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A NEW (1+1)-DIMENSIONAL MATRIX K-CONSTRAINED KP HIERARCHY

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We introduce a new generalization of matrix (1+1)-dimensional k-constrained KP hierarchy. The new hierarchy contains matrix generalizations of stationary DS systems, (2+1)-dimensional modified Korteweg-de Vries equation and the Nizhnik equation. A binary Darboux transformation method is proposed for integration of systems from this hierarchy.

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Введено новое обобщение матричной КП иерархии. Новая иерархия включает матричные обобщения стационарной системы Деви-Стюартсона, (2+1)-модифицированного уравнения Кортевега-де Фриза и Нижника. Предложен метод интегрирования систем из этой иерархии при помощи бинарных преобразований Дарбу.

1. Introduction. Algebraic methods are of great importance in the soliton equations theory (see [1–7]). In particular, they make possible to omit analytical difficulties that arise in the investigation of corresponding direct and inverse scattering problems for nonlinear equations. One of the important objects arising from algebraic approach in the soliton theory is scalar and matrix hierarchies for nonlinear integrable systems of Kadomtsev-Petviashvili type (KP hierarchy, [8–12]).

The KP hierarchy and its generalizations play an important role in mathematical physics. One of such generalizations is the so-called "KP equation with self-consistent sources" (KPSCS), discovered by V. K. Melnikov ([13–16]). In [17–21], k-symmetry constraints of the KP hierarchy (k-cKP hierarchy) which have connections with KPSCS were investigated. k-cKP hierarchy contains physically relevant systems like the nonlinear Schrödinger equation, the Yajima-Oikawa system, a generalization of the Boussinesq equation, and the Melnikov system. Multicomponent k-constraints of the KP hierarchy were introduced in [22] and investigated in [23–28].

The modified k-constrained KP (k-cmKP) hierarchy was proposed in [20, 29, 30]. It contains, for example, the vector Chen-Lee-Liu and the modified KdV (mKdV) equation. Multicomponent versions of the Kundu-Eckhaus and Gerdjikov-Ivanov equations were also obtained in [29], via gauge transformations of the k-cKP, respectively the k-cmKP hierarchy. Moreover, in [31, 32], (2+1)-dimensional extensions of the k-cKP hierarchy were introduced and dressing methods via differential transformations were investigated. Some systems of this

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hierarchy were investigated via binary Darboux transformations in [33, 34]. This hierarchy was also rediscovered recently in [35, 36].

In this paper our aim was to generalize (1+1)-dimensional matrix k-constrained KP hierarchy to the case of two integro-differential operators. As a result we obtain a new bidirectional (one of the operators in obtained hierarchy depends on two indices k and l) generalization of (1+1)-dimensional matrix k-constrained KP hierarchy that we will call (1+1)-BDk-cKP hierarchy (see formulae (10)).

This paper is organized as follows. In Section 2 we present a short survey of results on constraints for KP hierarchies. In Section 3 we introduce a new (1+1)-BDk-cKP hierarchy. Members of the obtained hierarchy are also listed there. (1+1)-BDk-cKP hierarchy contains matrix generalizations of stationary Davey-Stewartson hierarchy, new stationary Yajima-Oikawa and Melnikov hierarchies. In Section 4 we consider dressing via binary Darboux transformation for the (1+1)-BDk-cKP hierarchy. As an example, we consider construction of solutions for the matrix generalization of stationary DS system that was considered in Section 3. In the final section, Conclusions we discuss the obtained results and mention problems for further investigations. We also present an equation obtained from (1+1)-BDk-cKP hierarchy that generalize vector nonlinear Schrödinger system (the Manakov system).

2. k-constrained KP hierarchy and its extensions. To make this paper self-contained, we briefly introduce the KP hierarchy ([1]) and its multi-component k-symmetry constraints (k-cKP hierarchy). A Lax representation of the KP hierarchy is given by

$$L_{t_n} = [B_n, L], \quad n \geq 1, \tag{1}$$

where $L = D + U_1 D^{-1} + U_2 D^{-2} + \dots$ is a scalar pseudodifferential operator, $t_1 := x$, $D := \frac{\partial}{\partial x}$, and $B_n := (L^n)_+ := (L^n)_{\geq 0} = D^n + \sum_{i=0}^{n-2} u_i D^i$ is the differential operator part of L^n. The consistency condition (zero-curvature equations), arising from the commutativity of flows (1), is

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{2}$$

Let B_n^τ denote the formal transpose of B_n, i.e. $B_n^\tau := (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\top$, where $^\top$ denotes the matrix transpose. We will use curly brackets to denote the action of an operator on a function whereas, for example, $B_n\, q$ means the composition of the operator B_n and the operator of multiplication by the function q. The following formula holds for $B_n q$ and $B_n\{q\}$: $B_n\{q\} = B_n q - (B_n q)_{>0}$. In the case $k = 2$, $n = 3$ formula (2) presents a Lax pair for the Kadomtsev-Petviahvili equation ([37]). Its Lax pair was obtained in [38] (see also [2]).

The multicomponent k-constraints of the KP hierarchy is given by ([22])

$$L_{t_n} = [B_n, L], \tag{3}$$

with the k-symmetry reduction

$$L_k := L^k = B_k + \sum_{i=1}^{m} \sum_{j=1}^{m} q_i m_{ij} D^{-1} r_j = B_k + \mathbf{q} \mathcal{M}_0 D^{-1} \mathbf{r}^\top, \tag{4}$$

where $\mathbf{q} = (q_1, \dots, q_m)$ and $\mathbf{r} = (r_1, \dots, r_m)$ are vector functions, $\mathcal{M}_0 = (m_{ij})_{i,j=1}^m$ is a constant $m \times m$ matrix. In the scalar case ($m = 1$) we obtain k-constrained KP hierarchy ([17–21]). The hierarchy given by (3)–(4) admits the Lax representation (here $k \in \mathbf{N}$ is fixed)

$$[L_k, M_n] = 0, \quad L_k = B_k + \mathbf{q} \mathcal{M}_0 D^{-1} \mathbf{r}^\top, \quad M_n = \partial_{t_n} - B_n. \tag{5}$$

Lax equation (5) is equivalent to the following system

$$[L_k, M_n]_{\geq 0} = 0, \ M_n\{\mathbf{q}\} = 0, \ M_n^{\tau}\{\mathbf{r}\} = 0. \tag{6}$$

Below we will also use the formal adjoint $B_n^{*}{:=}\bar{B}_n^{\tau} = (-1)^n D^n + \sum_{i=0}^{n-2}(-1)^i D^i u_i^{*}$ of B_n, where * denotes the Hermitian conjugation (complex conjugation and transpose).

For $k=1$, the hierarchy given by (6) is a multi-component generalization of the AKNS hierarchy. For $k=2$ and $k=3$, one obtains vector generalizations of the Yajima-Oikawa and Melnikov ([14]) hierarchies, respectively.

In [20, 29, 30], a k-constrained modified KP (k-cmKP) hierarchy was introduced and investigated. It contains vector generalizations of the Chen-Lee-Liu, the modified multi-component Yajima-Oikawa and Melnikov hierarchies.

