
We present a unified approach to define dimension functions like $trind$, $trind_p$, $trt$ and $P$. We show how some similar facts on these functions can be proved similarly. Moreover, several new classes of infinite-dimensional spaces close to the classes of countable-dimensional and $\sigma$-hereditarily disconnected ones are introduced. We prove a compactification theorem for these classes.

1. Introduction. In [13] G. Steinke suggested and studied an integer valued inductive topological invariant, the separation dimension $t$. Recall that the separation dimension $t$ for a topological space $X$ is defined inductively as follows: $t X = -1$ if and only if $X = \emptyset$; $t X = 0$ if $|X| = 1$; let $|X| > 1$ and $n$ be an integer $\geq 0$, if for each subset $M$ of $X$ with $|M| > 1$ there exist distinct points $x, y$ of $M$ and a partition $L_M$ in the subspace $M$ of $X$ between $x$ and $y$ such that $t L_M \leq n - 1$ then we write $t X \leq n$. One of the main property of $t$ is the following. If $\{X_i: i \in I\}$ is the family of all connected components of a non-empty space $X$ then $t X = \sup\{t X_i: i \in I\}$. In particular, for any space $X$ we have $t X = 0$ if and only if $X$ is hereditarily disconnected.

Recall ([6]) that, the classes of strongly countable-dimensional metrizable compacta, countable-dimensional metrizable compacta and compact metrizable C-spaces are classical objects of infinite dimension theory. In [1] F. G. Arenas, V. A. Chatyrko and M. L. Puertas considered a natural transfinite extension of $t$, the topological invariant $trt$, and showed that each metrizable compact space $X$ with $trt X \neq \infty$ must be a C-space. Moreover, every strongly countable-dimensional metrizable compact space $X$ has $trt X \leq \omega_0$. However, there exist countable-dimensional metrizable compact spaces (namely, the infinite-dimensional Cantor manifolds) of dimension $trt > \omega_0$. Since the inequality $trt X \leq trind X$, where
trind is the small transfinite inductive dimension ([6]), holds for each $T_3$-space $X$, every countable-dimensional metrizable compact space $X$ satisfies $\text{trt } X < \omega_1$. Set

$$\alpha_0 = \sup \{ \text{trt } K : K \text{ is a countable-dimensional metrizable compact space} \}.$$ 

It is clear that $\alpha_0 \leq \omega_1$ but the exact value of $\alpha_0$ is still unknown.

In [10] T. M. Radul introduced an ordinal valued topological invariant, the dimension $\mathcal{P}$, by modifying the definition of trt: the subsets $M$ of the space $X$ are supposed to be compact. It is easy to see that for any space $X$ we have $\mathcal{P} X = \sup \{ \text{trt } K : K \text{ is a compact subset of } X \} \leq \text{trt } X$. In [10] T. M. Radul proved that each $\sigma$-hereditarily disconnected hereditarily normal space $X$ satisfies $\mathcal{P} X \neq \infty$. Recall (see [7] or [2]) that a space $X$ is $\sigma$-\textit{hereditarily disconnected} if $X$ is a countable union of hereditarily disconnected subspaces. Since each zero-dimensional space in the sense of the small inductive dimension ind is hereditarily disconnected, each countable-dimensional in the sense of ind space is $\sigma$-hereditarily disconnected. Let us observe that for the subspace $K^{\omega_0}$ of the Hilbert cube $I^{\omega_0}$ consisting of points with finitely many non-zero coordinates (and so being strongly countable dimensional) we have $\mathcal{P} K^{\omega_0} = \omega_0$. Recall that $\text{trt } K^{\omega_0} > \omega_0$ ([1]) but we do not know whether $\text{trt } K^{\omega_0} \neq \infty$.

It is still unclear if each metrizable compact space $X$ with $\text{trt } X \neq \infty$ has to be $\sigma$-hereditarily disconnected. The well known R. Pol’s example $P$ ([9]) of a weakly infinite-dimensional uncountable-dimensional metrizable compact space is a $\sigma$-hereditarily disconnected C-space, and hence by Radul’s result $\text{trt } P \neq \infty$. (In fact, $P$ can be constructed so that $\text{trt } P = \omega_0$, see a remark in [1].) But it is unknown whether every compact metrizable C-space $X$ is $\sigma$-hereditarily disconnected (resp. has $\text{trt } X \neq \infty$).

In this paper we show that the dimension trind (as well as the transfinite inductive invariant trind$_\mathcal{P}$ from ([3])) can also be defined similarly to the definition of trt. One of the subjects of the paper is to unify proofs of some facts about the invariants trind, trind$_\mathcal{P}$, trt, $\mathcal{P}$ and introduce new classes of infinite-dimensional spaces close to the classes of countable-dimensional and $\sigma$-hereditarily disconnected ones. We prove a compactification theorem for these new classes. In particular, we show that, for any hereditarily disconnected separable completely metrizable space $X$ there is a metrizable compactification $Y$ of $X$ such that $\text{trt } Y \leq \omega_0 + 1$. Furthermore, for Renska’s examples (see [11] (resp. [12])) of $\alpha$-dimensional metrizable Cantor trind (resp. trInd)-manifolds, where $\alpha$ is any isolated countable ordinal, we have the values of trt are equal to $\omega_0 + 1$.

Our terminology follows [5] and [6].

2. Definitions and common properties. All considered topological spaces are assumed to be $T_3$-spaces. Let us fix for each space $X$ a class $\mathcal{A}_X$ of subsets of $X$. The family of all classes $\mathcal{A}_X$ we denote by $\mathcal{A}$ and call it a family of classes of subsets of spaces (in short, an SSC-family).

Definition 1. Let $X$ be a space and $\mathcal{A}$ be an SSC-family.

