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DIFFERENTIAL EQUATIONS AND INTEGRAL CHARACTERIZATIONS OF TIMELIKE AND SPACELIKE SPHERICAL CURVES IN THE MINKOWSKI SPACE-TIME E_1^4

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In this paper we give differential equations characterizing timelike and spacelike curves lying on hyperbolic sphere H_0^3 and Lorentzian sphere S_1^3 in the Minkowski space-time E_1^4. Furthermore, we give the integral characterizations of these curves in E_1^4.

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В работе получены дифференциальные уравнения, описывающие временноподобные и пространственно подобные кривые, лежащые на гиперболической сфере H_0^3 и сфере Лоренца S_1^3 в пространстве Минковского E_1^4. Кроме того, получены интегральныя описания этих кривых в E_1^4.

1. Introduction. In the local differential geometry, the special curves i.e., the curves whose curvatures satisfy some relations or differential equations play an important role. One of the well-known of these special curve is spherical curve which is the curve lying fully on a sphere in the space. The problem of finding a differential equation for the spherical curves was thought by many mathematicians so far. E. Kreyzig ([4]) found a differential equation characterizing the spherical curves in the Euclidean 3-space E^3. Y. C. Wong ([13], [14]) gave differential equations and integral characterizations of spherical curves in E^3. S. Breuer and D. Gottlieb ([1]) studied spherical curves and gave explicit characterizations of spherical curves. E. Kreyszig and A. Pendl ([5]) studied spherical curves and their analogue in affine differential geometry. Characterizations of timelike and spacelike spherical curves in the Minkowski 3-space E_1^3 have been given in the ref. [6]–[9], [11].

V. Dannon ([2]) showed that spherical curves in E^4 can be given by Frenet analogue equations and then he gave an integral characterization for spherical curves in E^4. Similar characterizations of timelike and spacelike spherical curves lying on Lorentzian sphere have been given by M. Kazaz, H. H. Uğurlu and A. Özdemir ([3]) in the E_1^4. They have found differential equation systems characterizing the spherical curves in E_1^4. They have also showed that finding an integral characterization for a Lorentzian spherical E_1^4-timelike or spacelike curves is identical to finding it for E_1^3-timelike and spacelike curves. M. Sezer ([12]) has

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studied differential equations and integral characterizations of spherical curves in E^4 by using the differential equation system given by Dannon.

In this study, first we obtain differential equation system characterizing spacelike spherical curve with timelike second binormal T_4. Then by using the method given in [12], we find differential equations of spherical curves lying on hyperbolic sphere H_0^3. Later we give the corresponding characterizations of curves lying on the Lorentzian sphere S_1^3 in the Minkowski space-time E_1^4. We also give integral characterizations of these curves.

2. Preliminaries. The Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, where (x_1, x_2, x_3, x_4) is the rectangular coordinate system in E_1^4 ([15]).

Since this metric is an indefinite metric, an arbitrary vector $v \in E_1^4$ can have one of three causal characters: it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 is locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. A timelike or spacelike curve $\alpha(s)$ is said to be parameterized by arc length function s, if $\langle \alpha'(s), \alpha'(s)\rangle = \mp 1$ ([10]). Also, recall that the norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. Therefore, v is a unit vector if $\langle v, v\rangle = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $\langle v, w\rangle = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

The hyperbolic sphere and the Lorentzian sphere with center $m = (m_1, m_2, m_3, m_4) \in E_1^4$ and radius $r \in \mathbb{R}^+$ in the space-time E_1^4 are the hyper quadrics given by

$$H_0^3 = \left\{ a = (a_1, a_2, a_3, a_4) \in E_1^4 \,\middle|\, \langle a - m, a - m\rangle = -r^2 \right\}, \tag{1}$$

$$S_1^3 = \left\{ a = (a_1, a_2, a_3, a_4) \in E_1^4 \,\middle|\, \langle a - m, a - m\rangle = r^2 \right\}, \tag{2}$$

respectively ([3]).

Let us denote the moving Frenet frame along the curve $\alpha(s)$ which is parameterized by arc length function s in the space E_1^4 by $\{T_1, T_2, T_3, T_4\}$. Then T_1, T_2, T_3 and T_4 are the tangent, the principal normal, the first binormal and the second binormal vector fields, respectively. Then for the curve $\alpha(s)$ the following Frenet equations are given in [15].

