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ON A THEOREM OF JOHN AND ITS GENERALIZATIONS

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The purpose of this paper is to consider some generalizations of the class of functions having zero integrals over balls of a fixed radius. We obtain an analog of John's uniqueness theorem for weighted spherical means on sphere.

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Цель данной статьи состоит в том, чтобы рассмотреть некоторые обобщения класса функций, имеющих нулевые интегралы по шарам фиксированного радиуса. Получено аналог теоремы единственности Йона для взвешенных сферических средних на сфере.

1. Introduction. Let $\mathbb{R}^n$ be the real Euclidean space of dimension $n \geq 2$ with the Euclidean norm $|\cdot|$, and let $\mathcal{B}_R = \{x \in \mathbb{R}^n \colon |x| < R\},\ R > 0$. The classical theorem of John (see [1], [2, Ch. 6]) asserts that if a function $f \in C^\infty(\mathcal{B}_R)$ is such that its integral over each sphere of fixed radius r contained in $\mathcal{B}_R$ is zero and $f = 0$ in $\mathcal{B}_r$, then $f = 0$ in $\mathcal{B}_R$. An analogous statement holds also for the class of functions with zero mean over balls. It was shown in [3], [4] that the smoothness of the function f in the condition of the above theorem cannot be weakened (see also [1], [2], where the cases $n = 2$ and 3 were considered).

The results in [3] essentially sharpen and generalize the theorem of John in various directions. For example, the relationship between the order of smoothness of functions of the given class and the set of nonzero coefficients in their Fourier expansions with respect to spherical harmonics was discussed in [3]. A further development of tools offered in [3] made it possible to obtain similar results on symmetric spaces (see [5]).

Uniqueness theorems for the indicated classes of functions play a key role in the solution of a number of problems related to spherical means. We enumerate the most remarkable results obtained by means of their use (see [4]): the final version of the local two-radii theorem, the solution of the support problem for certain function classes, new two-radii theorems in the theory of harmonic functions, the extreme variants of the Pompeiu problem, uniqueness theorems for multiple lacunary trigonometric series, and others. These results were the concluding stages in the series of investigations started by John, Delsarte more half a century ago, and continued by Zalcman, Berenstein and others (see the monographs [3], [5] with extensive bibliographies).

In this paper we obtain an analog of John's uniqueness theorem for weighted spherical means on two-dimensional sphere. A similar result on an Euclidean space was established

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before by V. V. Volchkov in [6]. However, the methods in [6] use the vector structure of $\mathbb{R}^n$ and do not work for spaces with nonzero curvature. Also we note that the case under consideration cannot be investigated by means of the general theory of transmutation operators (see [5]) which is a powerful tool for study of convolution equations with radial distributions on various homogeneous spaces.

2. Statement of the main result. Let $\mathbb{S}^2$ be the standard unit sphere in $\mathbb{R}^3$ with the inner metric d and area measure $d\xi$, $B_R = \{\xi \in \mathbb{S}^2 \colon d(o,\xi) < R\}$ be the open geodesic ball (spherical cap) of radius R with center at the point $o = (0,0,1) \in \mathbb{S}^2$. In what follows we consider that r be a fixed number belonging to the interval $(0;\pi)$ and $r < R$. Denote by $\overline{B}_r$ the closure of B_r, $S_r = \{\xi \in \mathbb{S}^2 \colon d(o,\xi) = r\}$.

Let $SO(3)$ be the group of rotations in $\mathbb{R}^3$. As usual we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_+$ the sets of positive integers, integers, and non-negative integers numbers, respectively. For fixed $M \in \mathbb{Z}_+$ we put

$$U_{r,M}(B_R) = \Big\{ f \in C(B_R) \colon \int_{S_r} f(\tau\xi)(\xi_1 + i\xi_2)^M dl(\xi) = 0 \ \forall \tau \in SO(3) \colon \tau\overline{B}_r \subset B_R \Big\},$$

where ξ_1, ξ_2, ξ_3 are the Cartesian coordinates of a point $\xi \in \mathbb{S}^2$ and $dl(\xi)$ is the length element on $\mathbb{S}^2$. For $s \in \mathbb{Z}_+ \cup \{\infty\}$ we define $U^s_{r,M}(B_R) = U_{r,M}(B_R) \cap C^s(B_R)$.