(a) The small inductive invariant $\mathcal{A}(0)$-ind of the space $X$, denoted by $\mathcal{A}(0)$-ind$(X)$ is defined inductively as follows. $\mathcal{A}(0)$-ind$(X) = -1$ if and only if $X = \emptyset$; $\mathcal{A}(0)$-ind$(X)$ = 0 if $|X| = 1$. Let $|X| > 1$ and $n$ be an integer $\geq 0$, if for each element $M$ of $\mathcal{A}_X$ with $|M| > 1$ and for every pair $(A, x)$, where $A$ is a closed subset of $M$ and $x \in M - A$, there is a partition $L_M$ in the space $M$ between $x$ and $A$ such that $\mathcal{A}(0)$-ind$L_M \leq n - 1$ then we write $\mathcal{A}(0)$-ind $X \leq n$. 

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Remark 1.

Proposition 1. (a) The small inductive invariant $A(1)$-ind of the space $X$, denoted by $A(1)$-ind$(X)$ is defined inductively as follows. $A(1)$-ind$(X) = -1$ if and only if $X = \emptyset$; $A(1)$-ind$(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and $n$ be an integer $\geq 0$, if for each element $M$ of $A_X$ with $|M| > 1$ and for every pair $(x, y)$ of distinct points of $M$ there is a partition $L_M$ in the space $M$ between $x$ and $y$ such that $A(1)$-ind$L_M \leq n - 1$ then we write $A(1)$-ind$X \leq n$.

(b) The small inductive invariant $A(2)$-ind of the space $X$, denoted by $A(2)$-ind$(X)$ is defined inductively as follows. $A(2)$-ind$(X) = -1$ if and only if $X = \emptyset$; $A(2)$-ind$(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and $n$ be an integer $\geq 0$, if for each element $M$ of $A_X$ with $|M| > 1$ there exists a point $x \in M$ possessing the following property: for every closed subset $A$ of the space $M$ with $x \notin A$ there is a partition $L_M$ in the space $M$ between $x$ and $A$ such that $A(2)$-ind$L_M \leq n - 1$ then we write $A(2)$-ind$X \leq n$.

(c) The small inductive invariant $A(3)$-ind of the space $X$, denoted by $A(3)$-ind$(X)$ is defined inductively as follows. $A(3)$-ind$(X) = -1$ if and only if $X = \emptyset$; $A(3)$-ind$(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and $n$ be an integer $\geq 0$, if for each element $M$ of $A_X$ with $|M| > 1$ there exists a proper closed subset $A$ of the space $M$ possessing the following property: for every point $x \in M - A$ there is a partition $L_M$ in the space $M$ between $x$ and $A$ such that $A(3)$-ind$L_M \leq n - 1$ then we write $A(3)$-ind$X \leq n$.

(d) The small inductive invariant $A(4)$-ind of the space $X$, denoted by $A(4)$-ind$(X)$ is defined inductively as follows. $A(4)$-ind$(X) = -1$ if and only if $X = \emptyset$; $A(4)$-ind$(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and $n$ be an integer $\geq 0$, if for each element $M$ of $A_X$ with $|M| > 1$ there exist distinct points $x, y$ of $M$ and a partition $L_M$ in the space $M$ between $x$ and $y$ such that $A(4)$-ind$L_M \leq n - 1$ then we write $A(4)$-ind$X \leq n$.

The transfinite extension $A(i)$-trind of the invariant $A(i)$-trind is defined in the standard fashion, $i \in \{0, 1, 2, 3, 4\}$.

Let us introduce SSC-families $A^j$, $j \in \{1, 2, 3, 4\}$, as follows: for every space $X$ put $A_X^1 = \{X\}$, $A_X^2 = 2^X$, $A_X^3 = 2^X_{\text{comp}}$, $A_X^4 = 2^X_{\text{cl}}$; where $2^X$ (resp. $2^X_{\text{comp}}$ or $2^X_{\text{cl}}$) is the family of all (resp. compact or closed) subsets of $X$. Note that one can suggest many other SSC-families $A$ different from $A^j$, $j \in \{1, 2, 3, 4\}$.

Remark 1. Note that

(a) $A^1(0)$-trind$X \leq \text{trind}X$ and $A^1(1)$-trind$X = \text{trind}_pX$ ([3]);

(b) $A^2(4)$-trind$X = A^4(4)$-trind$X = \text{trt}X$ ([1] or Corollary 2);

(c) $A^3(4)$-trind$X = \text{p}X$ ([10]).

The following statement is evidently valid for every SSC-family $A$ and every space $X$.

Proposition 1. (a) $A(0)$-trind$X \geq A(1)$-trind$X \geq A(3)$-trind$X \geq A(4)$-trind$X$ and $A(0)$-trind$X \geq A(2)$-trind$X \geq A(3)$-trind$X$.

(b) $A(i)$-trind$X \leq \text{sup}\{A(i)$-trind$A : A \in A_X\}$ for every $i \in \{0, 1, 2, 3, 4\}$, whenever $A \in A_0$ for each $A \in A_X$ with $|A| > 1$.

(Note that the SSC-families $A^j$, $j \in \{1, 2, 3, 4\}$, satisfy this condition.)

Definition 2. We will say that an SSC-family $A$ possesses property $(*)_1$ (resp. $(*)_2$ or $(*)_3$) if for every space $X$ and every subspace $Y$ of $X$ the following assertions hold:

$(*)_1$: for each $A_Y \in A_Y$ there exists an element $A_X \in A_X$ such that $A_Y \subset A_X$;
((*)_2): for each \( A_Y \in \mathcal{A}_Y \) we have \( \text{Cl}_X(A_Y) \in \mathcal{A}_X \), or \((*)_3): \mathcal{A}_Y \subset \mathcal{A}_X \).

Note that if an SSC-family \( \mathcal{A} \) possesses property \((*)_2\) or property \((*)_3\) then it possesses also property \((*)_1\).