An essential extension of the k-cKP hierarchy is its (2+1)-dimensional generalization introduced in [31, 32] and rediscovered in [35, 36].

3. A new bidirectional extension of (1+1)-dimensional k-constrained KP ((1+1)-BDk-cKP) hierarchy. In this section we introduce a new generalization of the (1+1)-dimensional k-constrained KP hierarchy given by (5) to the case of two integro-differential operators. One of them (the operator $L_{k,l}$ (10) generalizes the corresponding operator L_k (5) and depends on two independent indices l and k. It leads to a generalization of (1+1)-dimensional k-cKP hierarchy (5) in an additional direction l ($l \in \{1, 2, \dots\}$). For further purposes we will use the following well-known formulae for integral operator $h_1 D^{-1} h_2$ constructed by matrix-valued functions h_1 and h_2 and the differential operator A with matrix-valued coefficients in the algebra of pseudodifferential operators

$$A h_1 \mathcal{D}^{-1} h_2 = (A h_1 \mathcal{D}^{-1} h_2)_{\geq 0} + A\{h_1\}\mathcal{D}^{-1} h_2, \tag{7}$$

$$h_1 \mathcal{D}^{-1} h_2 A = (h_1 \mathcal{D}^{-1} h_2 A)_{\geq 0} + h_1 \mathcal{D}^{-1}[A^{\tau}\{h_2^{\top}\}]^{\top}, \tag{8}$$

$$h_1 \mathcal{D}^{-1} h_2 h_3 \mathcal{D}^{-1} h_4 = h_1 D^{-1}\{h_2 h_3\}\mathcal{D}^{-1} h_4 - h_1 \mathcal{D}^{-1} D^{-1}\{h_2 h_3\} h_4. \tag{9}$$

It is known that the Matrix KP hierarchy can be formulated by a pseudodifferential operator

$$W = I + w_1 D + w_2 D^2 + \dots$$

with $N \times N$-matrix-valued coefficients w_i. Consider the differential operators $\mathcal{J}_k D^k$ and $\alpha_n \partial_{t_n} - \tilde{\mathcal{J}}_n D^n$, $\alpha_n \in \mathbb{C}$, $n, k \in \mathbb{N}$, where $\mathcal{J}_k$ and $\tilde{\mathcal{J}}_n$ are $N \times N$ commuting matrices (i.e., $[\tilde{\mathcal{J}}_n, \mathcal{J}_k] = 0$). It is evident that the dressed operators

$$L_k{:=}W\mathcal{J}_k D^k W^{-1} = \mathcal{J}_k D^k + u_{k-1} D^{k-1} + \dots + u_0 + u_{-1} D^{-1} + \dots,$$

$$M_n{:=}W(\alpha_n \partial_{t_n} - \tilde{\mathcal{J}}_n D^n)W^{-1} = \alpha_n \partial_{t_n} - \tilde{\mathcal{J}}_n D^n - v_{n-1} D^{n-1} + \dots + v_0 + v_{-1} D^{-1} + \dots,$$

with $N \times N$-matrix coefficients u_i and v_j commute: $[L_k, M_n] = 0$. We shall impose the following reduction on operators L_k and M_n

$$(L_k)_{<0}{:=}(L_{k,l})_{<0} = c_l \sum_{j=0}^{l} \mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^{\top}[l-j], \ (M_n)_{<0} = -\gamma \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^{\top}, \ \gamma, \ c_l \in \mathbb{C},$$

where $\mathbf{q}$ and $\mathbf{r}$ are $N \times m$ matrix functions; $\mathbf{q}[j]$ and $\mathbf{r}[j]$ are matrix functions of the following form $\mathbf{q}[j]{:=}(M_n)^j\{\mathbf{q}\}$, $\mathbf{r}^{\top}[j]{:=}((M_n^{\tau})^j\{\mathbf{r}\})^{\top}$.

As a result, we obtain the following bi-directional k-cKP hierarchy

$$L_{k,l} = B_k + c_l \sum_{j=0}^{l} \mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j],$$
$$B_k = \mathcal{J}_k D^k + \sum_{j=0}^{k-1} u_j D^j,\ u_j = u_j(x,t_n),\ l \in \{0,\ldots\}, \tag{10}$$
$$M_n = \alpha_n \partial_{t_n} - A_n - \gamma \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top,\ A_n = \tilde{\mathcal{J}}_n D^n + \sum_{i=0}^{n-1} v_i D^i,\ v_i = v_i(x,t_n),\ \alpha_n \in \mathbb{C},$$

where u_j and v_i, are $N \times N$ matrix functions.

The following theorem holds.

Theorem 1. *The Lax equation* $[L_{k,l}, M_n] = 0$ *is equivalent to the system*

$$[L_{k,l}, M_n]_{\geq 0} = 0,\ \gamma L_{k,l}\{\mathbf{q}\} + c_l (M_n)^{l+1}\{\mathbf{q}\} = 0,\ \gamma L_{k,l}^{\tau}\{\mathbf{r}\} + c_l (M_n^{\tau})^{l+1}\{\mathbf{r}\} = 0. \tag{11}$$

Proof. From the equality $[L_{k,l}, M_n] = [L_{k,l}, M_n]_{\geq 0} + [L_{k,l}, M_n]_{<0}$ we obtain that the Lax equation $[L_{k,l}, M_n] = 0$ is equivalent to the following one

$$[L_{k,l}, M_n]_{\geq 0} = 0,\ [L_{k,l}, M_n]_{<0} = 0. \tag{12}$$

Thus, it is sufficient to prove that $[L_{k,l}, M_n]_{<0} = 0 \iff \gamma L_{k,l}\{\mathbf{q}\} + c_l (M_n)^{l+1}\{\mathbf{q}\} = 0$, $\gamma L_{k,l}^{\tau}\{\mathbf{r}\} + c_l (M_n^{\tau})^{l+1}\{\mathbf{r}\} = 0$. Using bi-linearity of the commutator and explicit form (10) of operators $L_{k,l}$ and M_n we obtain

$$[L_{k,l}, M_n]_{<0} = c_l \sum_{j=0}^{l} [\mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j], \alpha_n \partial_{t_n} - A_n]_{<0} -$$
$$-c_l \sum_{j=0}^{l} [\mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j], \gamma \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top]_{<0} - [B_k, \gamma \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top]_{<0}. \tag{13}$$

After direct computations of each of the three summands on the right-hand side of formula (13) we obtain

1. Equality (14) is a consequence of formulae (7)–(8).

$$c_l \sum_{j=0}^{l} [\mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j], \alpha_n \partial_{t_n} - A_n]_{<0} = -c_l \sum_{j=0}^{l} \left(\alpha_n \mathbf{q}_{t_n}[j] - A_n\{\mathbf{q}[j]\}\right) \times$$
$$\times \mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j] - c_l \sum_{j=0}^{l} \mathbf{q}[j]\mathcal{M}_0 D^{-1}\left(\alpha_n \mathbf{r}_{t_n}^\top[l-j] + A_n^{\tau}\{\mathbf{r}^\top[l-j]\}\right). \tag{14}$$