For $M = 0$ the class $U_{r,M}(B_R)$ coincides with the class of functions $f \in C(B_R)$ satisfying in B_{R-r} the equation $f * \sigma_r = 0$, where σ_r is the delta function concentrated on S_r, and "$*$" designates the convolution on a sphere. This case and its analogues for various homogeneous spaces were investigated by many authors (see [5, part 3]). If $M > 0$ then the equation

$$\int_{S_r} f(\tau\xi)(\xi_1 + i\xi_2)^M dl(\xi) = 0$$

is not reduced to the convolution equation with a radial distribution that makes impossible application of the general theory from [5].

The main result of the present paper is

Theorem 1. *Let $f \in U^\infty_{r,M}(B_R)$ and $f = 0$ in B_r. Then $f = 0$ in B_R.*

The proof of Theorem 1 is based on methods of harmonic analysis and integral equations, and also uses some important results from the theory of special functions. We note that Theorem 1 becomes invalid for the class $U^s_{r,M}(B_R)$, $s \in \mathbb{Z}_+$. In addition, the radius of the zero set of function f cannot be decreased in general.

3. Auxiliary statements. Let $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$, φ, θ be the spherical coordinates of a point ξ ($0 \le \varphi \le 2\pi$, $0 \le \theta \le \pi$ and $\xi_1 = \sin\theta\sin\varphi$, $\xi_2 = \sin\theta\cos\varphi$, $\xi_3 = \cos\theta$). We associate with each function $f \in C(B_R)$ the Fourier series

$$f(\xi) \sim \sum_{k=-\infty}^{\infty} f_k(\theta) e^{ik\varphi}, \quad \theta \in (0,R), \tag{1}$$

where

$$f_k(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f^o(\varphi,\theta) e^{-ik\varphi} d\varphi, \quad f^o(\varphi,\theta) = f(\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta).$$

Lemma 1. *Let $f \in U^s_{r,M}(B_R)$. Then $f_k(\theta)e^{ik\varphi} \in U^s_{r,M}(B_R)$ for any $k \in \mathbb{Z}$.*

Proof. For brevity we set $f^k(\xi) = f_k(\theta)e^{ik\varphi}$. Denote by g_α the rotation of $\mathbb{R}^3$ through the angle α in the plane (x_1, x_2). It follows from (1) that

$$f^k(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(g_\alpha\xi)e^{ik\alpha}d\alpha. \tag{2}$$

In particular, $f^k \in C^s(B_R)$. Next, let $\tau \in SO(3)$ and $\tau\overline{B}_r \subset B_R$. According to (2) one has

$$\int_{S_r} f^k(\tau\xi)(\xi_1 + i\xi_2)^M dl(\xi) = \frac{1}{2\pi}\int_0^{2\pi}\int_{S_r} f(g_\alpha\tau\xi)(\xi_1 + i\xi_2)^M dl(\xi)e^{ik\alpha}d\alpha.$$

Hence, taking into account that $g_\alpha\tau\overline{B}_r \subset B_R$ for any $\alpha \in [0, 2\pi]$ we obtain the desired assertion. □

Lemma 2. *Let $s \in \mathbb{N}$ and assume that $f \in U^s_{r,M}(B_R)$. Then*

$$\cos\varphi\frac{\partial f^o}{\partial\theta} - \sin\varphi\,\mathrm{ctg}\,\theta\frac{\partial f^o}{\partial\varphi} \in U^{s-1}_{r,M}(B_R).$$

