j(x,\lambda):=\frac{1}{j!}\frac{\partial^j y(x,z)}{\partial z^j}\Big|_{z=\lambda}$, $j\ge 1$. Set $\tau(\lambda)y:=\lambda^2 y-2\lambda p y-\ell(y)$. Straightforward verification shows that y_j satisfy equalities (26) with τ instead of T. Moreover, since λ is a zero of $y(1,z)$ of order m, we have $y_1(1)=\dots=y_{m-1}(1)=0$, and all the functions y_j, $j\in\{0,\dots,m-1\}$, belong to the domain of T and so form a chain of eigen- and associated vectors of T corresponding to λ. Therefore m does not exceed the algebraic multiplicity of λ as an eigenvalue of T.

Assume that $v_0,\dots,v_l$ is a chain of eigen- and associated vectors corresponding to an eigenvalue λ of T. Then v_0 solves the equation $\tau(\lambda)y=0$ and satisfies boundary conditions (2), and thus coincides with y_0, up to a scalar factor. Without loss of generality, we assume that $v_0=y_0$ and then show by induction that there exists a sequence $(c_k)_{k=1}^l$ such that

$$v_k-y_k=\sum_{j=1}^{k}c_j v_{k-j}. \tag{27}$$

To start with, note that $\tau'(\lambda)(v_0-y_0)+\tau(\lambda)(v_1-y_1)=0$. But $v_0-y_0=0$ and $(v_1-y_1)(0)=0$. Therefore $v_1-y_1=c_1v_0$ giving the base of induction. Next suppose that the statement holds for all $k<n$ and prove it for $k=n$. Observe that

$$\tau(\lambda)(v_n-y_n)=-\sum_{j=1}^{n}\frac{1}{j!}\tau^{(j)}(v_{n-j}-y_{n-j}).$$

By assumption we obtain

$$\tau(\lambda)(v_n-y_n)=-\sum_{j=1}^{n}\frac{1}{j!}\tau^{(j)}\Big(\sum_{i=1}^{n-j}c_i v_{n-j-i}\Big)=-\sum_{i=1}^{n-1}c_i\Big(\sum_{j=1}^{n-i}\frac{1}{j!}\tau^{(j)}v_{n-j-i}\Big)=\sum_{j=1}^{n-1}c_j\tau(\lambda)v_{n-j}.$$

Therefore,

$$\tau(\lambda)\Big(v_n-y_n-\sum_{j=1}^{n-1}c_jv_{n-j}\Big)=0,\ \Big(v_n-y_n-\sum_{j=1}^{n-1}c_jv_{n-j}\Big)(0)=0,$$

giving that $v_n-y_n-\sum_{j=1}^{n-1}c_jv_{n-j}=c_nv_0$, i.e. (27) holds.

Assuming that $l\ge m$, we see that $v_m-y_m=\sum_{j=1}^{m}c_jv_{m-j}$ and so $y_m(1)=0$. This contradicts the fact that λ is a zero of $\varphi(z)$ of order m. $\square$

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