2. Formula (15) follows from (9).

$$\gamma c_l\sum_{j=0}^{l}[\mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top,\mathbf{q}[j]\mathcal{M}_0D^{-1}\mathbf{r}^\top[l-j]]_{<0}=\gamma c_l\sum_{j=0}^{l}\mathbf{q}\mathcal{M}_0D^{-1}\{\mathbf{r}^\top\mathbf{q}[j]\}\times$$
$$\times\mathcal{M}_0D^{-1}\mathbf{r}^\top[l-j]-\gamma c_l\sum_{j=0}^{l}\mathbf{q}\mathcal{M}_0D^{-1}D^{-1}\{\mathbf{r}^\top\mathbf{q}[j]\}\mathcal{M}_0\mathbf{r}^\top[l-j]+$$
$$-\gamma c_l\sum_{j=0}^{l}\mathbf{q}[j]\mathcal{M}_0D^{-1}\{\mathbf{r}^\top[l-j]\mathbf{q}\}\mathcal{M}_0D^{-1}\mathbf{r}^\top+\gamma c_l\sum_{j=0}^{l}\mathbf{q}[j]\mathcal{M}_0D^{-1}D^{-1}\{\mathbf{r}^\top[l-j]\mathbf{q}\}\mathcal{M}_0\mathbf{r}^\top. \tag{15}$$

3. Equality (16) is obtained via (7)–(8).

$$-[B_k,\gamma\mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top]_{<0}=-\gamma B_k\{\mathbf{q}\}\mathcal{M}_0D^{-1}\mathbf{r}^\top+\gamma\mathbf{q}\mathcal{M}_0D^{-1}\left(B_k^\tau\{\mathbf{r}\})^\top\right). \tag{16}$$

From formulae (13)–(16) we have

$$[L_{k,l},M_n]_{<0}=c_l\left(\sum_{j=0}^{l}\mathbf{q}[j]\mathcal{M}_0D^{-1}M_n^\tau\{\mathbf{r}[l-j]\}-\sum_{j=0}^{l}M_n\{\mathbf{q}[j]\}\mathcal{M}_0D^{-1}\mathbf{r}^\top[l-j]\right)-$$
$$-\gamma L_{k,l}\{\mathbf{q}\}\mathcal{M}_0D^{-1}\mathbf{r}^\top+\gamma\mathbf{q}\mathcal{M}_0D^{-1}(L_{k,l}^\tau\{\mathbf{r}\})^\top=c_l\sum_{j=0}^{l}\mathbf{q}[j]\mathcal{M}_0D^{-1}\mathbf{r}^\top[l-j+1]-$$
$$-c_l\sum_{j=0}^{l}\mathbf{q}[j+1]\mathcal{M}_0D^{-1}\mathbf{r}^\top[l-j]-\gamma L_{k,l}\{\mathbf{q}\}\mathcal{M}_0D^{-1}\mathbf{r}^\top+\gamma\mathbf{q}\mathcal{M}_0D^{-1}L_{k,l}^\tau\{\mathbf{r}\}^\top=$$
$$=-(\gamma L_{k,l}+c_l(M_n)^{l+1})\{\mathbf{q}\}\mathcal{M}_0D^{-1}\mathbf{r}^\top+\mathbf{q}\mathcal{M}_0D^{-1}((\gamma L_{k,l}^\tau+c_l(M_n^\tau)^{l+1})\{\mathbf{r}\})^\top.$$

From the latter equality we obtain the equivalence of the equation $[L_k,M_{n,l}]=0$ and (11). □

New hierarchy (10) includes Matrix k-constrained KP-hierarchy ($\gamma=0$, $l=0$).
The following corollary immediately follows from Theorem 1.

Corollary 1. *The Lax equation* $[\tilde{L}_{k,l},M_n]=0$, *where*

$$\tilde{L}_{k,l}=\gamma L_{k,l}+c_l(M_n)^{l+1} \tag{17}$$

and the operators $\tilde{L}_{k,l}$ *and* M_n *are defined by (10), is equivalent to the system*

$$[L_{k,l},M_n]_{\geq 0}=0,\ \tilde{L}_{k,l}\{\mathbf{q}\}=0,\ \tilde{L}_{k,l}^\tau\{\mathbf{r}\}=0. \tag{18}$$

The BDk-cKP hierarchy (10) admits an essential generalization

$$P_{k,s}=B_k+\sum_{l=0}^{s}c_l\sum_{j=0}^{l}\mathbf{q}[j]\mathcal{M}_0D^{-1}\mathbf{r}^\top[l-j],\ B_k=\mathcal{J}_kD^k+\sum_{j=0}^{k-1}u_jD^j,\ u_j=u_j(x,t_n),\ l\in\{0,\dots\},$$
$$M_n=\alpha_n\partial_{t_n}-A_n-\gamma\mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top,\ A_n=\tilde{\mathcal{J}}_nD^n+\sum_{i=0}^{n-1}v_iD^i,\ v_i=v_i(x,t_n),\ \alpha_n\in\mathbb{C}. \tag{19}$$

Corollary 2. *Lax equation* $[P_{k,s}, M_n] = 0$ *is equivalent to the system*

$$[P_{k,s}, M_n] = 0, \ \left(\gamma P_{k,s} + \sum_{l=0}^{s} c_l (M_n)^{l+1}\right)\{\mathbf{q}\} = 0, \ \left(\gamma P^{\tau}_{k,s} + \sum_{l=0}^{s} c_l (M^{\tau}_n)^{l+1}\right)\{\mathbf{r}\} = 0.$$

We do not tend to consider precisely the case $\gamma = 0$ in hierarchy (10) in this paper. Thus, without loss of generality we put $\gamma = 1$.

For further convenience we will consider the Lax pairs consisting of the operators $\tilde{L}_{k,l}$ (17) and M_n (10) (the operator $\tilde{L}_{k,l}$ is involved in equations for functions $\mathbf{q}$ and $\mathbf{r}$; see formulae (18)). Consider examples of equations given by operators $\tilde{L}_{k,l}$ (17) and M_n (10) that can be obtained under certain choice of (k, n, l). For simplicity, we will also introduce the notation $t{:=}t_0$, $\alpha{:=}\alpha_0$.