**Remark 2.** The families \( \mathcal{A}^j, j \in \{1, 2, 3, 4\} \), possess property \((*)_1\). The families \( \mathcal{A}^j, j \in \{2, 3, 4\} \), possess property \((*)_2\). The families \( \mathcal{A}^j, j \in \{2, 3\} \), possess property \((*)_3\).

**Proposition 2.** Let an SSC-family \( \mathcal{A} \) possess the property \((*)_1\) (resp. \((*)_2\) or \((*)_3\)). Then for every space \( X \) and every subspace \( Y \) of \( X \) we have
\[
\mathcal{A}(i)\text{-trind} X \geq \mathcal{A}(i)\text{-trind} Y \quad \text{for } i \in \{0, 1\} \quad (\text{resp. } i \in \{0, 1, 4\} \text{ or } i \in \{0, 1, 2, 3, 4\}).
\]

**Proof.** The case \((*)_1\). We will prove the statement only for \( i = 0 \).

Let \( \mathcal{A}(0)\text{-trind} X = \alpha \geq -1 \). Note that for \( \alpha = -1 \) or for \(|Y| = 1\) the statement is valid. Apply induction on \( \alpha \). Consider the case: \( \alpha \geq 0 \) and \(|Y| \geq 2 \). Let \( M_Y \in \mathcal{A}_Y \) with \(|M_Y| \geq 2 \). By the property \((*)_1\) there exists an element \( M_X \in \mathcal{A}_X \) such that \( M_Y \subset M_X \). Consider a point \( x \in M_Y \) and a closed subset \( A_Y \) of the space \( M_Y \) such that \( x \notin A_Y \). Choose any closed subset \( A_X \) of the space \( M_X \) such that \( A_Y = A_X \cap M_Y \) and note that \( x \notin A_X \). Since \( \mathcal{A}(0)\text{-trind} X = \alpha \) there is a partition \( L_{M_X} \) in the space \( M_X \) between \( x \) and \( A_X \) such that \( \mathcal{A}(0)\text{-ind} L_{M_Y} < \alpha \). Note that the set \( L_{M_Y} = L_{M_X} \cap M_Y \) is a partition in the space \( M_Y \) between \( x \) and \( A_Y \). By the inductive assumption we have \( \mathcal{A}(0)\text{-ind} L_{M_Y} = \mathcal{A}(0)\text{-ind} L_{M_X} < \alpha \) and so \( \mathcal{A}(0)\text{-trind} Y \leq \alpha \).

The case \((*)_2\). We need to prove the statement only for \( i = 4 \) (see the sentence after Definition 2). Let \( \mathcal{A}(4)\text{-trind} X = \alpha \geq -1 \). Note that for \( \alpha = -1 \) or for \(|Y| = 1\) the statement is valid. Apply induction on \( \alpha \). Consider the case: \( \alpha \geq 0 \) and \(|Y| \geq 2 \). Let \( M_Y \in \mathcal{A}_Y \) with \(|M_Y| \geq 2 \). By the property \((*)_2\) the set \( M_X = \text{Cl}_X(M_Y) \) is an element of \( \mathcal{A}_X \). Since \( \mathcal{A}(4)\text{-trind} X = \alpha \) there exist distinct points \( x, y \) of \( M_X \) and a partition \( L_{M_X} \) in the space \( M_X \) between \( x \) and \( y \) such that \( \mathcal{A}(4)\text{-ind} L_{M_X} < \alpha \). It also implies that there exist open disjoint subsets \( U_X \) and \( V_X \) of the space \( M_X \) such that \( x \in U_X, y \in V_X \) and \( L_{M_X} = M_X \setminus (U_X \cup V_X) \). Choose a point \( a \in U_X \cap M_Y \) and a point \( b \in V_X \cap M_Y \) and note that the set \( L_{M_Y} = L_{M_X} \cap M_Y \) is a partition between the points \( a \) and \( b \) in the space \( M_Y \). By the inductive assumption we have \( \mathcal{A}(4)\text{-ind} L_{M_Y} = \mathcal{A}(4)\text{-ind} L_{M_X} < \alpha \) and so \( \mathcal{A}(4)\text{-ind} Y \leq \alpha \).

The case \((*)_3\) is trivial. \( \square \)

Applying Propositions 1 \((b)\) and 2 we get such a statement.

**Corollary 1.** Let an SSC-family \( \mathcal{A} \) possess property \((*)_1\) (resp. \((*)_2\) or \((*)_3\)), \( X \) be a space and \( M(\mathcal{A}, X, i) = \sup \{ \mathcal{A}(i)\text{-trind} A : A \in \mathcal{A}_X \} \). Then \( \mathcal{A}(i)\text{-trind} X \geq M(\mathcal{A}, X, i) \) for \( i \in \{0, 1\} \) (resp. \( i \in \{0, 1, 4\} \) or \( i \in \{0, 1, 2, 3, 4\} \)). Moreover, \( \mathcal{A}(i)\text{-trind} X = M(\mathcal{A}, X, i) \) for \( i \in \{0, 1\} \) (resp. \( i \in \{0, 1, 4\} \) or \( i \in \{0, 1, 2, 3, 4\} \)), whenever \( A \in \mathcal{A}_A \) for each \( A \in \mathcal{A}_X \) with \(|A| > 1\).

**Definition 3.** We will say that an SSC-family \( \mathcal{A} \) is an upper bound of degree 1 (resp. degree 2 or degree 3) for an SSC-family \( \mathcal{A}' \), in short, \( \mathcal{A} \geq_1 \mathcal{A}' \) (resp. \( \mathcal{A} \geq_2 \mathcal{A}' \) or \( \mathcal{A} \geq_3 \mathcal{A}' \)) if for every space \( X \) the following conditions hold:

1. for each \( A' \in \mathcal{A}'_X \) there exists an element \( A \in \mathcal{A}_X \) such that \( A' \subset A \);
2. for each \( A' \in \mathcal{A}'_X \) we have \( \text{Cl}_X(A') \in \mathcal{A}_X \), or \( \mathcal{A}'_X \subset \mathcal{A}_X \).