If T_1 is timelike and the others are spacelike, then the Frenet formulae has the form

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix},$$

where $k_1 = \langle T_1', T_2\rangle, k_2 = \langle T_2', T_3\rangle, k_3 = \langle T_3', T_4\rangle$.

If T_2 is timelike and the others are spacelike, then the Frenet equations are given by

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix},$$

where $k_1 = -\langle T_1', T_2\rangle, k_2 = \langle T_2', T_3\rangle, k_3 = \langle T_3', T_4\rangle$.

If T_3 is timelike and the others are spacelike, then

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix},$$

where $k_1 = \langle T_1', T_2 \rangle, k_2 = -\langle T_2', T_3 \rangle, k_3 = \langle T_3', T_4 \rangle$.

Finally, if T_4 is timelike and the others are spacelike, then

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix},$$

where $k_1 = \langle T_1', T_2 \rangle, k_2 = \langle T_2', T_3 \rangle, k_3 = -\langle T_3', T_4 \rangle$.

If the curve $\alpha(s)$ is timelike we get $\langle T_1, T_1 \rangle = -1,\ \langle T_i, T_i \rangle = 1,\ (i \in \{2,3,4\})$.

If the curve $\alpha(s)$ is spacelike then $\langle T_1, T_1 \rangle = 1$. In this case, one of the vectors T_2, T_3 or T_4 is timelike.

3. Differential equations and integral characterizations of spacelike curves lying on hyperbolic sphere H_0^3. In this section, we give differential equations and integral characterizations of spacelike curves lying on the hyperbolic sphere H_0^3 in the Minkowski space-time E_1^4. First, we give the following proposition characterizing the spacelike spherical curves with timelike vector T_4 and lying on hyperbolic sphere H_0^3 in E_1^4.

Proposition 1. *Let $\alpha(s)\colon I \subset \mathbb{R} \to E_1^4$ be a unit speed regular spacelike curve with smooth curvature functions $k_1(s), k_2(s), k_3(s)$. Then the following conditions are equivalent.*

i) $\alpha(s)$ lies on a H_0^3 sphere.

ii) $k_1(s) \neq 0$ and there are two C^2-functions $f(s)$ and $g(s)$ such that

$$\frac{d\rho}{ds} = k_2 f,\ \frac{df}{ds} = -k_2\rho + k_3 g,\ \frac{dg}{ds} = k_3 f, \tag{3}$$

where $\rho = 1/k_1$.

Proof. Assume that $\alpha(s)$ lies on a hyperbolic sphere H_0^3 with a radius a and a center x_0. Let the center x_0 be chosen as origin. Then we can write $\langle \alpha, \alpha \rangle = -a^2$. For the position vector of the curve we have $\alpha = \sum_{i=1}^4 f_i T_i$, where $f_i = f_i(s)$ are the functions of arc length parameter s and defined by $f_i = \langle \alpha, T_i \rangle ; (1 \leq i \leq 3)$, $f_4 = -\langle \alpha, T_4 \rangle$; and $T_i = T_i(s)$ are the Frenet vectors of the curve. Differentiating equality $\langle \alpha, \alpha \rangle = -a^2$ with respect to s and by using Frenet formulae, we obtain $f_1 = \langle \alpha, T_1 \rangle = 0$. Differentiating again gives $k_1 \neq 0$ and $f_2 = \langle \alpha, T_2 \rangle = -\rho$. Then we have $\rho' = -k_2 f_3$ and writing $f_3 = -f$ we obtain $\rho' = k_2 f$.