1. $k = 2$, $l = 1$, $n = 0$. In this case we obtain the following Lax pair in (10)

$$\begin{gathered}\tilde{L}_{2,1} = L_{2,1} + c_1(M_0)^2 = D^2 + v_0 + c_1\alpha^2\partial_t^2 - 2c_1\alpha\mathbf{q}\mathcal{M}_0 D^{-1}\partial_t \mathbf{r}^\top, \ c \in \mathbb{C}, \\ M_0 = \alpha\partial_t - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top.\end{gathered} \tag{20}$$

The commutator equation $[\tilde{L}_{2,1}, M_0] = 0$ is equivalent to the system

$$\begin{gathered}\mathbf{q}_{xx} + c_1\alpha^2\mathbf{q}_{tt} + v_0\mathbf{q} + c_1\mathbf{q}\mathcal{M}_0 S = 0, \ \mathbf{r}^\top_{xx} + c_1\alpha^2\mathbf{r}^\top_{tt} + \mathbf{r}^\top v_0 + c_1 S\mathcal{M}_0\mathbf{r}^\top = 0, \\ \alpha v_{0t} = -2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)_x, \ S_x = -2\alpha(\mathbf{r}^\top\mathbf{q})_t.\end{gathered} \tag{21}$$

Equation (21) and its integro-differential Lax representation in (2+1)-dimensional case was investigated in [39]. Consider additional reductions of pair of the operators $\tilde{L}_{2,1}$ and M_0 (20) and system (21). After the reduction $c_1 \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\mathbf{r}^\top = \mathbf{q}^*$, $\mathcal{M}_0 = \mathcal{M}_0^*$, the operators $\tilde{L}_{2,1}$ and M_0 are Hermitian and skew-Hermitian respectively, and (21) takes the form

$$\mathbf{q}_{xx} + c_1\alpha^2\mathbf{q}_{tt} + v_0\mathbf{q} + c_1\mathbf{q}\mathcal{M}_0 S = 0, \ \alpha v_{0t} = -2(\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x, \ S_x = -2\alpha(\mathbf{q}^*\mathbf{q})_t. \tag{22}$$

Let us consider (22) in the case where $\alpha = 1$, $u{:=}\mathbf{q}$ and $\mu{:=}\mathcal{M}_0$ are scalars. Then (22) can be rewritten as

$$u_{xx} + c_1 u_{tt} + v_0 u + \mu c_1 S u = 0, \ v_{0,t} = -2\mu(|u|^2)_x, \ S_x = -2(|u|^2)_t. \tag{23}$$

As a consequence of (23) we obtain

$$u_{xx} + c_1 u_{tt} + \mu S_1 u = 0, \ \ S_{1,xt} = -2(|u|^2)_{xx} - 2c_1(|u|^2)_{tt}, \tag{24}$$

where $S_1 = \mu^{-1}v_0 + c_1 S$. This is the well-known stationary Davey-Stewartson system (DS-I) and (22) is therefore a matrix (noncommutative) generalization. The interest in noncommutative versions of DS systems and some other noncommutative nonlinear equations (in particular, solution generating technique) has also arisen recently in [40–43].

2. $k = 2$, $l = 2$, $n = 0$

$$\begin{gathered}\tilde{L}_{2,2} = L_{2,2} + c_2(M_0)^3 = c_2\alpha^3\partial_t^3 + D^2 + v_0 - 3\alpha^2 c_2\mathbf{q}_t\mathcal{M}_0 D^{-1}\mathbf{r}^\top_t + \\ +3\alpha c_2\mathbf{q}\mathcal{M}_0\partial_t D^{-1}\mathbf{r}^\top\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - 3\alpha c_2\mathbf{q}\mathcal{M}_0 D^{-1}\{\mathbf{r}^\top\mathbf{q}\}_t\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \\ -3c_2\alpha^2\partial_t\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top\partial_t, \ M_0 = \alpha\partial_t - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top.\end{gathered} \tag{25}$$

In the vector case ($m=1$) after setting $\mu{:=}\mathcal{M}_0 \in \mathbb{C}$, $\alpha=1$ the commutator equation $[\tilde{L}_{2,2}, M_0]=0$ is equivalent to the system

$$\begin{array}{c} -\mathbf{q}_{xx} - c_2\mathbf{q}_{ttt} - v_0\mathbf{q} + 3c_2\mu(\mathbf{q}S_1)_t - 3c_2\mu\mathbf{q}S_2 = 0, \\ -\mathbf{r}^\top_{xx} - c_2\mathbf{r}^\top_{ttt} - \mathbf{r}^\top v_0 + 3c_2\mu S_1\mathbf{r}^\top_t + 3c_2\mu S_2\mathbf{r}^\top = 0, \\ v_{0t} = -2\mu(\mathbf{q}\mathbf{r}^\top)_x, \ S_{1x} = (\mathbf{r}^\top\mathbf{q})_t, \quad S_{2x} = (\mathbf{r}^\top_t\mathbf{q})_t. \end{array} \tag{26}$$

Thus, the equation given by the commutator $[\tilde{L}_{2,2}, M_0] = 0$ generalizes (26) to the matrix case, but we do not present it because of its rather complicated structure.

3. $k=3, l=1, n=0$.
   In this case the operator $\tilde{L}_{3,1}$ has the form

$$\begin{array}{c} \tilde{L}_{3,1} = L_{3,1} - c_1(M_0)^2 = D^3 + v_1 D + v_0 + c_1\alpha^2\partial_t^2 - 2c_1\alpha\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top_t - \\ -2c_1\alpha\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top\partial_t, \ M_0 = \alpha\partial_t - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top. \end{array} \tag{27}$$

   In the vector case ($N=1$) the equation $[\tilde{L}_{3,1}, M_0]=0$ is equivalent to the system

$$\begin{array}{c} \mathbf{q}_{xxx} + c_1\alpha^2\mathbf{q}_{tt} + v_1\mathbf{q}_x + v_0\mathbf{q} + c_1\mathbf{q}\mathcal{M}_0 S_1 = 0, \\ -\mathbf{r}^\top_{xxx} + c_1\alpha^2\mathbf{r}^\top_{tt} - (\mathbf{r}^\top v_1)_x + \mathbf{r}^\top v_0 + c_1 S_1\mathcal{M}_0\mathbf{r}^\top = 0, \\ \alpha v_{0,t} = -3(\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top)_x, \ \alpha v_{1,t} = -3(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)_x, \ S_{1x} = -2\alpha(\mathbf{r}^\top\mathbf{q})_t. \end{array} \tag{28}$$

4. $k=3, l=2, n=0$.

$$\begin{array}{c} \tilde{L}_{3,2} = D^3 + c_2\alpha^3\partial_t^3 - v_1 D + v_0 - 3\alpha^2 c_2\mathbf{q}_t\mathcal{M}_0 D^{-1}\mathbf{r}^\top_t + \\ +3\alpha c_2\mathbf{q}\mathcal{M}_0\partial_t D^{-1}\mathbf{r}^\top\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - 3\alpha c_2\mathbf{q}\mathcal{M}_0 D^{-1}\{\mathbf{r}^\top\mathbf{q}\}_t\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \\ -3c_2\alpha^2\partial_t\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top\partial_t, \ M_0 = \alpha\partial_t - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top. \end{array} \tag{29}$$

   The Lax equation $[\tilde{L}_{3,2}, M_0]=0$ results in the (1+1)-dimensional matrix mKdV-type system that has rather complicated form. For this reason, we will consider only some special cases of it (matrix generalization of (2+1)-dimensional mKdV system and its Lax representation can be found in [39]):

   (a) Consider the scalar case of the pair (29) (i.e., $N=m=1$), setting $\mathbb{R} \ni \mu{:=}\mathcal{M}_0$, $q(x,t){:=}\mathbf{q}(x,t)$, $r(x,t){:=}\mathbf{r}(x,t)$ and $\alpha=1$. Under the additional Hermitian conjugation reduction: $c_2 \in \mathbb{R}$, $r=\bar{q}$, the Lax equation $[\tilde{L}_{3,2}, M_0]=0$ is equivalent to the equation

$$\begin{array}{c} q_{xxx} + c_2 q_{ttt} - 3\mu q_x \int |q|^2_x dt - 3c_2\mu q_t \int |q|^2_t dx - \\ -3c_2\mu q \int (\bar{q}q_t)_t dx - 3\mu q \int (q_x q)_x dt = 0. \end{array} \tag{30}$$

   after setting $t=x$, $q=\bar{q}$ and $c_2=-2$ (30) takes the form

$$q_{xxx} - 6\mu q^2 q_x = 0, \tag{31}$$

   which is the stationary mKdV equation. System (30) is its complex spatially two-dimensional generalization.

