УДК 517.98

Yu. G. Kovalev

1D NONNEGATIVE SCHRÖDINGER OPERATORS WITH POINT INTERACTIONS

Yu. G. Kovalev. *1D nonnegative Schrödinger operators with point interactions*, Mat. Stud. **39** (2013), 150–163.

Let Y be an infinite discrete set of points in $\mathbb{R}$, satisfying the condition $\inf\{|y-y'|,\ y,y'\in Y,\ y'\neq y\}>0$. In the paper we prove that the systems $\{\delta(x-y)\}_{y\in Y}$, $\{\delta'(x-y)\}_{y\in Y}$, $\{\delta(x-y),\ \delta'(x-y)\}_{y\in Y}$ form Riesz bases in the corresponding closed linear spans in the Sobolev spaces $W_2^{-1}(\mathbb{R})$ and $W_2^{-2}(\mathbb{R})$. As an application, we prove the transversalness of the Friedrichs and Kreĭn nonnegative selfadjoint extensions of the nonnegative symmetric operators A_0, A', and H_0 defined as restrictions of the operator $A=-\frac{d^2}{dx^2}$, $\mathrm{dom}(A)=W_2^2(\mathbb{R})$ to the linear manifolds $\mathrm{dom}(A_0)=\left\{f\in W_2^2(\mathbb{R})\colon f(y)=0,\ y\in Y\right\}$, $\mathrm{dom}(A')=\{g\in W_2^2(\mathbb{R})\colon g'(y)=0,\ y\in Y\}$, and $\mathrm{dom}(H_0)=\left\{f\in W_2^2(\mathbb{R})\colon f(y)=0,\ f'(y)=0,\ y\in Y\right\}$, respectively. Using the divergence forms, the basic nonnegative boundary triplets for A_0^*, A'^*, and H_0^* are constructed.

Ю. Г. Ковалев. *1D неотрицательные операторы Шрёдингера с точечными взаимодействиями* // Мат. Студії. – 2013. – Т.39, №2. – C.150–163.

Пусть Y бесконечное дискретное множество точек в $\mathbb{R}$, удовлетворяющее условию $\inf\{|y-y'|,\ y,y'\in Y,\ y'\neq y\}>0$. Мы показываем, что системы $\{\delta(x-y)\}_{y\in Y}$, $\{\delta'(x-y)\}_{y\in Y}$, $\{\delta(x-y),\ \delta'(x-y)\}_{y\in Y}$ образуют базисы Рисса в соответствующих замкнутых линейных оболочках в пространствах Соболева $W_2^{-1}(\mathbb{R})$ и $W_2^{-2}(\mathbb{R})$. В приложении мы доказываем трансверсальность неотрицательных самосопряженных расширений Фридрихса и Крейна неотрицательных симметрических операторов A_0, A' и H_0, определенных как сужение оператора $A=-\frac{d^2}{dx^2}$, $\mathrm{dom}(A)=W_2^2(\mathbb{R})$ на линейные многообразия $\mathrm{dom}(A_0)=\left\{f\in W_2^2(\mathbb{R})\colon f(y)=0,\ y\in Y\right\}$, $\mathrm{dom}(A')=\left\{g\in W_2^2(\mathbb{R})\colon g'(y)=0,\ y\in Y\right\}$ и $\mathrm{dom}(H_0)=\left\{f\in W_2^2(\mathbb{R})\colon f(y)=0,\ f'(y)=0,\ y\in Y\right\}$ соответственно. Используя дивергентную форму, построены базисные неотрицательные граничные тройки для A_0^*, A'^* и H_0^*.

1. Introduction. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_-=\{j\in\mathbb{Z},\ j\le -1\}$, $\mathbb{Z}_+=\{j\in\mathbb{Z},\ j\ge 1\}$. By $\mathbb{J}$ we will denote one of the sets $\mathbb{Z}$, $\mathbb{Z}_-$, $\mathbb{Z}_+$. Let Y be a finite or infinite monotone sequence of points in $\mathbb{R}$. When Y is infinite we will suppose that

$$\inf\{|y_j-y_k|,\ j\neq k\}=d>0. \tag{1}$$

For an infinite Y, the following three cases are possible

$$Y=\{y_j,\ j\in\mathbb{Z}\},\ \text{if}\ \inf\{Y\}=-\infty\ \text{and}\ \sup\{Y\}=+\infty,$$
$$Y=\{y_j,\ j\in\mathbb{Z}_-\},\ \text{if}\ y_{-1}=\sup\{Y\}<+\infty,\ Y=\{y_j,\ j\in\mathbb{Z}_+\},\ \text{if}\ y_1=\inf\{Y\}>-\infty.$$

Clearly, the notation $Y=\{y_j,:j\in\mathbb{J}\}$ serves all these cases.

2010 *Mathematics Subject Classification:* 34L40, 46E35, 47A20, 47B25, 81Q10.
Keywords: point interaction; Riesz basis; boundary triplet; the Friedrichs extension; the Kreĭn extension.

Let $W_2^{\pm 1}(\mathbb{R})$, $W_2^{\pm 2}(\mathbb{R})$ be Sobolev spaces. Define in the Hilbert space $L_2(\mathbb{R})$ the linear operators

$$\operatorname{dom}(A_0) = \{f \in W_2^2(\mathbb{R})\colon f(y) = 0,\ y \in Y\},\ A_0 := -\frac{d^2}{dx^2}, \tag{2}$$

$$\operatorname{dom}(A') = \{g \in W_2^2(\mathbb{R})\colon g'(y) = 0,\ y \in Y\},\ A' := -\frac{d^2}{dx^2}, \tag{3}$$

$$\operatorname{dom}(H_0) = \{f \in W_2^2(\mathbb{R})\colon f(y) = 0,\ f'(y) = 0,\ y \in Y\},\ H_0 := -\frac{d^2}{dx^2}. \tag{4}$$

The operators A_0, A', and H_0 are basic for investigations of Hamiltonians on the real line corresponding to the δ, δ' and $\delta - \delta'$ interactions, respectively ([1]). They are symmetric, densely defined, closed, and nonnegative ([1]), and are restrictions of the selfadjoint and nonnegative operator A defined by

$$\operatorname{dom}(A) = W_2^2(\mathbb{R}),\ A = -\frac{d^2}{dx^2}. \tag{5}$$

In addition, the operators A_0 and A' are symmetric extensions of the operator H_0. The adjoint operators are given by

$$\begin{gathered} \operatorname{dom}(A_0^*) = W_2^1(\mathbb{R}) \cap W_2^2(\mathbb{R} \setminus Y),\ A_0^* = -\frac{d^2}{dx^2}, \\ \operatorname{dom}(A'^*) = \{g \in W_2^2(\mathbb{R})\colon g'(y+) = g'(y-),\ y \in Y\},\ A'^* = -\frac{d^2}{dx^2}, \\ \operatorname{dom}(H_0^*) = W_2^2(\mathbb{R} \setminus Y),\ H_0^* = -\frac{d^2}{dx^2}. \end{gathered} \tag{6}$$

It is well known ([1]) that

$$\delta_y = \delta(x-y) \in W_2^{-1}(\mathbb{R}) \setminus L_2(\mathbb{R}),\ (\delta_y)' = \delta'(x-y) \in W_2^{-2}(\mathbb{R}) \setminus W_2^{-1}(\mathbb{R}), \tag{7}$$

where $\delta(x-y)$ and $\delta'(x-y)$ are the delta-function and its derivative.

We have the following chain of Hilbert spaces $W_2^2(\mathbb{R}) \subset W_2^1(\mathbb{R}) \subset L_2(\mathbb{R}) \subset W_2^{-1}(\mathbb{R}) \subset W_2^{-2}(\mathbb{R})$ The triplets $W_2^2(\mathbb{R}) \subset L_2(\mathbb{R}) \subset W_2^{-2}(\mathbb{R})$ and $W_2^1(\mathbb{R}) \subset L_2(\mathbb{R}) \subset W_2^{-1}(\mathbb{R})$ are rigged Hilbert spaces, i.e., the Hilbert space $W_2^{-2}(\mathbb{R})$ ($W_2^{-1}(\mathbb{R})$, respectively) is the set of all continuous anti-linear functionals on $W_2^2(\mathbb{R})$ (on $W_2^1(\mathbb{R})$, respectively, [6]).