Similarly, differentiating the equality $f_3 = \langle \alpha, T_3 \rangle$ and using the obtained results we have $f' = -k_2\rho + k_3 g$, where $g = f_4$. Differentiating $g = f_4 = -\langle \alpha, T_4 \rangle$ gives $g' = k_3 f$, and finally we write

$$\frac{d\rho}{ds} = k_2 f, \frac{df}{ds} = -k_2\rho + k_3 g, \frac{dg}{ds} = k_3 f.$$

Conversely, assume that (3) holds. Then using (3) and Frenet formulae we have $\frac{d}{ds}(\alpha - \sum_2^4 f_i T_i) = 0$. Thus, $\alpha - \sum_2^4 f_i T_i = \text{constant} \equiv x_0$ and it is obtained that $\|\alpha - x_0\|^2 = f_2^2 + f_3^2 - f_4^2$ and $\frac{d}{ds}\|\alpha - x_0\|^2 = 0$. Therefore, $\|\alpha - x_0\|^2 = \text{constant} = a^2$, i.e. α lies on the hyperbolic sphere H_0^3 of radius a about x_0. $\square$

Theorem 1. *Let $\alpha(s)\colon I\subset\mathbb{R}\to E_1^4$ be a unit speed regular spacelike curve with non-zero smooth curvature functions $k_1(s), k_2(s), k_3(s)$ and let k_2 and k_3 be of a fixed sign. Then, the condition for the spacelike curve $\alpha(s)$ be a H_0^3-spherical curve is that $\rho(s)=1/k_1(s), k_2(s)$ and $k_3(s)$ satisfy one of the following differential equations which are equivalent.*

i) $\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{1}{k_2}\frac{d\rho}{ds}\right)+\frac{k_2}{k_3}\rho\right]-\frac{k_3}{k_2}\frac{d\rho}{ds}=0.$

ii) $\frac{d}{d\xi}\left[\frac{k_2}{k_3}\left(\frac{d^2\rho}{d\xi^2}+\rho\right)\right]-\frac{k_3}{k_2}\frac{d\rho}{d\xi}=0$, *where* $\xi(s)=\int_0^s k_2(u)du$.

iii) $h^2\left(\frac{d^2\rho}{d\xi^2}+\rho\right)^2-\left(\frac{d\rho}{d\xi}\right)^2-\rho^2=K^2$, *where* $k_2/k_3=h(\xi)$, $\xi(s)=\int_0^s k_2(u)du$ *and* K *is constant.*

iv) $\frac{d}{d\theta}\left[\frac{d}{d\theta}\left(\frac{k_3}{k_2}\frac{d\rho}{d\theta}\right)+\frac{k_2}{k_3}\rho\right]-\frac{k_3}{k_2}\frac{d\rho}{d\theta}=0$, *where* $\theta(s)=\int_0^s k_3(u)du$.

Proof. Let us consider the system of linear differential equations in (3). According to (3), we can write

$$\rho\frac{d\rho}{ds}=k_2f\rho,\quad f\frac{df}{ds}=-k_2\rho f+k_3gf,\quad g\frac{dg}{ds}=k_3fg,$$

and thus we obtain the following differential equation $f\frac{df}{ds}+\rho\frac{d\rho}{ds}-g\frac{dg}{ds}=0$, which gives

$$f^2+\rho^2-g^2=C^2, \tag{4}$$

where C is an arbitrary constant. From Proposition 1 we have $\alpha=-\rho T_2-fT_3+gT_4$ and then $\langle\alpha,\alpha\rangle=f^2+\rho^2-g^2$. Using (4) it is obtained that $\langle\alpha,\alpha\rangle$ is constant. Then the spacelike curve $\alpha(s)$ lies on hyperbolic sphere H_0^3 with radius C. Eliminating f, g and their derivatives from system (3), we find the following linear differential equation of third order in ρ

$$\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{1}{k_2}\frac{d\rho}{ds}\right)+\frac{k_2}{k_3}\rho\right]-\frac{k_3}{k_2}\frac{d\rho}{ds}=0, \tag{5}$$

which represents the differential equation characterizing all spacelike curves lying on H_0^3 of radius C.

By using the transformation $\xi(s)=\int_0^s k_2(u)du$, $(s\in I)$ from (5) we obtain the following equation

$$\frac{d}{d\xi}\left[\frac{k_2}{k_3}\left(\frac{d^2\rho}{d\xi^2}+\rho\right)\right]-\frac{k_3}{k_2}\frac{d\rho}{d\xi}=0. \tag{6}$$

On the other hand, substituting $k_2/k_3=h(\xi)$ in equation (6), we obtain Bernoulli's equation with the unknown function $h(\xi)$ and from this equation we get the nonlinear differential equation

$$h^2\left(\frac{d^2\rho}{d\xi^2}+\rho\right)^2-\left(\frac{d\rho}{d\xi}\right)^2-\rho^2=K^2, \tag{7}$$

where K is arbitrary constant, $h=k_2/k_3$ and $\xi(s)=\int_0^s k_2(u)du$.