Note that if for SSC-families \( \mathcal{A} \) and \( \mathcal{A}' \) we have \( \mathcal{A} \geq_2 \mathcal{A}' \) or \( \mathcal{A} \geq_3 \mathcal{A}' \) then we have also \( \mathcal{A} \geq_1 \mathcal{A}' \).
Remark 3. For every SSC-family $\mathcal{A}$ we have that $\mathcal{A}^1 \geq_1 \mathcal{A}$, $\mathcal{A}^4 \geq_2 \mathcal{A}$ and $\mathcal{A}^2 \geq_3 \mathcal{A}$.

The following statement is evidently valid.

**Proposition 3.** Let $\mathcal{A}$ and $\mathcal{A}'$ be SSC-families such that $\mathcal{A} \geq_3 \mathcal{A}'$. Then $\mathcal{A}(i)$-trind $X \geq \mathcal{A}'(i)$-trind $X$ for every space $X$ and every $i \in \{0, 1, 2, 3, 4\}$.

**Corollary 2.** For every space $X$ and every $i \in \{0, 1, 2, 3, 4\}$ we have $\mathcal{A}^1(i)$-trind $X \leq \mathcal{A}^4(i)$-trind $X \leq \mathcal{A}^2(i)$-trind $X \leq \mathcal{A}^4(i)$-trind $X$.

**Proposition 4.** Let $\mathcal{A}$ and $\mathcal{A}'$ be SSC-families such that the family $\mathcal{A}$ possesses property $(\ast)_1$ (resp. $(\ast)_2$) and $\mathcal{A} \geq_1 \mathcal{A}'$ (resp. $\mathcal{A} \geq_2 \mathcal{A}'$). Then $\mathcal{A}(i)$-trind $X \leq \mathcal{A}'(i)$-trind $X$ for every space $X$.

**Proof.** The case $(\ast)_1$ and $\mathcal{A} \geq_1 \mathcal{A}'$. We will prove the statement only for $i = 0$. Let $\mathcal{A}(0)$-trind $X = \alpha \geq -1$. Note that for $\alpha = -1$ or for $|X| = 1$ the statement is valid. Apply induction on $\alpha$. Consider the case: $\alpha \geq 0$ and $|X| \geq 2$. Let $M' \in \mathcal{A}'_X$ with $|M'| \geq 2$. Since $\mathcal{A} \geq_1 \mathcal{A}'$, there exists $M \in \mathcal{A}_X$ such that $M' \subset M$. Consider a point $x \in M'$ and a closed subset $A'$ of the space $M'$ such that $x \notin A'$. Choose any closed subset $A$ of the space $M$ such that $A' = A \cap M'$ and note that $x \notin A$. Since $\mathcal{A}(0)$-trind $X = \alpha$, there is a partition $L_M$ in the space $M$ between $x$ and $A$ such that $\mathcal{A}(0)$-ind $L_M < \alpha$. Note that the set $L_{M'} = L_M \cap M'$ is a partition in the space $M'$ between $x$ and $A'$. It follows from Proposition 2 that $\mathcal{A}(0)$-ind $L_{M'} \leq \mathcal{A}(0)$-ind $L_M$. Then by the inductive assumption we have $\mathcal{A}'(0)$-ind $L_{M'} \leq \mathcal{A}(0)$-ind $L_M < \alpha$ and so $\mathcal{A}'(0)$-trind $X \leq \alpha$.

The case $(\ast)_2$ and $\mathcal{A} \geq_2 \mathcal{A}'$. We need to prove the statement only for $i = 4$ (see the sentences after Definitions 2 and 3). Let $\mathcal{A}(4)$-trind $X = \alpha \geq -1$. Note that for $\alpha = -1$ or for $|X| = 1$ the statement is valid. Apply induction on $\alpha$. Consider the case: $\alpha \geq 0$ and $|X| \geq 2$. Let $M' \in \mathcal{A}'$ with $|M'| \geq 2$. Since $\mathcal{A} \geq_2 \mathcal{A}'$ the set $M = \text{Cl}_X(M')$ is an element of $\mathcal{A}_X$. Recall that $\mathcal{A}(4)$-trind $X = \alpha$. So there exist distinct points $x, y$ of $M$ and a partition $L_M$ in the space $M$ between $x$ and $y$ such that $\mathcal{A}(4)$-ind $L_M < \alpha$. This also implies that there exist open disjoint subsets $U$ and $V$ of the space $M$ such that $x \in U$, $y \in V$ and $L_M = M \setminus (U \cup V)$. Choose a point $a \in M' \cap U$ and a point $b \in M' \cap V$ and note that the set $L_{M'} = L_M \cap M'$ is a partition between the points $a$ and $b$ in the space $M'$. It follows from Proposition 2 that $\mathcal{A}(4)$-ind $L_{M'} \leq \mathcal{A}(4)$-ind $L_M$. By the inductive assumption we have $\mathcal{A}'(4)$-ind $L_{M'} \leq \mathcal{A}(4)$-ind $L_M < \alpha$ and so $\mathcal{A}'(4)$-trind $X \leq \alpha$. □

Applying additionally Remarks 2 and 3 we get the following statement.

**Corollary 3.** For every SSC-family $\mathcal{A}$ and for every space $X$ we have the following inequalities:

(a) $\mathcal{A}^1(i)$-trind $X \geq \mathcal{A}(i)$-trind $X$, where $i \in \{0, 1\}$;

(b) $\mathcal{A}^4(i)$-trind $X \geq \mathcal{A}(i)$-trind $X$, where $i \in \{0, 1, 4\}$;

(c) $\mathcal{A}^2(i)$-trind $X \geq \mathcal{A}(i)$-trind $X$, where $i \in \{0, 1, 2, 3, 4\}$.