Let $Y = \{y_j \in \mathbb{R},\ j \in \mathbb{J}\}$ be a discrete set in $\mathbb{R}$ satisfying (1). Define the following subspaces

$$\begin{gathered} \Phi = \overline{\operatorname{span}}_{W_2^{-2}(\mathbb{R})} \{\delta'(x-y),\ y \in Y\} \quad \text{(the closure in} \quad W_2^{-2}(\mathbb{R})), \\ \Psi_{-1} = \overline{\operatorname{span}}_{W_2^{-1}(\mathbb{R})} \{\delta(x-y),\ y \in Y\} \quad \text{(the closure in} \quad W_2^{-1}(\mathbb{R})), \\ \Psi_{-2} = \overline{\operatorname{span}}_{W_2^{-2}(\mathbb{R})} \{\delta(x-y), \quad y \in Y\} \quad \text{(the closure in} \quad W_2^{-2}(\mathbb{R})), \\ \Omega = \overline{\operatorname{span}}_{W_2^{-2}(\mathbb{R})} \{\delta(x-y),\ \delta'(x-y), \quad y \in Y\} \quad \text{(the closure in} \quad W_2^{-2}(\mathbb{R})). \end{gathered}$$

Clearly, $\Psi_{-1} \subseteq \Psi_{-2}$. It is known ([1]) that $\Phi \cap L_2(\mathbb{R}) = \{0\}$, $\Psi_{-2} \cap L_2(\mathbb{R}) = \{0\}$, $\Omega \cap L_2(\mathbb{R}) = \{0\}$. Therefore, the operators A', A_0, and H_0 are densely defined and

$$\operatorname{dom}(A') = \{f \in W_2^2(\mathbb{R})\colon (f,\varphi) = 0,\ \varphi \in \Phi\}, \tag{8}$$

$$\operatorname{dom}(A_0) = \{f \in W_2^2(\mathbb{R})\colon (f,\psi) = 0,\ \psi \in \Psi_{-2}\}, \tag{9}$$

$$\operatorname{dom}(H_0) = \{f \in W_2^2(\mathbb{R})\colon (f,\omega) = 0,\ \omega \in \Omega\}. \tag{10}$$

In this paper we establish some new connections between the Sobolev spaces $W_2^1(\mathbb{R})$, $W_2^2(\mathbb{R})$ and the Hilbert space ℓ_2. Using these connections we prove that

- $\Psi_{-1} = \Psi_{-2}$;
- the systems $\{\delta(x-y_j)\}_{j\in\mathbb{J}}$, $\{\delta'(x-y_j)\}_{j\in\mathbb{J}}$, $\{\delta(x-y_j),\ \delta'(x-y_j)\}_{j\in\mathbb{J}}$ form the Riesz bases of the subspaces Ψ_{-2}, Φ, and Ω, respectively;
- the Friedrichs and Kreĭn extensions of A', A_0, and H_0 are mutually transversal.

Finally, we construct *basic positive boundary triplets* ([2], [3]) for A'^*, A_0^*, and H_0^* and give descriptions of all nonnegative selfadjoint extensions.

2. The Sobolev spaces $W_2^1(\mathbb{R})$, $W_2^2(\mathbb{R})$ and the Hilbert space ℓ_2. In this Section we establish some connections between the Hilbert spaces $W_2^1(\mathbb{R})$, $W_2^2(\mathbb{R})$ and the Hilbert space $\ell_2(\mathbb{J})$.

Proposition 1. *Suppose Y is infinite and* (1) *holds. Then*

1) *If $g \in W_2^2(\mathbb{R})$ then the sequences $\{g(y_j),\ y_j \in Y\}$ and $\{g'(y_j),\ y_j \in Y\}$ belong to $\ell_2(\mathbb{J})$. Moreover, there exists a positive constants c such that*

$$\|\{g(y_j)\}\|_{\ell_2(\mathbb{J})} \le c\|g\|_{W_2^2(\mathbb{R})},\ \|\{g'(y_j)\}\|_{\ell_2(\mathbb{J})} \le c\|g\|_{W_2^2(\mathbb{R})},\ \forall g \in W_2^2(\mathbb{R}).$$

2) *If $\{a_j,\ j\in\mathbb{J}\}$, $\{b_j,\ j\in\mathbb{J}\} \in \ell_2(\mathbb{J})$ then there exists a function $g \in W_2^2(\mathbb{R})$ such that $g(y_j) = a_j$, $g'(y_j) = b_j$, $\forall j \in \mathbb{J}$.*

Proof. 1) Let $g \in W_2^2(\mathbb{R})$. One can verify that the equalities

$$g(y_j)=\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y_j|}(g(x)-\operatorname{sgn}(x-y_j)g'(x))dx,\ g'(y_j)=\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y_j|}(g'(x)-\operatorname{sgn}(x-y_j)g''(x))dx$$

hold. Further

$$|g(y_j)| \le \frac{1}{2}\sum_{n\in\mathbb{J}}\left(\int_{y_{n-1}}^{y_n} e^{-2|x-y_j|}dx\right)^{1/2}\left(\int_{y_{n-1}}^{y_n} |g(x)-\operatorname{sgn}(x-y_j)g'(x)|^2\,dx\right)^{1/2} = \frac{1}{2}\sum_{n\in\mathbb{J}} M_{jn}h_n,$$

where $\{h_n,\ n\in\mathbb{J}\} \in \ell_2(\mathbb{J})$ because

$$\sum_{n\in\mathbb{J}} h_n^2 = \sum_{n\in\mathbb{J}}\int_{y_{n-1}}^{y_n} |g(x)-\operatorname{sgn}(x-y_j)g'(x)|^2\,dx \le$$

$$\le 2\sum_{n\in\mathbb{J}}\int_{y_{n-1}}^{y_n}\left(|g(x)|^2+|g'(x)|^2\right)dx \le 2\|g\|^2_{W_2^2(\mathbb{R})} < \infty,$$

$$\sum_{n\in\mathbb{J}} M_{jn} = \sum_{n\in\mathbb{J}}\left(\int_{y_{n-1}}^{y_n} e^{-2|x-y_j|}dx\right)^{1/2} \le \sum_{n\in\mathbb{J}}\frac{1}{\sqrt{2}}\left\{\begin{array}{ll} e^{-|y_n-y_j|}, & n\le j,\\ e^{-|y_{n-1}-y_j|}, & n\ge j+1,\end{array}\right\} \le$$

$$\le \sqrt{2}\sum_{m\in\mathbb{Z}} e^{-|m|d} = \sqrt{2}\,\frac{e^d+1}{e^d-1}.$$

Let M be the linear operator in $\ell_2(\mathbb{J})$ given by the matrix $\|M_{jn}\|_{j,n\in\mathbb{J}}$. Then the Holmgren bound of M ([1, Appendix C]) satisfies

$$\|M\|_H = \left(\sup_{j\in\mathbb{J}}\sum_{n\in\mathbb{J}}|M_{jn}|\right)^{1/2}\left(\sup_{n\in\mathbb{J}}\sum_{j\in\mathbb{J}}|M_{jn}|\right)^{1/2} \le \sqrt{2}\,\frac{e^d+1}{e^d-1} < \infty.$$

It follows that M is bounded in $\ell_2(\mathbb{J})$. Hence

$$\sum_{j\in\mathbb{J}}|g(y_j)|^2 \le \frac{1}{4}\sum_{j\in\mathbb{J}}\left(\sum_{n\in\mathbb{J}}M_{jn}h_n\right)^2 = \frac{1}{4}\|Mh\|^2_{\ell_2(\mathbb{J})} \le$$

$$\le \frac{1}{4}\|M\|_H^2\|g\|^2_{W_2^2(\mathbb{R})} \le \left(\frac{1}{\sqrt{2}}\frac{e^d+1}{e^d-1}\right)^2\|g\|^2_{W_2^2(\mathbb{R})} = c_1^2\|g\|^2_{W_2^2(\mathbb{R})} < \infty. \tag{11}$$

Similarly $\sum_{j\in\mathbb{J}}|g'(y_j)|^2 \le c_2^2\|g\|^2_{W_2^2(\mathbb{R})} < \infty$. So, $\{g(y_j),\ y_j\in Y\}$, $\{g'(y_j),\ y_j\in Y\}\in\ell_2(\mathbb{J})$.