(b) Consider the scalar case ($N = 1$, $m = 1$) of the Lax pair given by (29) under the additional reduction $\beta = 1$, $\mu{:=}\mathcal{M}_0 = 1$, $r{:=}\mathbf{r} = \nu$ with a constant $\nu \in \mathbb{R}$. In terms of $u{:=}\mathbf{q}\nu$ the Lax equation $[\tilde{L}_{3,2}, M_0] = 0$ is equivalent to the following one

$$u_{xxx} + c_2 u_{ttt} - 3D\left\{\left(\int u_x dt\right) u\right\} - 3c_2\partial_t\left\{u\left(\int u_t dx\right)\right\} = 0, \tag{32}$$

which is the stationary case of Nizhnik equation ([44]).

4. Dressing methods for the new bidirectional (1+1)-dimensional k-constrained KP hierarchy. In this section our aim is to consider hierarchy of equations given by the Lax pair (10) in the case of $\gamma = 1$ We suppose that the operators $L_{k,l}$ and M_n in (10) satisfy the commutator equation $[L_{k,l}, M_n] = 0$. At first we recall some results from [45]. Let $N \times K$-matrix functions φ and ψ be solutions of the linear problems

$$M_n\{\varphi\} = \varphi\Lambda,\ M_n^\tau\{\psi\} = \psi\tilde{\Lambda},\ \Lambda, \tilde{\Lambda} \in \mathrm{Mat}_{K\times K}(\mathbb{C}). \tag{33}$$

Introduce binary Darboux transformation (BDT) in the following way

$$W = I - \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1} D^{-1}\psi^\top, \tag{34}$$

where C is a $K \times K$-constant nondegenerate matrix. The inverse operator W^{-1} has the form

$$W^{-1} = I + \varphi D^{-1}\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top. \tag{35}$$

The following theorem is proven in [45].

Theorem 2 ([45])**.** *The operator $\hat{M}_n{:=}WM_nW^{-1}$ obtained from M_n in (10) via BDT (34) has the form*

$$\hat{M}_n{:=}WM_nW^{-1} = \alpha_n\partial_{t_n} - \hat{A}_n - \hat{\mathbf{q}}\mathcal{M}_0 D^{-1}\hat{\mathbf{r}}^\top + \Phi\mathcal{M}_1 D^{-1}\Psi^\top,\ \hat{A}_n = \tilde{\mathcal{J}}_n D^n + \sum_{j=0}^{n-1}\hat{v}_j D^j, \tag{36}$$

where

$$\begin{gathered}\mathcal{M}_1 = C\Lambda - \tilde{\Lambda}^\top C,\ \Phi = \varphi\Delta^{-1},\ \Psi = \psi\Delta^{-1,\top},\ \Delta = C + D^{-1}\{\psi^\top\varphi\},\\ \hat{\mathbf{q}} = W\{\mathbf{q}\},\ \hat{\mathbf{r}} = W^{-1,\tau}\{\mathbf{r}\}.\end{gathered} \tag{37}$$

$\hat{v}_j$ are $N \times N$-matrix coefficients depending on functions φ, ψ and v_j.

Exact forms of all coefficients $\hat{v}_j$ are given in [45].

The following corollary follows from Theorem 2.

Corollary 3. *The functions $\Phi = \varphi\Delta^{-1} = W\{\varphi\}C^{-1}$ and $\Psi = \psi\Delta^{-1,\top} = W^{-1,\tau}\{\psi\}C^{\top,-1}$ satisfy the equations*

$$\hat{M}_n\{\Phi\} = \Phi C\Lambda C^{-1},\ \hat{M}_n^\tau\{\Psi\} = \Psi C^\top\tilde{\Lambda}C^{\top,-1}. \tag{38}$$

For further purposes we will need the following lemmas.

Lemma 1. *Let $\mathcal{M}_{l+1}$ be a matrix of the form $\mathcal{M}_{l+1} = C\Lambda^{l+1} - (\tilde{\Lambda}^\top)^{l+1}C$, $l \in \mathbb{N}$. The following formula holds*

$$\mathcal{M}_{l+1} = \sum_{s=0}^{l} C\Lambda^s C^{-1}\mathcal{M}_1 C^{-1}(\tilde{\Lambda}^\top)^{l-s}C. \tag{39}$$

Proof. The following recurrent formulae that can be easily checked by direct calculation

$$\mathcal{M}_2 = C\Lambda C^{-1}\mathcal{M}_1 + \mathcal{M}_1 C\tilde{\Lambda}^\top C^{-1}, \tag{40}$$

$$\mathcal{M}_{l+1} = C\Lambda C^{-1}\mathcal{M}_l + \mathcal{M}_l C^{-1}\tilde{\Lambda}^\top C - C\Lambda C^{-1}\mathcal{M}_{l-1}C^{-1}\tilde{\Lambda}^\top C. \tag{41}$$

Using formulae (40)–(41) and induction on k, we can prove that the following formula holds

$$\mathcal{M}_{l+1} = \sum_{s=0}^{k} C\Lambda^s C^{-1}\mathcal{M}_{l-k+1}C^{-1}(\tilde{\Lambda}^\top)^{k-s}C - \sum_{s=1}^{k} C\Lambda^s C^{-1}\mathcal{M}_{l-k}C^{-1}(\tilde{\Lambda}^\top)^{k-s+1}C, \tag{42}$$

for some $k \le l-2$. After the substitution of $k = l-2$ in (42) and using (40) we can obtain formula (39). This finishes the proof of formula (39) and Lemma 1. □

Lemma 2. *The following formula* $\Phi\mathcal{M}_{l+1}D^{-1}\Psi^\top = \sum_{s=0}^{l}\Phi[s]\mathcal{M}_1 D^{-1}\Psi^\top[l-s]$ *holds, where*

$$\Phi[j]{:=}(\hat{M}_n)^j\{\Phi\},\ \Psi[j]{:=}(\hat{M}_n^\tau)^j\{\Psi\}. \tag{43}$$