Also, changing the variable as $\theta(s)=\int_0^s k_3(u)du$, from the equation (5) we have

$$\frac{d}{d\theta}\left[\frac{d}{d\theta}\left(\frac{k_3}{k_2}\frac{d\rho}{d\theta}\right)+\frac{k_2}{k_3}\rho\right]-\frac{k_3}{k_2}\frac{d\rho}{d\theta}=0. \tag{8}$$

Each of equations (6), (7) and (8) is equivalent to equation (5). $\square$

Theorem 2. *Let $\alpha(s)\colon I\subset\mathbb{R}\to E_1^4$ be a unit speed regular spacelike curve with smooth non-zero curvature functions $k_1(s),k_2(s),k_3(s)$. Then $\alpha(s)$ is a H_0^3-spherical curve if and only if ρ,k_2 and k_3 satisfy the integral relation*

$$\rho^2+\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)^2-\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)^2=C^2,$$

where K_1,K_2,C are arbitrary constants. Moreover, a spacelike curve satisfying this condition lies on a hyperbolic sphere H_0^3 of radius C in E_1^4.

Proof. Consider the second and third equations of (3). Eliminating g and its derivatives from these equations, we have

$$k_3\frac{d^2f}{ds^2}-\frac{dk_3}{ds}\frac{df}{ds}+k_3^3f=-k_3\frac{d}{ds}\left(k_2\rho\right)+\frac{dk_3}{ds}k_2\rho. \tag{9}$$

If we change variables in the above equation as $\theta(s)=\int_0^s k_3(u)du$ which is assumed non-degenerate, we get

$$\frac{d^2f}{d\theta^2}+f=-\frac{d}{d\theta}\left(\frac{k_2\rho}{k_3}\right). \tag{10}$$

Following the same way for f, we have

$$k_3\frac{d^2g}{ds^2}-\frac{dk_3}{ds}\frac{dg}{ds}-k_3^3g=-k_3^2k_2\rho, \tag{11}$$

and making use of the transformation $\theta(s)=\int_0^s k_3(u)du$, it follows that

$$\frac{d^2g}{d\theta^2}-g=-\frac{k_2\rho}{k_3}. \tag{12}$$

Using the method of variation of parameters, the general solutions of linear differential equations (10) and (12) with constant coefficients are obtained as

$$\begin{cases} f(\theta)=\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)\sinh\theta+\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)\cosh\theta; \\ g(\theta)=\left(K_3-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)\sinh\theta+\left(K_4+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)\cosh\theta, \end{cases} \tag{13}$$

respectively, where K_1,K_2,K_3 and K_4 are arbitrary constants, and $\theta(s)=\int_0^s k_3(u)du$. Writing (13) into second or third equation of (3), it is seen that $K_3=K_2,K_4=K_1$. Thus we obtain the functions f and g satisfying the second and the third equation of (3), simultaneously as

$$\begin{cases} f(\theta)=\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)\sinh\theta+\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)\cosh\theta; \\ g(\theta)=\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)\cosh\theta+\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)\sinh\theta, \end{cases} \tag{14}$$

where K_1 and K_2 are arbitrary constants and $\theta(s)=\int_0^s k_3(u)du$.

Substitution (14) into the first equation of (3) we have

$$\frac{d\rho}{d\theta}=\frac{k_2}{k_3}\left[\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)\sinh\theta+\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)\cosh\theta\right]. \tag{15}$$

Then the triple (14) and (15) is equivalent to system (3). Furthermore, it is observed that when arbitrary constants K_1 and K_2 are eliminated from (15), equation (8) and therefore equation (5) are obtained.

On the other hand, it follows from (14) that the expression $-f^2+g^2$ can be written in the form

$$-f^2+g^2=\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)^2-\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)^2. \tag{16}$$

Substituting (16) into (4), we obtain the relation

$$\rho^2+\left(K_1+\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)^2-\left(K_2-\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)^2=C^2, \tag{17}$$

which satisfies the differential equation (5) characterizing H_0^3-spherical spacelike curves.