2) Let

$$f_\alpha(t) = \begin{cases} e\cdot\exp\left(\frac{\alpha^2}{t^2-\alpha^2}\right)\frac{-\alpha^2(a+bt)}{t^2-\alpha^2}, & |t|\le\alpha;\\ 0, & \text{otherwise.}\end{cases}$$

Clearly $f_\alpha(t)\in W_2^2(\mathbb{R})$ and $f_\alpha(0)=a$. Further

$$f'_\alpha(t) = \begin{cases} e\cdot\exp\left(\frac{\alpha^2}{t^2-\alpha^2}\right)\frac{\alpha^2}{(t^2-\alpha^2)^3}\left(bt^4+2at^3+2b\alpha^2t^2+b\alpha^4\right), & |t|\le\alpha;\\ 0, & \text{otherwise,}\end{cases}$$

and $f'_\alpha(0)=b$.

$$f''_\alpha(t) = \begin{cases} e\cdot\exp\left(\frac{\alpha^2}{t^2-\alpha^2}\right)\frac{\alpha^2(-2bt^7-6at^6-12b\alpha^2t^5-4a\alpha^2t^4-2b\alpha^4t^3+6a\alpha^4t^2+8b\alpha^6t)}{(t^2-\alpha^2)^5}, & |t|\le\alpha;\\ 0, & \text{otherwise.}\end{cases}$$

Let $\{a_k,\ k\in\mathbb{J}\}$, $\{b_k,\ k\in\mathbb{J}\}\in\ell_2(\mathbb{J})$,

$$g_k(x) = f_{d/2}(x-y_k) = \begin{cases} e\cdot\exp\left(\frac{(d/2)^2}{(x-y_k)^2-(d/2)^2}\right)\frac{-(d/2)^2(a_k+b_k(x-y_k))}{(x-y_k)^2-(d/2)^2}, & |x-y_k|\le d/2;\\ 0, & \text{otherwise,}\end{cases}$$

and $g(x)=\sum_{k\in\mathbb{J}}g_k(x)$, then $g(y_k)=a_k$, $g'(y_k)=b_k$. Now we show that the function $g(x)$ belongs to $W_2^2(\mathbb{R})$.

$$\int_{\mathbb{R}}|g(x)|^2dx = \int_{\mathbb{R}}\sum_{k\in\mathbb{J}}|g_k(x)|^2dx \le$$

$$\le \sum_{k\in\mathbb{J}}\int_{y_k-d/2}^{y_k+d/2}e^2\cdot\exp\left(\frac{2(d/2)^2}{(x-y_k)^2-(d/2)^2}\right)\frac{(d/2)^4|a_k+b_k(x-y_k)|^2}{((x-y_k)^2-(d/2)^2)^2}dx =$$

$$= \sum_{k\in\mathbb{J}}e^2(d/2)^4\int_{-d/2}^{d/2}\exp\left(\frac{2(d/2)^2}{t^2-(d/2)^2}\right)\frac{|a_k+b_kt|^2}{(t^2-(d/2)^2)^2}dt \le$$

$$\le 2e^2(d/2)^4\sum_{k\in\mathbb{J}}\left[|a_k|^2\int_{-d/2}^{d/2}\exp\left(\frac{2(d/2)^2}{t^2-(d/2)^2}\right)\frac{dt}{(t^2-(d/2)^2)^2}+\right.$$

$$+ |b_k|^2 \int_{-d/2}^{d/2} \exp\left(\frac{2(d/2)^2}{t^2-(d/2)^2}\right) \frac{t^2 dt}{(t^2-(d/2)^2)^2}\Bigg].$$

Set

$$I_1 = \int_{-d/2}^{d/2} \exp\left(\frac{2(d/2)^2}{t^2-(d/2)^2}\right) \frac{dt}{(t^2-(d/2)^2)^2}, \quad I_2 = \int_{-d/2}^{d/2} \exp\left(\frac{2(d/2)^2}{t^2-(d/2)^2}\right) \frac{t^2 dt}{(t^2-(d/2)^2)^2},$$

then we obtain $\int_{\mathbb{R}} |g(x)|^2 dx \leq 2e^2 (d/2)^4 \left(\|a\|^2_{\ell_2(\mathbb{J})} I_1 + \|b\|^2_{\ell_2(\mathbb{J})} I_2\right) < \infty$. Similarly

$$\int_{\mathbb{R}} |g'(x)|^2 dx \leq e^2 (d/2)^4 \left(\|a\|^2_{\ell_2(\mathbb{J})} P_1 + \|b\|^2_{\ell_2(\mathbb{J})} P_2\right) < \infty,$$
$$\int_{\mathbb{R}} |g''(x)|^2 dx \leq e^2 (d/2)^4 \left(\|a\|^2_{\ell_2(\mathbb{J})} S_1 + \|b\|^2_{\ell_2(\mathbb{J})} S_2\right) < \infty.$$

So, $g(x) \in W_2^2(\mathbb{R})$. □

Corollary 1. *If $f \in W_2^1(\mathbb{R})$ then the sequence $\{f(y_j),\ y_j \in Y\}$ belongs to $\ell_2(\mathbb{J})$.*

Proof. Due to inequality (11) we have

$$\|\{f(y_j),\ \ y_j \in Y\}\|^2_{\ell_2(\mathbb{J})} \leq \left(\frac{1}{\sqrt{2}} \frac{e^d+1}{e^d-1}\right)^2 \|f\|^2_{W_2^1(\mathbb{R})} < \infty.$$ □

Proposition 2. *If $f \in W_2^1(\mathbb{R} \setminus Y)$ then the sequence $\{f(y_j+) - f(y_j-),\ \ y_j \in Y\}$ belongs to $\ell_2(\mathbb{J})$.*

Proof. Let $g(x)$ from $W_2^1(\mathbb{R} \setminus Y)$ be real, then the equalities

$$g^2(y_j-) - g^2(y_{j-1}+) e^{-(y_j - y_{j-1})} = \int_{y_{j-1}+}^{y_j-} e^{-|x-y_j|}(g^2(x) + 2g(x)g'(x))dx,$$
$$g^2(y_{j-1}+) - g^2(y_j-) e^{-(y_j - y_{j-1})} = \int_{y_{j-1}+}^{y_j-} e^{-|x-y_{j-1}|}(g^2(x) - 2g(x)g'(x))dx \tag{12}$$

hold. From (12) we have

$$(g^2(y_j-) + g^2(y_{j-1}+))(1 - e^{-(y_j-y_{j-1})}) =$$
$$= \int_{y_{j-1}+}^{y_j-} \left[g^2(x)(e^{-|x-y_j|} + e^{-|x-y_{j-1}|}) + 2g(x)g'(x)(e^{-|x-y_j|} - e^{-|x-y_{j-1}|})\right] dx \leq$$
$$\leq \int_{y_{j-1}+}^{y_j-} \left[2g^2(x) + 4|g(x)g'(x)|\right] dx \leq \int_{y_{j-1}+}^{y_j-} \left[4g^2(x) + 2g'^2(x)\right] dx.$$

Since $1 - e^{-(y_j - y_{j-1})} \geq 1 - e^{-d}$, we obtain

$$\sum_{j \in \mathbb{J}} (g^2(y_j-) + g^2(y_{j-1}+))(1 - e^{-d}) \leq \int_{\mathbb{R}\setminus Y} \left[4g^2(x) + 2g'^2(x)\right] dx,$$

and hence

$$\sum_{j\in\mathbb{J}}(g^2(y_j-)+g^2(y_j+))<\infty. \tag{13}$$

Consider $f(x)=f_R(x)+if_I(x)$ from $W_2^1(\mathbb{R}\setminus Y)$, then for $f_R(x)$ and $f_I(x)$ inequality (13) holds and hence $\sum_{j\in\mathbb{J}}(|f(y_j-)|^2+|f(y_j+)|^2)<\infty$.