Proof. Lemma 2 is a consequence of Corollary 1 and formula (39) of Lemma 1. Namely, the following relations hold

$$\Phi\mathcal{M}_{l+1}D^{-1}\Psi^\top = \sum_{s=0}^{l}\Phi C\Lambda^s C^{-1}\mathcal{M}_1 C^{-1}D^{-1}(\tilde{\Lambda}^\top)^{l-s}C\Psi^\top = \sum_{s=0}^{l}\Phi[s]\mathcal{M}_1 D^{-1}\Psi^\top[l-s]. \quad \square$$

Now we assume that the functions φ and ψ in addition to equations (33) satisfy the equations

$$L_{k,l}\{\varphi\} = -c_l\varphi\Lambda^{l+1} = -c_l M_n^{l+1}\{\varphi\},\ L^\tau_{k,l}\{\psi\} = -c_l\psi\tilde{\Lambda}^{l+1} = -c_l(M_n^\tau)^{l+1}\{\psi\}. \tag{44}$$

Problems (44) can be rewritten via the operator $\tilde{L}_{k,l}$ (17) as $\tilde{L}_{k,l}\{\varphi\} = 0,\ \tilde{L}^\tau_{k,l}\{\psi\} = 0$.

The following theorem for the operators $L_{k,l}$ (10) and $\tilde{L}_{k,l}$ (17) holds.

Theorem 3. *Let $N \times K$-matrix functions φ, ψ be solutions of problems (33) and (44). The transformed operator $\hat{L}_{k,l}{:=}WL_{k,l}W^{-1}$ obtained via BDT W (34) has the form*

$$\hat{L}_{k,l}{:=}WL_{k,l}W^{-1} = \hat{B}_k + c_l\sum_{j=0}^{l}\hat{\mathbf{q}}[j]\mathcal{M}_0 D^{-1}\hat{\mathbf{r}}^\top[l-j] +$$
$$+ c_l\sum_{s=0}^{l}\Phi[s]\mathcal{M}_1 D^{-1}\Psi^\top[l-s],\ \hat{B}_k = \mathcal{J}_k D^k + \sum_{i=0}^{k-1}\hat{u}_i D^i, \tag{45}$$

where the matrix $\mathcal{M}_n$ and the functions $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$, $\Phi[s]$, $\Psi[l-s]$ are defined by formulae (37), (43) and $\hat{\mathbf{q}}[j]$, $\hat{\mathbf{r}[j]}$ have the form

$$\hat{\mathbf{q}}[j] = (\hat{M}_n^j)\{\hat{\mathbf{q}}\},\ \ \hat{\mathbf{r}}[j] = (\hat{M}_n^j)^\tau\{\hat{\mathbf{r}}\},$$

$\hat{v}_i$ are $N \times N$-matrix coefficients that depend on the functions φ, ψ and v_i. The transformed operator $\hat{\tilde{L}}_{k,l} = W\tilde{L}_{k,l}W^{-1}$ has the form

$$\hat{\tilde{L}}_{k,l} = W\tilde{L}_{k,l}W^{-1} = \hat{L}_{k,l} + c_l(\hat{M}_n)^{l+1}, \tag{46}$$

where $\hat{M}_n$ is given by (36).

Proof. We shall rewrite the operator $L_{k,l}$ (10) in the form

$$L_{k,l} = \mathcal{J}_k D^k + \sum_{i=0}^{k-1} u_i D^i + c_l \tilde{\mathbf{q}} \tilde{\mathcal{M}}_0 D^{-1} \tilde{\mathbf{r}}^\top, \tag{47}$$

where $\tilde{\mathcal{M}}_0$ is an $m(l+1)\times m(l+1)$-block-diagonal matrix with entries of $\mathcal{M}_0$ at the diagonal; $\tilde{\mathbf{q}}$:=$(\mathbf{q}[0], \mathbf{q}[1], \ldots, \mathbf{q}[l])$, $\tilde{\mathbf{r}}$:=$(\mathbf{r}[l], \mathbf{r}[l-1], \ldots, \mathbf{r}[0])$. Using Theorem 2 we obtain that

$$\hat{L}_{k,l} = \mathcal{J}_k D^k + \sum_{i=0}^{k-1} \hat{u}_i D^i + c_l \hat{\tilde{\mathbf{q}}} \tilde{\mathcal{M}}_0 D^{-1} \hat{\tilde{\mathbf{r}}}^\top + \Phi \mathcal{M}_{l+1} D^{-1} \Psi^\top, \tag{48}$$

where $\hat{\tilde{\mathbf{q}}} = W\{\tilde{\mathbf{q}}\}$, $\hat{\tilde{\mathbf{q}}} = W^{-1,\tau}\{\tilde{\mathbf{r}}\}$. Using the exact form of $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ we have

$$\hat{\tilde{\mathbf{q}}} = W\{\tilde{\mathbf{q}}\} = (W\{\mathbf{q}[0]\}, \ldots, W\{\mathbf{q}[l]\}),\ \hat{\tilde{\mathbf{r}}} = W^{-1,\tau}\{\tilde{\mathbf{r}}\} = (W^{-1,\tau}\{\mathbf{r}[l]\}, \ldots, W^{-1,\tau}\{\mathbf{r}[0]\}).$$

We observe that $W\{\mathbf{q}[i]\} = WL^i\{\mathbf{q}\} = WL^iW^{-1}\{W\{\mathbf{q}\}\} = \hat{L}^i\{\hat{\mathbf{q}}\} =: \hat{\mathbf{q}}[i]$. It can be shown analogously that $W^{-1,\tau}\{\mathbf{r}[i]\} = \hat{L}^{\tau,i}\{W^{-1,\tau}\{\mathbf{r}\}\} = \hat{L}^{\tau,i}\{\hat{\mathbf{r}}\} =: \hat{\mathbf{r}}[i]$. Thus we have

$$\hat{\tilde{\mathbf{q}}} \tilde{\mathcal{M}}_0 D^{-1} \hat{\tilde{\mathbf{r}}}^\top = \sum_{j=0}^{l} \hat{\mathbf{q}}[j] \mathcal{M}_0 D^{-1} \hat{\mathbf{r}}^\top [l-j]. \tag{49}$$

For the last item in (48) from Lemma 2 we have

$$\Phi \mathcal{M}_{l+1} D^{-1} \Psi^\top = \sum_{s=0}^{l} \Phi[s] \mathcal{M}_1 D^{-1} \Psi^\top [l-s]. \tag{50}$$

Using formulae (48)–(50) we obtain that the operator $\hat{M}_{n,l}$ has form (45). The exact form of the operator $\hat{\hat{M}}_{n,l}$ follows from formula (45) and Theorem 2. □

From Theorem 3 we obtain the following corollary.