Proof. Let $\tau \in SO(3)$ and $\tau\overline{B}_r \subset B_R$. Denote by a_t the rotation of $\mathbb{R}^3$ through the angle $(-t)$ in the plane (x_2, x_3). If $|t|$ is sufficiently small then

$$\int_{\tau S_r} F(a_t\xi)\mathcal{P}_M(\tau^{-1}\xi)dl(\xi) = 0, \tag{3}$$

where $F(x) = f(x/|x|)$, $\mathcal{P}_M(\xi) = (\xi_1 + i\xi_2)^M$. Differentiating (3) with respect to t and putting $t = 0$ one finds

$$\int_{\tau S_r} h(\xi)\mathcal{P}_M(\tau^{-1}\xi)dl(\xi) = 0, \text{ where } h(\xi) = \xi_3\frac{\partial F}{\partial x_2}(\xi) - \xi_2\frac{\partial F}{\partial x_3}(\xi).$$

This finishes the proof since

$$h^o(\varphi, \theta) = \cos\varphi\frac{\partial f^o}{\partial\theta} - \sin\varphi\,\mathrm{ctg}\,\theta\frac{\partial f^o}{\partial\varphi}.$$ □

Lemma 3. *Let $s \in \mathbb{N}$ and $u(\theta)e^{ik\varphi} \in U^s_{r,M}(B_R)$ for some $k \in \mathbb{Z}$. Then the functions $(u'(\theta) - k\,\mathrm{ctg}\,\theta u(\theta))e^{i(k+1)\varphi}$ and $(u'(\theta) + k\,\mathrm{ctg}\,\theta u(\theta))e^{i(k-1)\varphi}$ belong to $U^{s-1}_{r,M}(B_R)$.*

Proof. Setting $f(\xi) = u(\theta)e^{ik\varphi}$ we find

$$2\left(\cos\varphi\frac{\partial f^o}{\partial\theta} - \sin\varphi\,\mathrm{ctg}\,\theta\frac{\partial f^o}{\partial\varphi}\right) = (u'(\theta) - k\,\mathrm{ctg}\,\theta u(\theta))e^{i(k+1)\varphi} + (u'(\theta) + k\,\mathrm{ctg}\,\theta u(\theta))e^{i(k-1)\varphi}.$$

Now Lemma 3 follows from Lemmas 1 and 2. □

For the further references we present now some properties of the Legendre functions P_ν^μ (see [7, Ch. 3, §3, i.3.4, formula (6)]). If $\nu = l \in \mathbb{Z}_+$, $\mu = m \in \mathbb{Z}$ then (see [8, Ch. 3, §3, i.9, formula (11)])

$$P_l^m(x) = \left(\frac{1+x}{1-x}\right)^{\frac{m}{2}} \sum_{\max(m,0)\leq j\leq l} \frac{(-1)^j (l+j)!}{(l-j)!(j-m)!j!} \left(\frac{1-x}{2}\right)^j. \tag{4}$$

As usual the sum in (4) is equal to zero if the set of indexes of summation is empty.

Legendre functions are closely related to the Gegenbauer polynomials $C_l^p(x)$ and the Chebyshev polynomials $T_l(x)$ (see [8, Ch. 9, §4, i.8, formulas (6′), (11′)]). We note the following differentiation formulas

$$\frac{d}{dx}\left(C_l^1(x)(1-x^2)^{\frac{1}{2}}\right) = -(l+1)T_{l+1}(x)(1-x^2)^{-\frac{1}{2}}, \tag{5}$$

$$\frac{d}{dx}\left(C_l^{\frac{n}{2}}(x)(1-x^2)^{\frac{n-1}{2}}\right) = \frac{(l+1)(n+l-1)}{2-n} C_{l+1}^{\frac{n-2}{2}}(x)(1-x^2)^{\frac{n-3}{2}}, \quad n \in \{3,4,\ldots\}. \tag{6}$$

These relations follows from [8, Ch. 9, §3, i.2, formulas (4), (5) and §4, i.8, formula (11′)].