Since $|f(y_j+)-f(y_j-)|^2\le 2(|f(y_j-)|^2+|f(y_j+)|^2)$, we obtain that $\{f(y_j+)-f(y_j-),\ j\in\mathbb{J}\}\in\ell_2(\mathbb{J})$. □

3.1. Applications. Let A be an unbounded self-adjoint operator in a Hilbert space H and let $H_{+2}\subset H_{+1}\subset H\subset H_{-1}\subset H_{-2}$ be the chain of rigged Hilbert spaces ([6]) constructed by means of A: $H_{+2}=\mathrm{dom}(A)$, $H_{+1}=\mathrm{dom}(|A|^{1/2})$ with norms $\|f\|_k=(\||A|^{k/2}f\|^2+\|f\|^2)^{1/2}$, $k\in\{1,2\}$. The "negative" Hilbert spaces H_{-k} $(k\in\{1,2\})$ are the completion of H with respect to the norms

$$\|f\|_{-k}=\sup_{g\in H_k,\ \|g\|_k=1}|(f,g)|.$$

The operator A has an extension $\mathbf{A}\in\mathcal{L}(H_k,H_{k-2})$, $k\in\{0,1\}$ $(H_0:=H)$ and $|\mathbf{A}|^{1/2}\in\mathcal{L}(H_k,H_{k-1})$, $k\in\{-1,0\}$ is an extension of $|A|^{1/2}$. The resolvent $\mathrm{R}_z=(A-zI)^{-1}$, $z\in\rho(A)$ has an extension $\mathbf{R}_z=(\mathbf{A}-zI)^{-1}\in\mathcal{L}(H_{-k},H_{-k+2})$, $k\in\{0,1,2\}$. Let Φ be a subspace in H_{-2} such that

$$\Phi\cap H=\{0\}, \tag{14}$$

then the operator A' defined by

$$\mathrm{dom}(A')=\Big\{f\in H_{+2}\colon\ (f,\varphi)=0\quad\text{for all}\quad\varphi\in\Phi\Big\},\ A'=A{\restriction}\,\mathrm{dom}(A') \tag{15}$$

is a closed, densely defined symmetric operator with the defect numbers equal to $\dim\Phi$. For the defect subspace $\mathfrak{N}_z(A')=\ker(A'^*-zI)$ the formula $\mathfrak{N}_z(A')=\mathbf{R}_z\Phi$ holds.

Suppose that A is a nonnegative operator. Then as it is well known, A is the Friedrichs extension of A' if and only if $\Phi\cap H_{-1}=\{0\}$.

The operator A given by (5) is nonnegative and self-adjoint in $H=L_2(\mathbb{R})$. Set for convenience

$$H_{+2}=\mathrm{dom}(A)=W_2^2(\mathbb{R}),\ H_{+1}=\mathrm{dom}(A^{1/2})=W_2^1(\mathbb{R}),\ H_{-1}=W_2^{-1}(\mathbb{R}),\ \ H_{-2}=W_2^{-2}(\mathbb{R}).$$

As mentioned above, (see (7)) one has $\delta_y=\delta(x-y)\in H_{-1}\setminus H$, $(\delta_y)'=\delta'(x-y)\in H_{-2}\setminus H_{-1}$. Let $Y=\{y_j\in\mathbb{R},\ j\in\mathbb{J}\}$ be a discrete set in $\mathbb{R}$ satisfying (1).

The defect subspaces of A', A_0, and H_0 are given by (see [1])

$$\mathfrak{N}_\lambda(A')=\overline{\mathrm{span}}\left\{\mathrm{sgn}(x-y_j)\exp(i\sqrt{\lambda}|x-y_j|),\ j\in\mathbb{J}\right\},$$
$$\mathfrak{N}_\lambda(A_0)=\overline{\mathrm{span}}\{\exp(i\sqrt{\lambda}|x-y_j|),\ j\in\mathbb{J}\},$$
$$\mathfrak{N}_\lambda(H_0)=\overline{\mathrm{span}}\{\exp(i\sqrt{\lambda}|x-y_j|),\ \mathrm{sgn}(x-y_j)\exp(i\sqrt{\lambda}|x-y_j|),\ j\in\mathbb{J}\},$$

respectively.

3.2. Riesz bases. Recall [8] that a countable set of vectors $\{g_j\}$ forms a *Riesz basis* in a separable Hilbert space $\mathfrak{H}$ if $\overline{\mathrm{span}}\{g_j\}=\mathfrak{H}$ and there exist two positive numbers a_1 and a_2

such that for each positive integer n and each collection of complex numbers $\{c_1, c_2, \dots c_n\}$ one has

$$a_2 \sum_{j=1}^{n} |c_j|^2 \le \left\| \sum_{j=1}^{n} c_j g_j \right\|_{\mathfrak{H}}^2 \le a_1 \sum_{j=1}^{n} |c_j|^2.$$

Since $\{e_j\}_{j\in\mathbb{J}}$ forms a Riesz basis $\mathfrak{H}$, every $f \in \mathfrak{H}$ has an expansion $f = \sum_{j\in\mathbb{J}} c_j e_j$ with $\sum_{j\in\mathbb{J}} |c_j|^2 < \infty$, and conversely, if $\sum_{j\in\mathbb{J}} |c_j|^2 < \infty$ then the series $\sum_{j\in\mathbb{J}} c_j e_j$ converges in $\mathfrak{H}$.

Proposition 3. *The systems $\{\delta(x-y_j)\}_{j\in\mathbb{J}}$, $\{\delta'(x-y_j)\}_{j\in\mathbb{J}}$ and $\{\delta(x-y_j),\ \delta'(x-y_j)\}_{j\in\mathbb{J}}$ form Riesz bases of the subspaces Ψ_{-2}, Φ and Ω, respectively.*

Proof. We will show that $\{\delta(x-y_j),\ \delta'(x-y_j)\}_{j\in\mathbb{J}}$ is a Riesz basis of the subspace Ω.

Let $f = \sum_j a_j\delta(x-y_j) + b_j\delta'(x-y_j) \in \Omega$, where $\{a_j\}_{j\in\mathbb{J}}$, $\{b_j\}_{j\in\mathbb{J}} \in l_2(\mathbb{J})$, then using the first statement of Propositions (1) we get

$$\left\| \sum_j a_j\delta(x-y_j) + b_j\delta'(x-y_j) \right\|_{H_{-2}}^2 = \sup_{g\in H_{+2},\ \|g\|_{H_{+2}}=1} |(f,g)|^2 =$$

$$= \sup_{g\in H_{+2},\ \|g\|_{H_{+2}}=1} \left| \sum_j a_j g(y_j) + b_j g'(y_j) \right|^2 \le$$

$$\le 2\left(\sup_{g\in H_{+2},\ \|g\|_{H_{+2}}=1} \sum_j |a_j|^2 \sum_j |g(y_j)|^2 + \sup_{g\in H_{+2},\ \|g\|_{H_{+2}}=1} \sum_j |b_j|^2 \sum_j |g'(y_j)|^2 \right) =$$

$$= C_1\|a\|_{\ell_2(\mathbb{J})}^2 + C_2\|b\|_{\ell_2(\mathbb{J})}^2 < \infty.$$

On the other hand, using the second statement of Proposition (1) we have

$$\sup_{g\in H_{+2},\ \|g\|_{H_{+2}}=1} \left| \sum_j a_j g(y_j) + b_j g'(y_j) \right|^2 \ge \left| \sum_j a_j \frac{\overline{a_j}}{\|a\|} + b_j \frac{\overline{b_j}}{\|b\|} \right|^2 = (\|a\|_{\ell_2(\mathbb{J})} + \|b\|_{\ell_2(\mathbb{J})})^2.$$

Therefore, the system $\{\delta(x-y_j),\ \delta'(x-y_j)\}_{j\in\mathbb{J}}$ forms a Riesz basis of the subspace Ω.

The other statements can be proved similarly. □

3.3. Transversalness of the Friedrichs and Kreĭn extensions. Let H be a separable Hilbert space and let $\mathcal{A}$ be a densely defined closed symmetric and nonnegative operator. Denote by $\mathcal{A}^*$ the adjoint to $\mathcal{A}$, by $\widetilde{\mathcal{A}}$ a nonnegative selfadjoint extension of $\mathcal{A}$. It is well known ([1]) that the operator $\mathcal{A}$ admits at least one nonnegative self-adjoint extension $\mathcal{A}_F$ called the *Friedrichs extension*, which is defined as follows. Denote by $\mathcal{A}[\cdot,\cdot]$ the closure of the sesquilinear form (see [10])

$$\mathcal{A}[f,g] = (\mathcal{A}f,g),\ f,g \in \mathrm{dom}(\mathcal{A}),$$

and let $\mathcal{D}[\mathcal{A}]$ be the domain of this closure. According to the first representation theorem ([10]) there exists a nonnegative self-adjoint operator $\mathcal{A}_F$ associated with $\mathcal{A}[\cdot,\cdot]$, i.e., $(\mathcal{A}_F h,\psi) = \mathcal{A}[h,\psi]$, $\psi \in \mathcal{D}[\mathcal{A}]$, $h \in \mathrm{dom}(\mathcal{A}_F)$. Clearly $\mathcal{A} \subset \mathcal{A}_F \subset \mathcal{A}^*$, where $\mathcal{A}^*$ is the adjoint operator to $\mathcal{A}$. It follows that $\mathrm{dom}(\mathcal{A}_F) = \mathcal{D}[\mathcal{A}] \cap \mathrm{dom}(\mathcal{A}^*)$. By the second representation theorem, the equalities $\mathcal{D}[\mathcal{A}] = \mathrm{dom}(\mathcal{A}_F^{1/2})$ and $\mathcal{A}[\phi,\psi] = (\mathcal{A}_F^{1/2}\phi, \mathcal{A}_F^{1/2}\psi)$, $\phi,\ \psi \in \mathcal{D}[\mathcal{A}]$ hold.