Corollary 4. *Assume that functions φ and ψ satisfy problems (33) and (44). Then the functions $\Phi = W\{\varphi\}C^{-1}$ and $\Psi = W^{-1,\tau}\{\psi\}C^{\top,-1}$ (see formulae (37)) satisfy the equations*

$$\hat{\hat{L}}_{k,l}\{\Phi\} = \hat{L}_{k,l}\{\Phi\} + c_l(\hat{M}_n)^{l+1}\{\Phi\} = 0,\ \hat{\hat{L}}^\tau_{k,l}\{\Psi\} = \hat{L}^\tau_{k,l}\{\Psi\} + c_l(\hat{M}^\tau_n)^{l+1}\{\Psi\} = 0, \tag{51}$$

where the operators $\hat{L}_{k,l}$, $\hat{\hat{L}}_{k,l}$ and $\hat{M}_n$ are defined by (36), (45) and (46).

As an example, we will consider dressing methods for equations connected with the operators $\tilde{L}_{2,1}$, M_0. Assume that φ and ψ are $N \times K$-matrix functions that satisfy the equations

$$M_0\{\varphi\} = \varphi\Lambda,\ M_0^\tau\{\psi\} = \psi\tilde{\Lambda},\ M_0 := \alpha\partial_t. \tag{52}$$

By Theorem 2, we obtain that the dressed operator $\hat{M}_0$ via BDT W (34) has the form

$$\hat{M}_0 = WM_0W^{-1} = \alpha\partial_t + \Phi\mathcal{M}_1 D^{-1}\Psi^\top. \tag{53}$$

Assume that $N\times K$-matrix functions φ and ψ in addition to equations (52) also satisfy the equations

$$L_{2,1}\{\varphi\} = -c_1\varphi\Lambda^2 = -c_1(M_0)^2\{\varphi\},\ L_{2,1}^{\tau}\{\psi\} = -c_1\psi\tilde{\Lambda}^2 = -c_1(M_0^{\tau})^2\{\psi\},\ L_{2,1}{:=}D^2. \quad (54)$$

By Theorem 3, we obtain that the transformed operator $\hat{L}_{2,1}$ has the form

$$\hat{L}_{2,1} = WL_{2,1}W^{-1} = D^2 + \hat{v}_0 + \hat{M}_0\{\Phi\}\mathcal{M}_1D^{-1}\Psi^{\top} + \Phi\mathcal{M}_1D^{-1}((\hat{M}_0^{\tau})\{\Psi\})^{\top}.$$

By direct calculations, it can be obtained that $\hat{v}_0 = 2(\varphi\Delta^{-1}\psi^{\top})_x$, $\Delta = C + D^{-1}\{\psi^{\top}\varphi\}$. It can be easily checked that

$$\begin{gathered}\alpha(\varphi\Delta^{-1}\psi^{\top})_t = \alpha\varphi_t\Delta^{-1}\psi^{\top} - \alpha\varphi\Delta^{-1}D^{-1}\{\psi^{\top}\varphi\}_t\Delta^{-1}\psi^{\top} + \alpha\varphi\Delta^{-1}\psi_t^{\top} = \\ = \varphi\Delta^{-1}(C\Lambda + \alpha D^{-1}\{\psi^{\top}\varphi_t\})\Delta^{-1}\psi^{\top} - \alpha\varphi\Delta^{-1}D^{-1}\{\psi^{\top}\varphi\}_t\Delta^{-1}\psi^{\top} + \\ +\varphi\Delta^{-1}(-\tilde{\Lambda}^{\top}C + \alpha D^{-1}\{\psi_t^{\top}\varphi\})\Delta^{-1}\psi^{\top} = \Phi\mathcal{M}_1\Psi^{\top}.\end{gathered} \quad (55)$$

From the latter formula we obtain that

$$\alpha\hat{v}_{0t} = 2\alpha(\varphi\Delta^{-1}\psi^{\top})_{xt} = 2(\Phi\mathcal{M}_1\Psi^{\top})_x. \quad (56)$$

From Corollary 4 we see that the functions $\Phi = \varphi\Delta^{-1}$ and $\Psi = \psi\Delta^{\top,-1}$ where $\Delta = C + D^{-1}\{\psi^{\top}\varphi\}$ (see formulae (37)) satisfy equations (51). After the change $\mathbf{q}{:=}\Phi$, $\mathbf{r}{:=}\Psi$, $\mathcal{M}_0{:=}-\mathcal{M}_1$, $v_0{:=}\hat{v}_0$ from formulae (51) and (56) we obtain that $N\times K$-matrix functions $\mathbf{q}$, $\mathbf{r}$, an $N\times N$-matrix function v_0, a $K\times K$-matrix function $S = 2\alpha(\Delta^{-1})_t$ and a $K\times K$-matrix $\mathcal{M}_0$ satisfy equations (21). It can be checked that in the case of additional reductions in formulae (52)–(54): $\alpha\in\mathbb{R}$, $c_1\in\mathbb{R}$, $\tilde{\Lambda} = -\bar{\Lambda}$, $\psi = \bar{\varphi}$ and $C = C^*$ in gauge transformation operator W (34) it can be checked by direct calculations that the functions $\mathbf{q}{:=}\Phi$, $S_1 = 2\alpha(\Delta^{-1})_t$ and $v_0 = \hat{v}_0 = 2(\varphi\Delta^{-1}\varphi^*)_x$ satisfy matrix DS system (21) with $\mathcal{M}_0 = -\mathcal{M}_1$.

From the previous considerations we obtain that in the scalar case ($N = 1$, $m = 1$), $\mu{:=}\mathcal{M}_0 = -\mathcal{M}_1 = -C(\Lambda + \bar{\Lambda})$ under the condition $\alpha = 1$ the functions

$$u = q = \frac{\exp(\theta)}{\Delta},\ S = -2\frac{\operatorname{Re}(\Lambda)\exp(2\operatorname{Re}(\theta))}{\operatorname{Re}(i\sqrt{c_1}\Lambda)\Delta^2},\ v_0 = -2\mu\frac{\operatorname{Re}(i\sqrt{c_1}\Lambda)\exp(2\operatorname{Re}(\theta))}{\operatorname{Re}(\Lambda)\ \Delta^2}, \quad (57)$$

where $\Delta = -\frac{\mu}{2\operatorname{Re}(\Lambda)} + \frac{1}{2\operatorname{Re}(i\sqrt{c_1}\Lambda)}\exp(2\operatorname{Re}(\theta))$ and $\theta = i\sqrt{c_1}\Lambda x + \Lambda t$, satisfy the scalar DS system (23) (see also (24))

$$u_{xx} + c_1u_{tt} + v_0u + \mu c_1Su = 0,\ v_{0t} = -2\mu|u|_x^2,\ S_x = -2|u|_t^2. \quad (58)$$

The functions u and $S_1 = \mu^{-1}v_0 + c_1S$ are therefore solutions of differential consequence (58)

$$u_{xx} + c_1u_{tt} + \mu S_1u = 0,\ \ S_{1,xt} = -2(|u|^2)_{xx} - 2c_1(|u|^2)_{tt}. \quad (59)$$

Consider special cases of (59) and its solutions

1. $c_1 = 1$.
   (a) $\mu = 1$. In this case the functions u and $S_1 = \mu^{-1}v_0 + c_1S$, where v_0 and S_1 are defined by (57), represent regular solutions of (59) in the case where $(\operatorname{Re}(\Lambda))(\operatorname{Im}(\Lambda)) > 0$ (in the case of $(\operatorname{Re}(\Lambda))(\operatorname{Im}(\Lambda)) < 0$ u and S_1 are singular)