In particular, $\mathcal{A}^1(i)$-trind $X = \mathcal{A}^2(i)$-trind $X = \mathcal{A}^4(i)$-trind $X$, where $i \in \{0, 1\}$, and $\mathcal{A}^2(4)$-trind $X = \mathcal{A}^4(4)$-trind $X$.

Now we have the following table of the functions $\mathcal{A}(i)$-trind, where $i \in \{0, 1, 2, 3, 4\}$, and $j \in \{1, 2, 3, 4\}$; the dots in the cells of coordinates $i, j$ replace the notations of invariants $\mathcal{A}(i)$-trind’s.
Let us mention some relationship between the table invariants. We start with equalities.

Proposition 5. For each metrizable locally compact space $X$ we have $\mathcal{A}^3(i)$-$\text{ind } X = \text{ind } X$
for every $j \in \{2, 4\}$ and every $i \in \{0, 1, 2, 3, 4\}$.

Proof. Recall [13] that for each metrizable locally compact space $X$ we have $t \ X = \text{ind } X$.
Hence the statement follows from Proposition 1.

Proposition 6. For every metrizable space $X$ with $p(X) < \omega_0$ and every $i \in \{0, 1, 2, 3, 4\}$, we have $\mathcal{A}^3(i)$-$\text{ind } X = p(X)$.

Proof. By Proposition 1 and Remark 1 it is sufficient to show that $\mathcal{A}^3(0)$-$\text{ind } X \leq p(X)$.
Recall [10] that $p(X) = \sup\{t(A) : A \text{ is compact subset of } X\}$. Since $p(X) < \omega_0$, there exists a compact subset $A$ of $X$ with $|A| \geq 2$ such that $p(X) = t(A)$. Note ([13]) that $\text{ind } A = t(A)$.
It follows from Remark 1 (a) and Corollary 3 (a) that $\text{ind } A = \mathcal{A}^1(0)$-$\text{ind } A \geq \mathcal{A}^3(0)$-$\text{ind } A$.
Hence $p(X) = t(A) = \text{ind } A \geq \mathcal{A}^3(0)$-$\text{ind } A$.

The following statement is obvious.

Proposition 7. For every compact space $X$ we have $\mathcal{A}^3(i)$-$\text{trind } X = \mathcal{A}^4(i)$-$\text{trind } X$, where
$i \in \{0, 1, 2, 3, 4\}$.

We continue with inequalities.

Remark 4. Recall ([8]) that for each integer $n \geq 1$ there exists a totally disconnected separable metrizable space $X_n$ such that $\text{ind } X_n = n$. Note that $\text{ind}_p X_n = \mathcal{A}^3(0)$-$\text{ind } X_n = 0$.

Remark 5. Recall ([4]) that there exists a compact space $Y$ with $\text{ind } Y = \text{ind}_p Y = 2$
such that each its component is homeomorphic to the closed interval $[0,1]$. Hence, $t \ Y = 1$.
Moreover, $\mathcal{A}^1(2)$-$\text{ind } Y = 1$ and $\mathcal{A}^1(3)$-$\text{ind } Y = 0$.

Proposition 8. For every strongly countable-dimensional compact metrizable space $X$ we have $\mathcal{A}^4(2)$-$\text{trind } X = \mathcal{A}^3(2)$-$\text{trind } X \leq \omega_0$.

Proof. Since every strongly countable-dimensional compact metrizable space has a non-empty open subset with $\text{ind } < \infty$, we get the statement.

Remark 6. Recall (cf. [6]) that for each $\alpha < \omega_1$ there exists a strongly countable-dimensional compact metrizable space $X_\alpha$ with $\text{trind } X_\alpha = \alpha$.

Let $X$ be a space and $X_{(k)}$ be the set of all points of $X$ that have arbitrary small neighborhoods with boundaries of dimension $\text{ind } \leq k - 1$, where $k$ is an integer $\geq 0$.
We call the space $X$ weakly $n$-dimensional in the sense of $\text{ind}$ if $\text{ind } X = n$ and $\text{ind}(X \setminus X_{(n-1)}) < n$.

Recall (cf. [8]) that for each integer $n \geq 1$ there exists a weakly $n$-dimensional in the sense of $\text{ind}$ separable metrizable space $Y_n$.
Note that the subset $Y_n \setminus (Y_n)_{n-1}$ of $Y_n$ can not be closed.
However there is a metrizable weakly 1-dimensional in the sense of $\text{ind}$ space $R$ such that $|R \setminus R_{(0)}| = 1$ (cf. [6, Problem 4.1.B]).
This implies that $\mathcal{A}^2(1)$-$\text{ind } R = \mathcal{A}^2(2)$-$\text{ind } R = 0$.  

\begin{table}
\begin{center}
\begin{tabular}{|c|c c c c c|}
\hline
\textbf{i} & 0 & 1 & 2 & 3 & 4 \\
\hline
\textbf{j} & trind & trind & \cdot & \cdot & \cdot \\
1 & trind & trind & \cdot & \cdot & trt \\
2 & \cdot & \cdot & \cdot & \cdot & p \\
3 & trind & trind & \cdot & \cdot & trt \\
4 & \end{tabular}
\end{center}
\end{table}
Proposition 9. Let $X$ be a weakly $n$-dimensional in the sense of $\text{ind}$ space, where $n \geq 1$. Then $A^2(2)$-ind $X \leq n - 1$.

Proof. Consider a subset $M$ of $X$ with $|M| \geq 2$.