M. G. Kreĭn in [14] discovered one more nonnegative self-adjoint extension of $\mathcal{A}$ having extremal property to be minimal (in the sense of the corresponding quadratic forms) among others nonnegative self-adjoint extensions of $\mathcal{A}$. This extension we will denote by $\mathcal{A}_K$ and call it the *Kreĭn extension* of $\mathcal{A}$.

Recall that two selfadjoint extensions $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ of a symmetric operator $\mathcal{A}$ are called disjoint if $\mathrm{dom}(\widetilde{\mathcal{A}}_1)\cap\mathrm{dom}(\widetilde{\mathcal{A}}_2)=\mathrm{dom}(\mathcal{A})$ and *transversal* if $\mathrm{dom}(\widetilde{\mathcal{A}}_1)+\mathrm{dom}(\widetilde{\mathcal{A}}_2)=\mathrm{dom}(\mathcal{A}^*)$. We need the following statement ([4], [16]).

Proposition 4. *The Friedrichs and Kreĭn extensions $\mathcal{A}_F$ and $\mathcal{A}_K$ are transversal if $\mathfrak{N}_z\subset\mathrm{dom}(\mathcal{A}_K^{1/2})$ at least for one (hence for all) $z\in\mathbb{C}\setminus[0,\infty)$.*

In what follows we will consider our operators (2)–(4) in the p-representation by means of the Fourier transform

$$\widehat{f}(p)=(\mathcal{F}f)(p)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)e^{-ipx}dx.$$

Note that

$$(\mathcal{F}\delta_y)(p)=\widehat{\delta}_y(p)=\frac{1}{\sqrt{2\pi}}\,e^{-ipy},\ (\mathcal{F}\delta_y')(p)=\widehat{\delta}_y'(p)=\frac{ipe^{-ipy}}{\sqrt{2\pi}},$$

and the Fourier transformation $\mathcal{F}$ is a unitary operator from $L_2(\mathbb{R},dx)$ onto $L_2(\mathbb{R},dp)$. In addition

$$\mathrm{dom}(\widehat{A})=\widehat{H}_{+2}=\left\{\hat{f}\in L_2(\mathbb{R},dp)\colon\int_{\mathbb{R}}|\widehat{f}(p)|^2(p^4+1)dp<\infty\right\},\ (\widehat{A}\widehat{f})(p)=p^2\widehat{f}(p),$$

$$\mathrm{dom}(\widehat{A}^{1/2})=\widehat{H}_{+1}=\left\{\hat{f}\in L_2(\mathbb{R},dp)\colon\int_{\mathbb{R}}|\widehat{f}(p)|^2(p^2+1)dp<\infty\right\},\ (\widehat{A}^{1/2}\widehat{f})(p)=|p|\widehat{f}(p).$$

$$\mathrm{dom}(\widehat{A}')=\left\{\hat{f}\in\widehat{H}_{+2}\colon\int_{\mathbb{R}}pe^{ipy_j}\hat{f}(p)dp=0,\ j\in\mathbb{J}\right\},\ (\widehat{A}'\widehat{f})(p)=p^2\widehat{f}(p)$$

$$\mathrm{dom}(\widehat{A}_0)=\left\{\hat{f}\in\widehat{H}_{+2}\colon\int_{\mathbb{R}}e^{ipy_j}\hat{f}(p)dp=0,\ j\in\mathbb{J}\right\},\ (\widehat{A}_0\widehat{f})(p)=p^2\widehat{f}(p),$$

$$\mathrm{dom}(\widehat{H}_0)=\left\{\hat{f}\in\widehat{H}_{+2}\colon\int_{\mathbb{R}}e^{ipy_j}\hat{f}(p)dp=0,\ \int_{\mathbb{R}}pe^{ipy_j}\hat{f}(p)dp=0\ j\in\mathbb{J}\right\},\ (\widehat{H}_0\widehat{f})(p)=p^2\widehat{f}(p).$$

The pairs of operators $\langle\widehat{A},A\rangle$, $\langle\widehat{A}',A'\rangle$, $\langle\widehat{A}_0,A_0\rangle$, and $\langle\widehat{H_0},H_0\rangle$ are unitary equivalent since $\mathcal{F}A=\widehat{A}\mathcal{F}$. Clearly, $\widehat{H}_{+2}=\mathcal{F}H_{+2}$, $\widehat{H}_{+1}=\mathcal{F}H_{+1}$,

$$\widehat{H}_{-1}=\mathcal{F}H_{-1}=\left\{\widehat{f}(p)\colon\frac{\hat{f}(p)}{p^2+1}\in\widehat{H}_{+1}\right\},\ \|\widehat{f}(p)\|^2_{\widehat{H}_{-1}}=\int_{\mathbb{R}}\frac{|\widehat{f}(p)|^2}{p^2+1}\,dp,$$

$$\widehat{H}_{-2}=\mathcal{F}H_{-2}=\left\{\widehat{f}(p)\colon\frac{\hat{f}(p)}{p^4+1}\in\widehat{H}_{+2}\right\},\ \|\widehat{f}(p)\|^2_{\widehat{H}_{-2}}=\int_{\mathbb{R}}\frac{|\widehat{f}(p)|^2}{p^4+1}\,dp,$$

$$\widehat{\mathbf{A}}\hat{f}=p^2\hat{f}(p),\quad\widehat{\mathbf{A}}\colon\widehat{H}_{+1}\to\widehat{H}_{-1},\quad L_2(\mathbb{R})\to\widehat{H}_{-2}.$$

Let $\widehat{\Phi}=\mathcal{F}\Phi,\ \widehat{\Psi}_{-1}=\mathcal{F}\Psi_{-1},\ \widehat{\Psi}_{-2}=\mathcal{F}\Psi_{-2},\ \widehat{\Omega}=\mathcal{F}\Omega$. Then

$$\widehat{\Phi}=\overline{\mathrm{span}}_{\widehat{H}_{-2}}\{pe^{-ipy_j},\ j\in\mathbb{J}\},\quad\widehat{\Psi}_{-2}=\overline{\mathrm{span}}_{\widehat{H}_{-2}}\{e^{-ipy_j},\ j\in\mathbb{J}\},$$

$$\widehat{\Psi}_{-1}=\overline{\mathrm{span}}_{\widehat{H}_{-1}}\{e^{-ipy_j},\ j\in\mathbb{J}\},\quad\widehat{\Omega}=\overline{\mathrm{span}}_{\widehat{H}_{-2}}\{e^{-ipy_j},pe^{-ipy_j},\ j\in\mathbb{J}\}.$$

Theorem 1. *The equality $\Psi_{-2} = \Psi_{-1}$ holds.*

Proof. Let $f \in \Psi_{-2}$, then $f = \sum_k c_k \delta(x - y_k)$, $\sum_{k\in\mathbb{J}} |c_k|^2 < \infty$. Using Corollary (1) we have

$$\|f\|^2_{H_{-1}} = \sup_{g\in H_1,\, \|g\|_1=1} |(f,g)|^2 = \sup_{g\in H_1,\, \|g\|_1=1} \left|\sum_{k\in\mathbb{J}} c_k g(y_k)\right|^2 \le \sum_{k\in\mathbb{J}} |c_k|^2 \sup_{g\in H_1,\, \|g\|_1=1} \sum_{k\in\mathbb{J}} |g(y_k)|^2 < \infty.$$

Therefore, $\Psi_{-2} \subset H_{-1}$ and $\Psi_{-2} = \Psi_{-1}$. □

Corollary 2. *The systems $\{e^{-ipy_j}\}_{j\in\mathbb{J}}$, $\{pe^{-ipy_j}\}_{j\in\mathbb{J}}$ and $\{\frac{e^{-ipy_j}}{p^2+1}\}_{j\in\mathbb{J}}$, $\{\frac{pe^{-ipy_j}}{p^2+1}\}_{j\in\mathbb{J}}$ form Riesz bases of the subspaces $\widehat{\Psi}_{-1}$, $\widehat{\Phi}$ and $\widehat{\mathfrak{N}}_{-1}(\widehat{A}_0)$, $\widehat{\mathfrak{N}}_{-1}(\widehat{A}')$, respectively.*