Assume that numbers θ_1, θ_2 and $\theta_1 + \theta_2$ belong to the interval $[0;\pi)$. The multiplication formula

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\varphi} P_l(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi)d\varphi = P_l^k(\cos\theta_1)P_l^{-k}(\cos\theta_2), \tag{7}$$

holds, where $P_l = P_l^0$ (see [8, Ch. 2, §4, i.3, formula (2)]). Relation (7) can be written in the form

$$\frac{1}{\pi}\int_{|\theta_1-\theta_2|}^{\theta_1+\theta_2} P_l(\cos\theta) T_k\left(\frac{\cos\theta_1\cos\theta_2 - \cos\theta}{\sin\theta_1\sin\theta_2}\right) \times$$
$$\times \frac{\sin\theta d\theta}{\sqrt{(\cos\theta - \cos(\theta_1+\theta_2))(\cos(\theta_1-\theta_2) - \cos\theta)}} = P_l^k(\cos\theta_1)P_l^{-k}(\cos\theta_2) \tag{8}$$

(see [8, Ch. 3, §4, i.3, formula (7)]).

Next, for brevity we set

$$a = a(\theta,t,r) = \frac{\cos\theta - \cos r\cos t}{\sin r\sin t}, \quad b = b(\theta,t,r) = (\cos\theta - \cos(r+t))(\cos(t-r) - \cos\theta).$$

Lemma 4. *Let $0 < r < R < \pi$, $k \in \mathbb{N}$. Assume that $\Phi \in C^{k+1}[0,R]$, $\Phi = 0$ on $[0,r]$ and*

$$\int_{|t-r|}^{t+r} \Phi(\theta)\sin\theta T_k(a) b^{-\frac{1}{2}} d\theta = 0 \tag{9}$$

for $0 < t < R - r$. Then $\Phi = 0$ on $[0,R]$.

Proof. First of all we note that

$$1 - a^2 = \frac{b}{(\sin r\sin t)^2}, \tag{10}$$

$$\frac{db}{d\theta} = 2\sin\theta(\cos\theta - \cos r\cos t). \tag{11}$$

Define the functions $\mathfrak{X}_{m,n}$ ($m \in \mathbb{Z}_+$, $n \in \{2, 3, \dots\}$) by the equality

$$\mathfrak{X}_{m,n}(\theta) = \begin{cases} \sin\theta T_m(a) b^{-\frac{1}{2}}, & n = 2; \\ \sin\theta C_m^{\frac{n-2}{2}}(a) b^{\frac{n-3}{2}}, & n \geq 3. \end{cases}$$

Relations (10), (11) and (5), (6) give the following differentiation formulas

$$\frac{d}{d\theta}\left(\frac{\mathfrak{X}_{m-1,4}(\theta)}{\sin\theta}\right) = m \sin r \sin t \mathfrak{X}_{m,2}(\theta), \tag{12}$$

$$\frac{d}{d\theta}\left(\frac{\mathfrak{X}_{m-1,n+2}(\theta)}{\sin\theta}\right) = \frac{m(m+n-2)}{n-2} \sin r \sin t \mathfrak{X}_{m,n}(\theta), \quad n \geq 3. \tag{13}$$

Integrate (9) by parts k times with use (12) and (13). As a result we have

$$\int_{|t-r|}^{t+r} (\mathrm{D}^k \Phi)(\theta) \sin\theta b^{k-\frac{1}{2}} d\theta = 0, \quad 0 < t < R - r, \tag{14}$$

where $\mathrm{D} = \frac{1}{\sin\theta}\frac{d}{d\theta}$. To study equation (14) first we consider the case where $R \leq 2r$. Then $0 < t < R - r \leq r$ and $|t - r| = r - t < r$. Since $\Phi = 0$ on $[0, r]$, from this and (14) it follows that

$$\int_{r}^{t+r} (\mathrm{D}^k \Phi)(\theta) \sin\theta b^{k-\frac{1}{2}} d\theta = 0, \quad 0 < t < R - r. \tag{15}$$