If $M \subseteq X \setminus X_{(n-1)}$ then $\text{ind} M \leq n - 1$. So for each point and any closed subset $A$ of $M$ such that $x \notin A$ there exists a partition $L_M$ in $M$ between $x$ and $A$ with $\text{ind} L_M \leq n - 2$. Note that $A^2(2)$-ind $L_M \leq \text{ind} L_M$ by Proposition 1.

If there is a point $x \in M \subseteq X \setminus X_{(n-1)}$, so $x$ has arbitrary small neighborhoods with at most $(n-2)$-dimensional in the sense of $\text{ind}$ boundaries. This implies that for every closed subset $A$ of $M$ such that $x \notin A$ there exists a partition $L_M$ in $M$ between $x$ and $A$ with $\text{ind} L_M \leq n - 2$. Recall again that $A^2(2)$-ind $L_M \leq \text{ind} L_M$.

Both cases imply that $A^2(2)$-ind $X \leq n - 1$. \hfill \Box

3. Zero-dimensionality with respect to $A(i)$-trind. In this section, let $A$ be any SSC-family, and put $C_\alpha(A(i)) = \{X : A(i)$-trind $X \leq \alpha\}$ for $i \in \{0, 1, 2, 3, 4\}$.

Question 1. Determine the class $C_0(A(i))$, where $i \in \{0, 1, 2, 3, 4\}$.

Proposition 1 (a) and Corollaries 2 and 3 easily imply the next statement.

Proposition 10. The following assertions hold.

(a) $C_0(A(0)) \subseteq C_0(A(1)) \subseteq C_0(A(3)) \subseteq C_0(A(4))$ and $C_0(A(0)) \subseteq C_0(A(2)) \subseteq C_0(A(4))$.

(b) For every $i \in \{0, 1, 2, 3, 4\}$ we have $C_0(A^2(i)) \subseteq C_0(A^4(i)) \subseteq C_0(A^1(i))$ and $C_0(A^4(i)) \subseteq C_0(A^3(i))$.

(c) $C_0(A^1(i)) \subseteq C_0(A(i))$, where $i \in \{0, 1\}$;

$C_0(A^4(i)) \subseteq C_0(A(i))$, where $i \in \{0, 1, 4\}$;

$C_0(A^2(i)) \subseteq C_0(A(i))$, where $i \in \{0, 1, 2, 3, 4\}$.

In particular, $C_0(A^1(i)) = C_0(A^2(i)) = C_0(A^4(i))$, where $i \in \{0, 1\}$, and $C_0(A^2(4)) = C_0(A^4(4))$.

Additionally, we have the following proposition.

Proposition 11. (a) $C_0(A(3)) = C_0(A(4))$.

(b) $C_0(A^3(0)) = C_0(A^3(1)) = C_0(A^3(2)) = C_0(A^3(3)) = C_0(A^3(4))$.

Proof. (a) By Proposition 3.1 (a), it is sufficient to show that $C_0(A^3(4)) \subseteq C_0(A^3(3))$. Let $X \in C_0(A^3(4))$ and $M \in A^3_X$. Note that the subspace $M$ of the space $X$ is disconnected. So there are clopen disjoint non-empty subsets $M_1$ and $M_2$ of $M$ such that $M = M_1 \cup M_2$. Put $A = M_1$ and observe that every $x \in M_2$ can be separated from $A$ in $M$ by the empty set. This implies that $A(3)$-ind $X = 0$.

(b) By Proposition 10 (a), it is sufficient to show that $C_0(A^3(4)) \subseteq C_0(A^3(0))$. Let $X \in C_0(A^3(4))$ and $M \in A^3_x$ with $|M| > 1$. Note that $X$ must be punctiform ([10]). Consider $M \in A^3_x$, i.e. $M$ is a compact subspace of $X$. Note that $M$ is punctiform too. By [6, Theorem 1.4.5] we have ind $M = 0$. This implies that $A^3(0)$-ind $X = 0$. \hfill \Box
We summarize the classes \( C_0(\mathcal{A}^j(i)) \setminus \{ X : |X| \leq 1 \} \), where \( i \in \{0,1,2,3,4\} \), and \( j \in \{1,2,3,4\} \) in the following table.

**Table 2.**

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>j</td>
<td>( \mathcal{Z} )</td>
<td>( \mathcal{D}_t )</td>
<td>( \mathcal{Z}_p )</td>
<td>( \mathcal{D} )</td>
<td>( \mathcal{D} )</td>
</tr>
<tr>
<td>1</td>
<td>( \mathcal{Z} )</td>
<td>( \mathcal{D}_t )</td>
<td>( \mathcal{X} )</td>
<td>( \mathcal{D}_h )</td>
<td>( \mathcal{D}_h )</td>
</tr>
<tr>
<td>2</td>
<td>( \mathcal{P} )</td>
<td>( \mathcal{P} )</td>
<td>( \mathcal{P} )</td>
<td>( \mathcal{P} )</td>
<td>( \mathcal{P} )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{Z} )</td>
<td>( \mathcal{D}_t )</td>
<td>( \mathcal{Y} )</td>
<td>( \mathcal{D}_h )</td>
<td>( \mathcal{D}_h )</td>
</tr>
</tbody>
</table>

where

- \( \mathcal{Z} \) is the class of zero-dimensional spaces in the sense of \( \text{ind} \) with \( |X| > 1 \) (see Remark 1);
- \( \mathcal{D}_t \) is the class of totally disconnected spaces with \( |X| > 1 \) (see [6, Definition 1.4.1] and Remark 1);
- \( \mathcal{D}_h \) is the class of hereditarily disconnected spaces with \( |X| > 1 \) (see [6, Definition 1.4.2], Remark 1 and [13]);
- \( \mathcal{P} \) is the class of punctiform spaces with \( |X| > 1 \) (see [6, Definition 1.4.3], Remark 1 and [10]);
- \( \mathcal{D} \) is the class of disconnected spaces;
- \( \mathcal{Z}_p \) is the class of non-trivial spaces having at least one point at which the dimension \( \text{ind} \) is zero.