(b) $\mu = -1$. In this case the functions u and $S_1 = \mu^{-1}v_0 + c_1 S$, where v_0 and S_1 are defined by (57), represent regular solutions of (59) in the case of $(\mathrm{Re}(\Lambda))(\mathrm{Im}(\Lambda)) < 0$ (in the case where $(\mathrm{Re}(\Lambda))(\mathrm{Im}(\Lambda)) > 0$ u and S_1 are singular)

2. $c_1 = -1$.
   (a) $\mu = 1$. In this case the functions u and $S_1 = \mu^{-1}v_0 + c_1 S$, where v_0 and S_1 are given by (57), represent regular solutions of (59).
   (b) $\mu = -1$. In this case the functions u and $S_1 = \mu^{-1}v_0 + c_1 S$, where v_0 and S_1 are defined by (57), represent singular solutions of (59).

The construction of wider classes of solutions (e.g., soliton solutions) for vector and matrix nonlinear systems from (1+1)-BDk-cKP hierarchy would take too much space in this paper. Corresponding ideas can be found in [33, 34, 46].

5. Conclusions. In this paper we introduced a new (1+1)-BDk-cKP hierarchy (10) that generalizes matrix k-constrained KP hierarchy given by (3) and (4) that was investigated in [17–21]. We point out that an important case of hierarchy (10) ($\gamma = 0$) is not precisely investigated in this paper. In particular, dressing methods for this case still have to be elaborated. As an example, let us consider the case where $\gamma = 0$, $k = 1$, $s = 1$, $n = 2$ of hierarchy (10).

Then operators $P_{k,s}$ and M_n (19) take the form

$$P_{1,1} = D + c_1\left(\alpha_2 \mathbf{q}_{t_2}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \alpha_2\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top_{t_2} - \mathbf{q}_{xx}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \right.$$
$$\left. -\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top_{xx} - u\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top u\right) + c_0\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top,\ M_2 = \alpha_2\partial_{t_2} - D^2 - u.$$

According to Corollary 2, the operator equation $[P_{1,1}, M_2] = 0$ is equivalent to the system

$$[P_{1,1}, M_2]_{\geq 0} = 0,\ c_1 M_2^2\{\mathbf{q}\} + c_0 M_2\{\mathbf{q}\} = 0,\ c_1(M_2^\tau)^2\{\mathbf{r}\} + c_0 M_2^\tau\{\mathbf{r}\} = 0. \qquad (60)$$

We also point out that the latter system can be rewritten in terms of variables $\mathbf{q}_0 := \mathbf{q}$, $\mathbf{r}_0 := \mathbf{r}$ and $\mathbf{q}_1 := M_2\{\mathbf{q}_0\}$, $\mathbf{r}_1 := M_2^\tau\{\mathbf{r}_0\}$

$$\begin{array}{c} [P_{1,1}, M_2]_{\geq 0} = 0,\ \mathbf{q}_1 = M_2\{\mathbf{q}_0\},\ \mathbf{r}_1 = M_2^\tau\{\mathbf{r}_0\}, \\ c_1 M_2\{\mathbf{q}_1\} + c_0\mathbf{q}_1 = 0,\ c_1 M_2^\tau\{\mathbf{r}_1\} + c_0\mathbf{r}_1 = 0. \end{array} \qquad (61)$$

System (60) in the vector case ($N = 1$) can be rewritten in the following form

$$\begin{array}{c} c_1(\alpha_2^2\mathbf{q}_{t_2t_2} - 2\alpha_2\mathbf{q}_{xxt_2} + \mathbf{q}_{xxxx} - \alpha_2(u\mathbf{q})_{t_2} + (u\mathbf{q})_{xx} - \alpha_2 u\mathbf{q}_{t_2} + u\mathbf{q}_{xx} + u^2\mathbf{q}) + \\ +c_0(\alpha_2\mathbf{q}_{t_2} - \mathbf{q}_{xx} - u\mathbf{q}) = 0, \\ c_1(\alpha_2^2\mathbf{r}_{t_2t_2} + 2\alpha_2\mathbf{r}_{xxt_2} + \mathbf{r}_{xxxx} + \alpha_2(u\mathbf{r})_{t_2} + (u\mathbf{r})_{xx} + \alpha_2 u\mathbf{r}_{t_2} + u\mathbf{r}_{xx} + u^2\mathbf{r}) + \\ +c_0(-\alpha_2\mathbf{r}_{t_2} - \mathbf{r}_{xx} - u\mathbf{r}) = 0, \\ u = 2\left(\frac{c_1(\alpha_2\mathbf{q}_{t_2}\mathcal{M}_0\mathbf{r}^\top - \alpha_2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top_{t_2} - \mathbf{q}_{xx}\mathcal{M}_0\mathbf{r}^\top - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top_{xx}) + c_0\mathbf{q}\mathcal{M}_0\mathbf{r}^\top}{1 + 4c_1\mathbf{q}\mathcal{M}_0\mathbf{r}^\top}\right). \end{array} \qquad (62)$$

System (62) can be rewritten in the following form via equations (61)

$$\begin{array}{c} \alpha_2\mathbf{q}_{0,t_2} = \mathbf{q}_1 + \mathbf{q}_{0,xx} + u\mathbf{q}_0,\ \ \alpha_2\mathbf{r}_{0,t_2} = -\mathbf{r}_1 - \mathbf{r}_{0,xx} - u\mathbf{r}_0, \\ \alpha_2 c_1\mathbf{q}_{1,t_2} = c_1(\mathbf{q}_{1,xx} + u\mathbf{q}_1) - c_0\mathbf{q}_1,\ \ \alpha_2 c_1\mathbf{r}_{1,t_2} = c_0\mathbf{r}_1 - c_1(\mathbf{r}_{1,xx} + u\mathbf{r}_1), \\ u = 2\left(c_1(\mathbf{q}_1\mathcal{M}_0\mathbf{r}_0^\top + \mathbf{q}_0\mathcal{M}_0\mathbf{r}_1^\top) + c_0\mathbf{q}_0\mathcal{M}_0\mathbf{r}_0^\top\right). \end{array} \qquad (63)$$

Systems (62) and (63) are different forms of the generalization of the vector nonlinear Schrödinger system (NLS) in l-direction ($l = 1$). NLS can be obtained from the latter systems in the particular case ($c_1 = 0$) after the additional Hermitian conjugation reduction $\mathbf{q} = \bar{\mathbf{r}}$, $\alpha_2 \in i\mathbb{R}$. Analogous generalizations in l-direction can be made for Yajima-Oikawa hierarchy ($k = 2$, $s = 0$, $\gamma = 0$ in the hierarchy (19)) and Melnikov hierarchy ($k = 3$, $s = 0$, $\gamma = 0$ in the hierarchy (19)). Investigations in this direction will be made in another paper.

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