The converse is also true. If (17) is satisfied for a spacelike curve, it is easily seen that (17) satisfied the equation (5) which characterizes the H_0^3-spherical spacelike curves. For this purpose, arbitrary constants K_1, K_2 and C are eliminated from (17) and the differential equation corresponding to (17) is established. Thus, we can say that integral relation (17) is the implicit solution of differential equations (5) or (8) such that $\theta(s)=\int_0^s k_3(u)du$. □

4. Differential equations and integral characterizations of timelike and spacelike curves lying on Lorentzian sphere S_1^3. In this section, we give differential equations and integral characterizations of timelike and spacelike curves lying on Lorentzian sphere S_1^3 in the Minkowski space-time E_1^4. For this purpose, we use the following proposition given by M. Kazaz, H. H. Uğurlu and A. Özdemir ([3]).

Proposition 2 ([3])**.** *Let $\alpha(s)$ be a unit speed regular curve with Frenet frame $\{T_1,T_2,T_3,T_4\}$ and smooth non-zero curvature functions $k_1(s),k_2(s),k_3(s)$. Then the following assertions are equivalent.*

i) $\alpha(s)$ lies on a S_1^3 sphere.

ii) $k_1(s)\neq 0$ and there are two C^2-functions $f(s)$ and $g(s)$ such that
$\frac{d\rho}{ds}=k_2f,\ \frac{df}{ds}=-k_2\rho+k_3g,\ \frac{dg}{ds}=-k_3f$, if $\vec{\alpha}(s)$ is timelike,
$\frac{d\rho}{ds}=k_2f,\ \frac{df}{ds}=k_2\rho+k_3g,\ \frac{dg}{ds}=-k_3f$, if $\vec{\alpha}(s)$ is spacelike with timelike T_2,
$\frac{d\rho}{ds}=k_2f,\ \frac{df}{ds}=k_2\rho+k_3g,\ \frac{dg}{ds}=k_3f$, if $\vec{\alpha}(s)$ is spacelike with timelike T_3,
where $\rho=1/k_1$.

Then differential equations and integral characterizations of the S_1^3-spherical curves in E_1^4 are given as follows. Proofs of the following theorems can be given by a similar way of the proofs of Theorem 1 and Theorem 2.

Theorem 3. *Let $\alpha(s)$ be a unit speed regular curve with smooth non-zero curvature functions $k_1(s),k_2(s),k_3(s)$. Then, the condition for the curve $\alpha(s)$ to be a S_1^3-spherical curve is that $\rho(s)=1/k_1(s),k_2(s)$ and $k_3(s)$ satisfy one of the differential equations which are equivalent.*

Case 1. *$\alpha(s)$ is a timelike curve.*

i) $\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{1}{k_2}\frac{d\rho}{ds}\right)+\frac{k_2}{k_3}\rho\right]+\frac{k_3}{k_2}\frac{d\rho}{ds}=0.$

ii) $\frac{d}{d\xi}\left[\frac{k_2}{k_3}\left(\frac{d^2\rho}{d\xi^2}+\rho\right)\right]+\frac{k_3}{k_2}\frac{d\rho}{d\xi}=0,$ *where* $\xi(s)=\int_0^s k_2(u)du.$

iii) $-h^2\left(\frac{d^2\rho}{d\xi^2}+\rho\right)^2-\left(\frac{d\rho}{d\xi}\right)^2-\rho^2=K^2,$ *where* $k_2/k_3=h(\xi)$ *and* $\xi(s)=\int_0^s k_2(u)du.$

iv) $\frac{d}{d\theta}\left[\frac{d}{d\theta}\left(\frac{k_3}{k_2}\frac{d\rho}{d\theta}\right)+\frac{k_2}{k_3}\rho\right]+\frac{k_3}{k_2}\frac{d\rho}{d\theta}=0,$ *where* $\theta(s)=\int_0^s k_3(u)du.$