**Remark 7.**  (a) Recall that \( \mathcal{Z} \subset \mathcal{D}_t \subset \mathcal{D}_h \subset \mathcal{P} \) and there are subspaces of the real plane which exhibit the difference between the classes (see [6, Examples 1.4.6-8]).

(b) Note that \( \mathcal{Z} \subset \mathcal{Z}_p \subset \mathcal{D} \) and \( \mathcal{D}_h \subset \mathcal{D} \).

(c) Let \( X \oplus Y \) be the free union of topological spaces \( X \) and \( Y \), \( I \) the closed interval \([0,1]\) and \( P \) a one-point space. Then observe that \( P \oplus I \in \mathcal{Z}_p \setminus \mathcal{P} \), \( I \oplus I \in \mathcal{D} \setminus \mathcal{Z}_p \), the space \( Z \) from [6, Example 1.4.8] is in \( \mathcal{P} \setminus \mathcal{D} \) and the Erdös’ space \( H_0 \) from [6, Example 1.2.15] is in \( \mathcal{D}_t \setminus \mathcal{Z}_p \).

(d) It follows from Proposition 10 (a) and (b) that \( \mathcal{Z} \subset \mathcal{X} \subset \mathcal{Y} \subset \mathcal{D}_h \) and \( \mathcal{Y} \subset \mathcal{Z}_p \).

Note also that every weakly 1-dimensional in the sense of \( \text{ind} \) space is in \( \mathcal{X} \setminus \mathcal{Z} \) (see Proposition 9), the Erdös’ space \( H_0 \) is in \( \mathcal{D}_t \setminus \mathcal{Y} \) and the space \( P \oplus I \) from (c) is in \( \mathcal{Z}_p \setminus \mathcal{Y} \).

We have the following additional facts about the classes \( \mathcal{X} \) and \( \mathcal{Y} \):

(i) if \( X \in \mathcal{X} \), \( X' \subset X \) and \( |X'| > 1 \) then \( X' \in \mathcal{X} \);

(ii) if \( Y \in \mathcal{Y} \), \( Y' \) is a closed subset of \( Y \) and \( |Y'| > 1 \) then \( Y' \in \mathcal{Y} \);

(iii) \( X = Y = Z \) in the realm of locally compact spaces.

**Problem 1.** Describe the classes \( \mathcal{X} \) and \( \mathcal{Y} \) in the realm of separable metrizable spaces (resp. metrizable spaces or topological \( T_3 \)-spaces).

4. **Countable unions of spaces of** \( \mathcal{A}(i) \)-trind \( \leq 0 \), \( i \in \{0,1,2,3,4\} \). Let \( \mathcal{A} \) be any SSC-family.
**Definition 4.** A space $X$ is said to be $\sigma$-$\mathcal{C}_0(\mathcal{A}(i))$, where $i \in \{0, 1, 2, 3, 4\}$, if $X = \bigcup_{j=1}^{\infty} X_j$, where $X_j \in \mathcal{C}_0(\mathcal{A}(i))$ for each $j$.

**Problem 2.** Describe the class $\sigma$-$\mathcal{C}_0(\mathcal{A}(i))$, where $i \in \{0, 1, 2, 3, 4\}$.

We will restrict now our discussion to the realm of separable metrizable spaces.

**Proposition 12.** Let $X$ be a separable completely metrizable $\sigma$-$\mathcal{C}_0(\mathcal{A}^j(i))$ space, where $i \in \{0, 1, 2, 3, 4\}$ and $j \in \{1, 2, 3, 4\}$. Then there is a metrizable compactification $Y$ of $X$ such that $Y$ is also $\sigma$-$\mathcal{C}_0(\mathcal{A}^j(i))$.

*Proof.* Recall ([6, Lemma 5.3.1]) that there is metrizable compactification $Y$ of $X$ such that the remainder $Y \setminus X$ is strongly countable-dimensional. Note that the space $Y \setminus X$ is $\sigma$-$\mathcal{C}_0(\mathcal{A}^j(i))$. Hence, $Y$ is also $\sigma$-$\mathcal{C}_0(\mathcal{A}^j(i))$. \[ \square \]

**Remark 8.** Let us recall [10] that the R. Pol’s metrizable compactum $P$ is a compactification of some complete $A$-strongly infinite-dimensional totally disconnected space $P_0$ with the reminder $P \setminus P_0 = \bigcup_{k=1}^{\infty} P_k$, where $P_k$ is a finite-dimensional compactum for each $k$. Note that $P$ is $\sigma$-$\mathcal{C}_0(\mathcal{A}^j(i))$ for every $i \in \{1, 3, 4\}$ and every $j \in \{1, 2, 3, 4\}$, and for the pair: $i = 0$ and $j = 3$. We note also that $P$ is not $\sigma$-$\mathcal{C}_0(\mathcal{A}^j(i))$ for $i = 0$ and every $j \in \{1, 2, 4\}$.

**Question 2.** Is $P$ $\sigma$-$\mathcal{C}_0(\mathcal{A}^2(2))$ (resp. $\sigma$-$\mathcal{C}_0(\mathcal{A}^4(2))$)?

The following statement is evident.

**Lemma 1.** Let $Y$ be a metrizable compact space, $X \subset Y$ and $Y \setminus X = \bigcup_{i=1}^{\infty} X_i$, where for each $i$ the set $X_i$ is compact and $\text{ind} X_i < \infty$. Assume that $M$ is a closed subset of $Y$. Then either $|M \cap X| > 1$ or $M$ is strongly countable dimensional.