We rewrite (15) in the form

$$\int_{\cos t}^{\cos r} h_1(x) \left((x - \cos t)(\cos(t - 2r) - x)\right)^{k-\frac{1}{2}} dx = 0, \quad r \leq t < R, \tag{16}$$

where $h_1(x) = \left(\mathrm{D}^k \Phi\right)(\arccos x)$. It follows from (16) that

$$\int_{r}^{t} h_2(x) \left(\sin\frac{x+t}{2} \sin\frac{t-x-2r}{2} (\cos(t-r) - \cos(x-r))\right)^{k-\frac{1}{2}} dx = 0, \quad r \leq t < R,$$

where $h_2(x) = h_1(\cos x)\sin x$. Hence

$$\int_{t}^{1} h_3(x)(x-t)^{k-\frac{1}{2}} g_1(x,t) dx = 0, \quad \cos(R - r) < t \leq 1, \tag{17}$$

where

$$h_3(x) = \frac{h_2(r + \arccos x)}{\sqrt{1 - x^2}}, \quad g_1(x,t) = \left(t + \sqrt{1-x^2}\sin 2r - x\cos 2r\right)^{k-\frac{1}{2}}.$$

Let $\cos(R - r) < y \leq 1$. We multiply (17) by $(t - y)^{k-\frac{1}{2}}$ and integrate with respect to t from y to 1. Changing the order of integration, we obtain

$$\int_{y}^{1} h_3(x) \int_{y}^{x} \left((x-t)(t-y)\right)^{k-\frac{1}{2}} g_1(x,t) dt dx = 0, \quad \cos(R - r) < y \leq 1.$$

The substitution $(x - y)z = x + y - 2t$ in the inner integral yields

$$\int_{y}^{1} h_3(x)(x-y)^{2k} g_2(x,y) dx = 0, \quad \cos(R - r) < y \leq 1, \tag{18}$$

where

$$g_2(x,y)=\int_{-1}^{1}(1-z^2)^{k-\frac{1}{2}}g_1\left(x,\frac{x+y-(x-y)z}{2}\right)dz.$$

Differentiating $2k+1$ times with respect to y, in (18), we have

$$h_3(y)-\int_y^1 h_3(x)\mathcal{K}(x,y)dx=0,\quad \cos(R-r)<y\leq 1,$$

where

$$\mathcal{K}(x,y)=\frac{\frac{\partial^{2k+1}}{\partial y^{2k+1}}\left((x-y)^{2k}g_2(x,y)\right)}{(2k)!g_2(y,y)}.$$

Thus, the function h_3 is a solution of the homogeneous integral Volterra equation of the second kind with the bounded kernel $\mathcal{K}(x,y)$. This means that $h_3=0$ on $(\cos(R-r),1)$. Bearing in mind that $\Phi=0$ on $[0,r]$ we complete the proof in the case $R\leq 2r$.

Next, suppose that the statement of Lemma 4 is valid for a radius $R\leq mr$, where $m\geq 2$ is a fixed positive integer. We prove it for $R\in(mr,(m+1)r]$. By the induction hypothesis $\Phi=0$ on $[0,mr]$. As $t+r<mr$ for $t<(m-1)r$, and $|t-r|<(m-1)r$ for $(m-1)r<t<R-r$ from (14) equation (15) again follows. As above we conclude that $\Phi=0$ on $[0,R]$. ☐

4. Proof of Theorem 1. First of all we establish the following statement.

Lemma 5. *Let $f(\xi)=u(\theta)\in U^{\infty}_{r,M}(B_R)$ and $f=0$ in B_r. Then $f=0$ in B_R.*