Proof. Since the operator $\mathcal{F}$ unitarily maps H_{-2} onto $\widehat{H}_{-2}$, by Proposition 3, the systems $\{e^{-ipy_j}\}_{j\in\mathbb{J}}$ and $\{pe^{-ipy_j}\}_{j\in\mathbb{J}}$ form Riesz bases of $\widehat{\Psi}_{-1}$ and $\widehat{\Phi}$, respectively. Let $\widehat{\mathfrak{N}}_{-1}(\widehat{A}') = \ker(\widehat{A'^*}+I)$, $\widehat{\mathfrak{N}}_{-1}(\widehat{A}_0) = \ker(\widehat{A_0^*}+I)$. Then $\widehat{\mathfrak{N}}_{-1}(\widehat{A}') = (\widehat{\mathbf{A}}+I)^{-1}\widehat{\Phi}$, $\quad \widehat{\mathfrak{N}}_{-1}(\widehat{A}_0) = (\widehat{\mathbf{A}}+I)^{-1}\widehat{\Psi}_{-1}$, and $\left\{\frac{pe^{-ipy_j}}{p^2+1}\right\}_{j\in\mathbb{J}}$ is a Riesz basis of $\widehat{\mathfrak{N}}_{-1}(\widehat{A}') \subset H$, $\left\{\frac{e^{-ipy_j}}{p^2+1}\right\}_{j\in\mathbb{J}}$ is a Riesz basis of $\widehat{\mathfrak{N}}_{-1}(\widehat{A}_0) \subset \widehat{H}_{+1}$. □

Theorem 2. *The equality $\Phi \cap H_{-1} = \{0\}$ holds.*

Proof. Let $g \in \widehat{\Phi}$, then $g(p) = \sum_k c_k pe^{-ipy_k}$, but by Corollary (2) $\int_{\mathbb{R}} \frac{1}{p^2+1} \left|\sum_k c_k pe^{-ipy_k}\right|^2 dp = \infty$, hence g does not belong to $\widehat{H}_{-1}$, i.e. $\widehat{\Phi} \cap \widehat{H}_{-1} = \{0\}$ and $\Phi \cap H_{-1} = \{0\}$. □

Corollary 3. *The Friedrichs and Kreĭn extensions of the operators H_0, A', A_0 are transversal.*

Proof. Let $u \in \widehat{\mathfrak{N}}_{-1}(\widehat{H}_0)$, then $u(p) = \sum\limits_k a_k \frac{e^{-ipy_k}}{p^2+1} + b_k \frac{pe^{-ipy_k}}{p^2+1}$. Using Corollary (2) we have

$$\sup_{f\in\operatorname{dom}(\widehat{H}_0)} \frac{|(\widehat{H}_0 f, u)|^2}{(\widehat{H}_0 f, f)} = \sup_{f\in\operatorname{dom}(\widehat{H}_0)} \frac{\left|\int_{\mathbb{R}} p^2 f(p)\overline{u(p)}dp\right|^2}{\int_{\mathbb{R}} p^2|f(p)|^2 dp} \le \sup_{f\in\operatorname{dom}(\widehat{H}_0)} \frac{\int_{\mathbb{R}} p^4|f(p)|^2 dp \int_{\mathbb{R}} |u(p)|^2 dp}{\int_{\mathbb{R}} p^2|f(p)|^2 dp} \le$$
$$\le \int_{\mathbb{R}} \left(\left|\sum_k \frac{a_k e^{-ipy_k}}{p^2+1}\right|^2 + \left|\sum_k \frac{b_k pe^{-ipy_k}}{p^2+1}\right|^2 \right) dp < \infty.$$

So, $\widehat{\mathfrak{N}}_{-1}(\widehat{H}_0) \subset \operatorname{dom}(\widehat{H}_{0K}^{1/2})$. Therefore, due to Proposition 4 the extensions $\widehat{H}_{0F}$ and $\widehat{H}_{0K}$ as well as H_{0F} and H_{0K} are transversal.

Transversalness of the Friedrichs and Kreĭn extensions of the operators A' and A_0 can be proved similarly. □

Corollary 4. *The operator A is the Friedrichs extension of the operator A'.*

Proof. Since $\Phi \cap H_{-1} = \{0\}$, we get that $A'_F = A$. □

3.4. Basic boundary triplets for operators A_0^*, A'^* and H_0^*. Let S be a closed densely defined symmetric operator with equal defect numbers in $\mathfrak{H}$. Let $\mathcal{H}$ be some Hilbert space, Γ_1 and Γ_2 be linear mappings of $\operatorname{dom}(S^*)$ into $\mathcal{H}$. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *boundary triplet for adjoint operator* S^* ([7], [11], [9]), if

$$(S^*x, y) - (x, S^*y) = (\Gamma_1 x, \Gamma_0 y)_{\mathcal{H}} - (\Gamma_0 x, \Gamma_1 y)_{\mathcal{H}} \quad \text{for all} \quad x, y \in \operatorname{dom}(S^*), \qquad (16)$$

and a mapping $\mathbf{\Gamma}\colon x \mapsto \{\Gamma_0 x, \Gamma_1 x\},\ x \in \operatorname{dom}(S^*)$ is a surjection of $\operatorname{dom}(S^*)$ onto $\mathcal{H} \oplus \mathcal{H}$.

Let S be a densely defined and nonnegative operator. Suppose that the Friedrichs and Kreĭn extensions of S are transversal. The boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* is called *basic* ([2], [3] rigid for positive definite S [17]) if $\ker(\Gamma_0) = \operatorname{dom}(S_F),\ \ker(\Gamma_1) = \operatorname{dom}(S_K)$. A basic boundary triplet is positive [2] and (see [3]) $S_K[x,y] = (S^*x, y) - (\Gamma_1 x, \Gamma_0 y)_{\mathcal{H}},\ x, y \in \operatorname{dom}(S^*)$.

Proposition 5 ([2])**.** *Let S be a densely defined and nonnegative operator with transversal Friedrichs and Kreĭn extensions and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a basic boundary triplet for S^*. Then the mapping*

$$\Theta \mapsto S_\Theta := S^* \upharpoonright \mathbf{\Gamma^{-1}}\Theta = S^* \upharpoonright \big\{ f \in \operatorname{dom}(S^*)\colon (\Gamma_0 f, \Gamma_1 f) \in \Theta \big\} \tag{17}$$

establishes a bijective correspondence between the set of all selfadjoint nonnegative linear relations Θ in $\mathcal{H}$ and the set of all nonnegative selfadjoint extensions of $S_\Theta \subseteq S^*$ of S.

Assume that

(**A**) L_1 and L_2 are two closed densely defined operators in the Hilbert space $\mathfrak{H}$ taking values in a Hilbert space H and such that $L_1 \subset L_2$.

Theorem 3 ([5])**.** *Let condition (**A**) be fulfilled. If the operator $\mathcal{A} = L_2^* L_1$ is densely defined and $\mathcal{A}^* = L_1^* L_2$, then*

1) *the operator $\mathcal{A}_F = L_1^* L_1$ is the Friedrichs extension of $\mathcal{A}$;*
2) *the Friedrichs and Kreĭn extensions of $\mathcal{A}$ are transversal;*
3) *the operator*

$$\begin{aligned} &\operatorname{dom} \mathcal{A}_K = \{ f \in \operatorname{dom}(L_2)\colon P_{\overline{\operatorname{ran}}(L_1)} L_2 f \in \operatorname{dom}(L_2^*) \}, \\ &\mathcal{A}_K f = L_2^* P_{\overline{\operatorname{ran}}(L_1)} L_2 f = L_1^* L_2 f,\ f \in \operatorname{dom}(\mathcal{A}_K) \end{aligned}$$

is the Kreĭn extension of $\mathcal{A}$ and

$$\mathcal{D}[\mathcal{A}_K] = \operatorname{dom}(L_2),\ \mathcal{A}_K[u,v] = (P_{\overline{\operatorname{ran}}(L_1)} L_2 u, P_{\overline{\operatorname{ran}}(L_1)} L_2 v),\ u, v = \operatorname{dom}(L_2).$$

The operator $\mathcal{A} = L_2^* L_1$ called an *operator in the divergence form.*

According to V. E. Lyantse and O. G. Storozh ([15]) a pair $\{\mathcal{H}, \Gamma\}$ is called a *boundary pair* for $L_1 \subset L_2$, if $\mathcal{H}$ is a Hilbert space, $\Gamma \in \mathcal{L}(\operatorname{dom}(L_2), \mathcal{H})$ and $\ker(\Gamma) = \operatorname{dom}(L_1),\ \operatorname{ran}(\Gamma) = \mathcal{H}$. Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for $L_1 \subset L_2$. Then there exists a linear operator $G \in \mathcal{L}(\operatorname{dom}(L_1^*), \mathcal{H})$ such that $\{\mathcal{H}, G\}$ is a boundary pair for $L_2^* \subset L_1^*$ and the Green identity

$$(L_1^* f, u)_H - (f, L_2 u)_{\mathfrak{H}} = (Gf, \Gamma u)_{\mathcal{H}},\ f \in \operatorname{dom}(L_1^*),\, u \in \operatorname{dom}(L_2) \tag{18}$$

holds. The set $\{\mathcal{H}, G, \Gamma\}$ is called the *boundary triplet for a pair of the operators* $L_1 \subset L_2$.