Case 2. *$\alpha(s)$ is a spacelike curve with timelike principal normal T_2.*

i) $\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{1}{k_2}\frac{d\rho}{ds}\right)-\frac{k_2}{k_3}\rho\right]+\frac{k_3}{k_2}\frac{d\rho}{ds}=0.$

ii) $\frac{d}{d\xi}\left[\frac{k_2}{k_3}\left(\frac{d^2\rho}{d\xi^2}-\rho\right)\right]+\frac{k_3}{k_2}\frac{d\rho}{d\xi}=0,$ *where* $\xi(s)=\int_0^s k_2(u)du.$

iii) $-h^2\left(\frac{d^2\rho}{d\xi^2}-\rho\right)^2-\left(\frac{d\rho}{d\xi}\right)^2+\rho^2=K^2,$ *where* $k_2/k_3=h(\xi)$ *and* $\xi(s)=\int_0^s k_2(u)du.$

iv) $\frac{d}{d\theta}\left[\frac{d}{d\theta}\left(\frac{k_3}{k_2}\frac{d\rho}{d\theta}\right)-\frac{k_2}{k_3}\rho\right]+\frac{k_3}{k_2}\frac{d\rho}{d\theta}=0,$ *where* $\theta(s)=\int_0^s k_3(u)du.$

Case 3. *$\alpha(s)$ is a spacelike curve with timelike first binormal T_3.*

i) $\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{1}{k_2}\frac{d\rho}{ds}\right)-\frac{k_2}{k_3}\rho\right]-\frac{k_3}{k_2}\frac{d\rho}{ds}=0.$

ii) $\frac{d}{d\xi}\left[\frac{k_2}{k_3}\left(\frac{d^2\rho}{d\xi^2}-\rho\right)\right]-\frac{k_3}{k_2}\frac{d\rho}{d\xi}=0,$ *where* $\xi(s)=\int_0^s k_2(u)du.$

iii) $h^2\left(\frac{d^2\rho}{d\xi^2}-\rho\right)^2-\left(\frac{d\rho}{d\xi}\right)^2+\rho^2=K^2,$ *where* $k_2/k_3=h(\xi)$ *and* $\xi(s)=\int_0^s k_2(u)du.$

iv) $\frac{d}{d\theta}\left[\frac{d}{d\theta}\left(\frac{k_3}{k_2}\frac{d\rho}{d\theta}\right)-\frac{k_2}{k_3}\rho\right]-\frac{k_3}{k_2}\frac{d\rho}{d\theta}=0,$ *where* $\theta(s)=\int_0^s k_3(u)du.$

Theorem 4. *Let $\alpha(s)$ be a unit speed regular curve with smooth non-zero curvature functions $k_1(s),k_2(s),k_3(s)$. Then $\alpha(s)$ is a spherical curve lying on S_1^3 if and only if ρ,k_2 and k_3 satisfy the following integral relation.*

Case 1. *$\alpha(s)$ is a timelike curve.*

$$\rho^2+\left(K_1-\int\frac{k_2}{k_3}\rho\sin\theta d\theta\right)^2+\left(K_2-\int\frac{k_2}{k_3}\rho\cos\theta d\theta\right)^2=C^2.$$

Case 2. *$\alpha(s)$ is a spacelike curve with timelike principal normal T_2.*

$$-\rho^2+\left(K_1+\int\frac{k_2}{k_3}\rho\sin\theta d\theta\right)^2+\left(K_2+\int\frac{k_2}{k_3}\rho\cos\theta d\theta\right)^2=C^2.$$

Case 3. *$\alpha(s)$ is a spacelike curve with timelike first binormal T_3.*

$$-\rho^2-\left(K_1-\int\frac{k_2}{k_3}\rho\sinh\theta d\theta\right)^2+\left(K_2+\int\frac{k_2}{k_3}\rho\cosh\theta d\theta\right)^2=C^2,$$

where K_1,K_2,C are arbitrary constants. Moreover, a curve satisfying this condition lies on a Lorentzian sphere S_1^3 of radius C.

5. Conclusion. Differential equations characterizing space curves play an important role in the curve theory. In this paper, some differential equations characterizing Lorentzian spherical and hyperbolic spherical curves are given in the Minkowski space-time E_1^4. According to the casual characters of the curves and their Frenet vectors, different conditions are found for timelike and spacelike curves to be spherical curves in E_1^4. Furthermore, integral characterizations of timelike and spacelike spherical curves are given in E_1^4.

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