**Lemma 2.** Let $X$ be a separable metrizable space, $M \subset X$, $x,y \in M$ and $L_M$ a partition of $M$ between the points $x,y$. Then there is a partition $L_X$ of $X$ between $x,y$ such that $L \cap M = L_M$.

*Proof.* Let $O_x$ and $O_y$ be disjoint open subsets of $M$ such that $x \in O_x$, $y \in O_y$ and $M \setminus (O_x \cup O_y) = L_M$. Put $L = \text{Cl}_X(L_M \cup O_x) \cap \text{Cl}_X(L_M \cup O_y)$. Note that $L$ is a partition of the subspace $\text{Cl}_X M$ of $X$ between the points $x,y$ such that $L \cap M = L_M$. By [6, Lemma 1.2.9] there is a partition $L_X$ of $X$ between $x,y$ such that $L_X \cap \text{Cl}_X M = L$. Note that $L_X \cap M = L_M$. \[ \square \]

**Proposition 13.** Let $Y$ be a metrizable compact space, $X \subset Y$ and $Y \setminus X = \bigcup_{i=1}^{\infty} X_i$, where for each $i$ the set $X_i$ is compact and $\text{ind} X_i < \infty$.

Assume that $\text{trt} X = \alpha \neq \infty$. Then

$$\text{trt} Y \leq \begin{cases} \omega_0 + \alpha + 1, & \text{if } \alpha < \omega_0^2; \\ \alpha + 1, & \text{if } \alpha \geq \omega_0^2. \end{cases}$$

*(One can omit 1 in the formula if $\alpha$ is an infinite limit ordinal.)*

*Proof.* Apply induction on $\alpha \geq 0$. Assume that $\alpha = 0$. Consider a closed subset $M$ of $Y$ with $|M| > 1$. By Lemma 1, we have two possibilities:

(a) $|M \cap X| > 1$ or (b) $M$ is strongly countable dimensional.

The case (a). Since $\text{trt} X = 0$, we have $\text{trt}(M \cap X) = 0$ and the empty set is a partition of $M \cap X$ between some points $x,y$ of $M \cap X$. By Lemma 2, there is a partition $L$ of $M$
between the points $x, y$ such that $L \subset M \setminus X$. Note that the space $L$ is strongly countable dimensional and hence $\text{trt } L \leq \omega_0$ (see [1]).

The case $(b)_0$. Since $M$ is strongly countable dimensional, we have $\text{trt } M \leq \omega_0$. Note that the both cases imply $\text{trt } X \leq \omega_0 + 1$.

Assume that the statement is valid for all $\alpha < \gamma \geq 1$. Let now $\alpha = \gamma$. Consider a closed subset $M$ of $Y$ with $|M| > 1$. Again by Lemma 2, we have two possibilities:

(a) $|M \cap X| > 1$ or (b) $M$ is strongly countable dimensional.

The case (a). Since $X = \gamma$, we have $\text{trt } (M \cap X) \leq \gamma$. Hence, there is a partition $L_{M \cap X}$ of $M \cap X \subset M$ between some points $x, y$ of $M \cap X$ such that $\text{trt } L_{M \cap X} < \gamma$. By Lemma 2 there is a partition $L_M$ of $M$ between the points $x, y$ such that $L_M \cap (M \cap X) = L_{M \cap X}$. Note that $L_{M \cap X} = L_M \cap X$ and the space $L_M \setminus L_{M \cap X} = \bigcup_{i=1}^{\infty} (L_M \cap X_i)$ is strongly countable dimensional. Hence, by induction, we have $\text{trt } L_M \leq \omega_0 + \text{trt } L_{M \cap X} + 1 < \omega_0 + \gamma + 1$. (Let us observe that if $\gamma$ is an infinite limit ordinal then $\omega_0 + \text{trt } L_{M \cap X} + 1 < \omega_0 + \gamma$.)

The case $(b)_\gamma$. Since $M$ is strongly countable dimensional, $\text{trt } M \leq \omega_0$. Note that both cases imply $\text{trt } X \leq \omega_0 + \gamma + 1$.

Let us recall that if $\alpha \geq \omega_0^2$ then $\omega_0 + \alpha = \alpha$.

\begin{corollary}
Let $X$ be a separable completely metrizable space and $\text{trt } X = \alpha \neq \infty$. Then there is a compactification $Y$ of $X$ such that

$$\text{trt } Y \leq \begin{cases} 
\omega_0 + \alpha + 1, & \text{if } \alpha < \omega_0^2; \\
\alpha + 1, & \text{if } \alpha \geq \omega_0^2.
\end{cases}$$

(One can omit 1 in the formula if $\alpha$ is an infinite limit ordinal.)
\end{corollary}

\begin{proof}
Recall ([6, Lemma 5.3.1]) that there is a metrizable compactification $Y$ of $X$ such that the remainder $Y \setminus X$ is strongly countable dimensional. Now, apply Proposition 13 to the space $Y$.
\end{proof}

\begin{corollary}
For any hereditarily disconnected separable completely metrizable space $X$ there is a metrizable compactification $Y$ of $X$ such that $\text{trt } Y \leq \omega_0 + 1$.
\end{corollary}

Recall (see [11] (resp. [12])) that for each isolated countable infinite ordinal $\alpha$ there exists an $\alpha$-dimensional metrizable Cantor trind-manifold $Y^\alpha$ (resp. trInd-manifold $Z^\alpha$) which is a disjoint union of countably many Euclidean cubes and the irrationals. It follows now from Proposition 13 that for each isolated countable infinite ordinal $\alpha$, $\text{trt } Y^\alpha = \text{trt } Z^\alpha = \omega_0 + 1$.

\begin{problem}
Is there a countable-dimensional separable metrizable space $X$ such that $\text{trt } X > \omega_0 + 1$ (and $\text{trt } X \neq \infty$)?
\end{problem}

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