Theorem 4. *Let condition (**A**) be fulfilled and let $\{\mathcal{H}, G, \Gamma\}$ be a boundary triplet for $L_1 \subset L_2$. If the operator $\mathcal{A} = L_2^* L_1$ is densely defined and $\mathcal{A}^* = L_1^* L_2$, then*

1. *the triplet $\Pi = \{\mathcal{H}, \Gamma, G P_{\overline{\operatorname{ran}}(L_1)} L_2\}$ is a basic for $\mathcal{A}^*$;*
2. *the mapping*

$$\Theta \mapsto \mathcal{A}_\Theta := \mathcal{A}^* \upharpoonright \Gamma^{-1}\Theta = \mathcal{A}^* \upharpoonright \big\{ f \in \operatorname{dom}(\mathcal{A}^*)\colon (\Gamma f, G P_{\overline{\operatorname{ran}}(L_1)} L_2 f) \in \Theta \big\} \tag{19}$$

establishes a bijective correspondence between all nonnegative selfadjoint extensions of the operator $\mathcal{A}$ and all nonnegative selfadjoint linear relations Θ in $\mathcal{H}$.

Proof. By Theorem 3, the Friedrichs and Kreĭn extensions of $\mathcal{A}$ are transversal, $\mathcal{D}[\mathcal{A}_K] = \mathrm{dom}(L_2)$, $\mathcal{D}[\mathcal{A}_F] = \mathrm{dom}(L_1)$. Hence, $\{\mathcal{H}, \Gamma\}$ is a boundary pair for $\mathcal{A}$. Let $x \in \mathrm{dom}(\mathcal{A}^*) = \mathrm{dom}(L_1^* L_2)$ and $y \in \mathrm{dom}(L_2)$. Then $P_{\overline{\mathrm{ran}}(L_1)} L_2 x = L_2 x - P_{\ker(L_1^*)} L_2 x \in \mathrm{dom}(L_1^*)$. Using Theorem 3 and (18) we get

$$\mathcal{A}_K[x,y] = (P_{\overline{\mathrm{ran}}\,(L_1)} L_2 x, L_2 u)_H = (L_1^* P_{\overline{\mathrm{ran}}(L_1)} L_2 x, y)_{\mathfrak{H}} - (G P_{\overline{\mathrm{ran}}(L_1)} L_2 x, \Gamma y)_{\mathcal{H}} = \\ = (L_1^* L_2 x, y)_{\mathfrak{H}} - (G P_{\overline{\mathrm{ran}}(L_1)} L_2 x, \Gamma y)_{\mathcal{H}} = (\mathcal{A}^* x, y)_{\mathfrak{H}} - (G P_{\overline{\mathrm{ran}}(L_1)} L_2 x, \Gamma y)_{\mathcal{H}}.$$

In particular, for $x, y \in \mathrm{dom}(\mathcal{A}^*)$ taking into account that the form $\mathcal{A}_K[x,y]$ is Hermitian, we have $(\mathcal{A}^* x, y) - (x, \mathcal{A}^* y) = (G P_{\overline{\mathrm{ran}}(L_1)} L_2 x, \Gamma y)_{\mathcal{H}} - (\Gamma x, G P_{\overline{\mathrm{ran}}(L_1)} L_2 y)_{\mathcal{H}}$. Thus, the triplet $\Pi = \{\mathcal{H}, \Gamma, G P_{\overline{\mathrm{ran}}(L_1)} L_2\}$ is basic for S^*. From Proposition 5 we get that statement (2) holds true. □

Consider in $L_2(\mathbb{R})$ the following operators

$$\mathrm{dom}(\mathcal{L}_0) = \{f \in W_2^1(\mathbb{R})\colon f(y) = 0, y \in Y\},\ \mathcal{L}_0 = i\frac{d}{dx}, \tag{20}$$

$$\mathrm{dom}(\mathcal{L}) = W_2^1(\mathbb{R}),\ \mathcal{L} = i\frac{d}{dx}. \tag{21}$$

From (20) it follows that $\mathcal{L}_0$ is a densely defined symmetric operator and its adjoint $\mathcal{L}_0^*$ is given by

$$\mathrm{dom}(\mathcal{L}_0^*) = W_2^1(\mathbb{R} \setminus Y),\ \mathcal{L}_0^* = i\frac{d}{dx}. \tag{22}$$

The operator $\mathcal{L}$ is a selfadjoint extension of $\mathcal{L}_0$. So, we have $\mathcal{L}_0 \subset \mathcal{L} \subset \mathcal{L}_0^*$. From (3)–(22) it follows that

$$A_0 = \mathcal{L}\mathcal{L}_0,\ A' = \mathcal{L}_0\mathcal{L},\ H_0 = \mathcal{L}_0^2,\ A = \mathcal{L}^2,\ A_0^* = \mathcal{L}_0^*\mathcal{L},\ A'^* = \mathcal{L}\mathcal{L}_0^*,\ H_0^* = \mathcal{L}_0^{*2}. \tag{23}$$

Using representation (23) and Theorem 3 the explicit expressions for the Friedrichs and Kreĭn extensions of A_0, A' and H_0 and their transversalness have been obtained in [5]. In the next statements for the operators A'^*, A_0^* and H_0^* explicit expressions for the basic boundary triplets and abstract boundary conditions for all nonnegative selfadjoint extensions are obtained.

Proposition 6. *Set*

$$\mathcal{H} = \begin{cases} \mathbb{C}^m, & Y \text{ consists of } m \text{ points};\\ \ell_2(\mathbb{J}), & Y \text{ is infinite}, \end{cases} \qquad \mathrm{dom}(\Gamma) = W_2^1(\mathbb{R} \setminus Y),$$

$$\Gamma u = \{i(u(y_j+) - u(y_j-)),\ j \in \mathbb{J}\},\ \mathrm{dom}(G) = W_2^1(\mathbb{R}),\ \ Gf = \{f(y_j),\ j \in \mathbb{J}\}.$$

Then

(i) *$\{\mathcal{H}, \Gamma, G\}$ is the boundary triplet for pair $\mathcal{L} \subset \mathcal{L}_0^*$;*

(ii) *the triplet $\Pi = \{\mathcal{H}, \Gamma, G\mathcal{L}_0^*\}$ is basic for A'^*, where $G\mathcal{L}_0^*$ is given by the relation $G\mathcal{L}_0^* f = \{if'(y_j),\ j \in \mathbb{J}\},\ f \in \mathrm{dom}(A'^*)$;*

(iii) *the mapping*

$$\Theta \mapsto A'_\Theta = A'^* \restriction \{f \in \mathrm{dom}(A'^*)\colon\ (\{i(f(y_j+) - f(y_j-)),\ j \in \mathbb{J}\}, \{if'(y_j),\ j \in \mathbb{J}\}) \in \Theta\}$$

establishes a one-to-one correspondence between all nonnegative selfadjoint extensions of the operator A' and all nonnegative selfadjoint linear relation Θ in $\mathcal{H}$.