Proof. Without loss of generality we may and do assume that $R\leq\pi$. Let $0<\varepsilon<R-r$. We consider a function w_ε satisfying the following conditions: 1) $\mathrm{w}_\varepsilon\in C^\infty[0,\pi]$; 2) $\mathrm{w}_\varepsilon=1$ on $[0,R-\varepsilon]$ and $\mathrm{w}_\varepsilon=0$ on $[R-\varepsilon/2,\pi]$. For $\theta\in[0,\pi]$ we set $\Phi(\theta)=u(\theta)\mathrm{w}_\varepsilon(\theta)$, where $u=0$ on $[R,\pi]$. Then $\Phi\in C^\infty[0,\pi]$ and

$$\Phi(\theta)=\sum_{l=0}^{\infty}\alpha_l P_l(\cos\theta),\tag{19}$$

where

$$\alpha_l=\frac{2l+1}{2}\int_0^\pi\Phi(\theta)P_l(\cos\theta)\sin\theta d\theta$$

(see [8, Ch. 3, §6, i.4, fomulas (21), (22)]). In addition, $\alpha_l=O(l^{-c})$ as $l\to+\infty$ for any fixed $c>0$. In what follows we use the map a_t (defined in the proof of Lemma 2), where $0<t<R-r-\varepsilon$. By the hypothesis we have

$$\int_{S_r}F(a_t\xi)(\xi_1+i\xi_2)^M dl(\xi)=0,\tag{20}$$

where $F(\xi)=\Phi(\arccos\xi_3)$. Expansion (19) shows that

$$F(a_t\xi)=\sum_{l=0}^{\infty}\alpha_l P_l(\xi_3\cos t-\xi_2\sin t).$$

Therefore (20) can be written in the form

$$\sum_{l=0}^{\infty}\alpha_l\int_0^{2\pi}P_l(\cos r\cos t-\sin r\sin t\cos\varphi)e^{-iM\varphi}d\varphi=0.$$

Hence by formula (7) we obtain

$$\sum_{l=0}^{\infty} \alpha_l P_l^{-M}(\cos r) P_l^M(\cos t) = 0. \tag{21}$$

Since $T_k(-x) = (-1)^k T_k(x)$, from (21), (8) and (9) we have the equation

$$\int_{|t-r|}^{t+r} \Phi(\theta) \sin\theta T_M(a(\theta,t,r))(b(\theta,t,r))^{\frac{-1}{2}} d\theta = 0, \quad 0 < t < R-r-\varepsilon.$$

From this and Lemma 4 we derive that $\Phi = 0$ on $[0, R-\varepsilon]$. In view of arbitrariness of $\varepsilon \in (0, R-r)$, $f = 0$ in B_R. □

We proceed to the proof of Theorem 1. It follows from the hypothesis, Lemma 1 and formula (2) that $f^k \in U^\infty_{r,M}(B_R)$ and $f^k = 0$ in B_r for any $k \in \mathbb{Z}$.

We prove by induction on k that $f^k = 0$ in B_R. If $k = 0$ then the statement follows from Lemma 5. Assume that the statement is valid for some $k \in \mathbb{Z}$ and establish it for $k+1$ and $k-1$. Using Lemma 3 we infer that

$$(\sin\theta)^{-k-1} \frac{d}{d\theta} \left((\sin\theta)^{k+1} f_{k+1}(\theta)\right) e^{ik\varphi} \in U^\infty_{r,M}(B_R),$$
$$(\sin\theta)^{k-1} \frac{d}{d\theta} \left((\sin\theta)^{1-k} f_{k-1}(\theta)\right) e^{ik\varphi} \in U^\infty_{r,M}(B_R).$$

Hence, keeping in mind that f^{k+1} and f^{k-1} are equal to zero in B_r we conclude that these functions are vanishing on B_R. Thus, $f^k = 0$ in B_R for all k and $f = 0$ in B_R.

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