Proof. By the definition of a boundary triplet for the pair $L_1 \subset L_2$, where $L_1 = \mathcal{L},\ L_2 = \mathcal{L}_0^*$ we get $\mathrm{dom}(\Gamma) = \mathrm{dom}(L_2) = \mathrm{dom}(\mathcal{L}_0^*) = W_2^1(\mathbb{R} \setminus Y)$ and $\ker(\Gamma) = \mathrm{dom}(\mathcal{L}) = W_2^1(\mathbb{R})$. Similarly, $\mathrm{dom}(\Phi) = \mathrm{dom}(L_1^*) = \mathrm{dom}(\mathcal{L}) = W_2^1(\mathbb{R})$, $\ker(\Phi) = \mathrm{dom}(L_2^*) = \mathrm{dom}(\mathcal{L}_0) = \{u \in W_2^1(\mathbb{R})\colon u(y) = 0,\ y \in Y\}$. Further, the Green identity

$$(L_1^* f, u)_{\mathfrak{H}} - (f, L_2 u)_H = \int_{\mathbb{R}} i f'(x)\overline{u(x)}dx - \int_{\mathbb{R}} f(x)\overline{iu'(x)}dx =$$
$$= i\sum_{j\in\mathbb{J}}\left(\int_{I_j} f'(x)\overline{u(x)}dx + \int_{I_j} f(x)\overline{u'(x)}dx\right) = i\sum_{j\in\mathbb{J}} f(x)\overline{u(x)}\,\Big|_{y_j}^{y_{j+1}} =$$
$$= i\sum_{j\in\mathbb{J}} f(y_j)\left(\overline{u(y_j-) - u(y_j+)}\right) = \sum_{j\in\mathbb{J}} f(y_j)\overline{i\,(u(y_j+) - u(y_j-))} = (Gf, \Gamma u)_{\mathcal{H}}$$

holds. Due to Propositions 1 and 2 the operators Γ and G are bounded. Hence the triplet $\{\mathcal{H}, G, \Gamma\}$ is the boundary triplet for the pair $\mathcal{L} \subset \mathcal{L}_0^*$.

Further, since $\ker(\mathcal{L}) = \{0\}$ and applying Theorem 4 we get (ii) and (iii). □

Recall [5], that

$$P_{\overline{\mathrm{ran}}(\mathcal{L}_0)}\mathcal{L}_0^* f = if' - i\sum_k \frac{1}{d_k}(f(y_{k+1} - 0) - f(y_k + 0))\chi_k,\ f \in \mathrm{dom}(\mathcal{L}_0^*),$$

where the functions $\{\frac{\chi_k}{\sqrt{d_k}}\}_{k\in\mathbb{J}}$ (χ_k is the characteristic function of the interval $[y_k, y_{k+1}]$, $d_k = |y_k - y_{k+1}|$) form an orthonormal basis of $\ker(\mathcal{L}_0^*)$ and $d_k = |y_k - y_{k+1}|$, $k \in \mathbb{J}$.

Proposition 7. *Set*

$$\mathcal{H} = \begin{cases} \mathbb{C}^m, & Y \text{ consists of } m \text{ points};\\ \ell_2(\mathbb{J}), & Y \text{ is infinite,}\end{cases} \qquad \mathrm{dom}(\Gamma) = W_2^1(\mathbb{R}),\ \ \Gamma u = \{iu(y_j),\ j \in \mathbb{J}\},$$
$$\mathrm{dom}(G) = W_2^1(\mathbb{R} \setminus Y),\ \ Gf = \{(f(y_j+) - f(y_j-)),\ j \in \mathbb{J}\},$$

then

(i) $\{\mathcal{H}, \Gamma, G\}$ *is a boundary triplet for the pair* $\mathcal{L}_0 \subset \mathcal{L}$;

(ii) the triplet $\Pi = \{\mathcal{H}, \Gamma, GP_{\overline{\mathrm{ran}}(\mathcal{L}_0)}\mathcal{L}\}$ *is a basic for* A_0^*, *where*

$$GP_{\overline{\mathrm{ran}}(\mathcal{L}_0)}\mathcal{L}f =$$
$$= \left\{if'(y_j+) - if'(y_j-) - i\frac{f(y_{j+1}-) - f(y_j+)}{y_{j+1} - y_j} + i\frac{f(y_j-) - f(y_{j-1}+)}{y_j - y_{j-1}},\ j \in \mathbb{J}\right\},$$

$f \in \mathrm{dom}(A_0^*)$;

(iii) the mapping

$$\Theta \mapsto A_{0\Theta} = A_0{}^* \restriction \Big\{f \in \mathrm{dom}(A_0{}^*)\colon \Big(\{iu(f_j),\ j \in \mathbb{J}\},$$
$$\left\{if'(y_j+) - if'(y_j-) - i\frac{f(y_{j+1}-) - f(y_j+)}{y_{j+1} - y_j} + i\frac{f(y_j-) - f(y_{j-1}+)}{y_j - y_{j-1}},\ j \in \mathbb{J}\right\}\Big) \in \Theta\Big\}$$

establishes a one-to-one correspondence between all nonnegative selfadjoint extensions of the operator A_0 *and all nonnegative selfadjoint linear relation* Θ *in* $\mathcal{H}$.

Proposition 8. *Set*

$$\mathcal{H} = \begin{cases} \mathbb{C}^{2m}, & Y \text{ consists of } m \text{ points;} \\ \ell_2(\mathbb{J}) \otimes \mathbb{C}^2, & Y \text{ is infinite,} \end{cases} \quad \operatorname{dom}(\Gamma) = W_2^1(\mathbb{R} \setminus Y),$$

$$\Gamma u = \{(iu(y_j-), iu(y_j+)) \,,\ j \in \mathbb{J}\},\ \operatorname{dom}(G) = W_2^1(\mathbb{R} \setminus Y), Gf = \{(f(y_j-), f(y_j+)) \,,\ j \in \mathbb{J}\}.$$

Then

(i) $\{\mathcal{H}, \Gamma, G\}$ *is a boundary triplet for pair* $\mathcal{L}_0 \subset \mathcal{L}_0^*$*;*

(ii) the triplet $\Pi = \{\mathcal{H}, \Gamma, GP_{\overline{\operatorname{ran}}(\mathcal{L}_0)}\mathcal{L}_0^*\}$ *is basic for* H_0^**, where*

$$GP_{\overline{\operatorname{ran}}(\mathcal{L}_0)}\mathcal{L}_0^* f = \\ = \left\{ \left(if'(y_j-) - i\frac{f(y_j-) - f(y_{j-1}+)}{y_j - y_{j-1}}, if'(y_j+) - i\frac{f(y_{j+1}-) - f(y_j+)}{y_{j+1} - y_j} \right) ,\ j \in \mathbb{J} \right\},$$

$f \in \operatorname{dom}(H_0^*)$;

(iii) the mapping

$$\Theta \mapsto H_{0\Theta} = H_0{}^* \upharpoonright \Big\{ f \in \operatorname{dom}(H_0{}^*) \colon \Big\{ \Big(-if(y_j-), if(y_j+) \Big) ,\ j \in \mathbb{J} \Big\} , \\ \left\{ \left(if'(y_j-) - i\frac{f(y_j-) - f(y_{j-1}+)}{y_j - y_{j-1}}, if'(y_j+) - i\frac{f(y_{j+1}-) - f(y_j+)}{y_{j+1} - y_j} \right) ,\ j \in \mathbb{J} \right\} \in \Theta \Big\}$$

establishes a one-to-one correspondence between all nonnegative selfadjoint extensions of the operator H_0 *and all nonnegative selfadjoint linear relations* Θ *in* $\mathcal{H}$*.*

Other boundary triplets for H_0^* have been constructed in [12] and in [13].

REFERENCES

1. Albeverio S., Gesztesy F., Hegh-Krohn R., Holden H., Solvable models in quantum mechanics. — Texts and Monographs in Physics. Springer-Verlag, New York.
2. Arlinskiĭ Yu.M. *Positive spaces of boundary values and sectorial extensions of a nonnegative symmetric operator*// Ukrain. Math. Journ. – 1988. – V.40, №1. – P. 8–14. (in Russian)
3. Arlinskiĭ Yu., Hassi S., Sebestyen Z., de Snoo H. *On the class of extremal extensions of a nonnegative operators*// Operator Theory: Advan. and Appl. – 2001. – V.127. – P. 41–81.
4. Arlinskiĭ Yu.M., Tsekanovskiĭ E.R. *The von Neumann problem for nonnegative symmetric operators*// Integral equation and operator theory. – 2005. – P. 315–356.
5. Arlinskiĭ Yu.M., Kovalev Yu.G. *Operators in divergence form and their Friedrichs and Kreĭn extensions*// Opuscula mathematica. – 2011. – V.31, №4. – P. 501–517.
6. Berezansky Yu.M., Expansions in eigenfunction of selfadjoint operators. – "Naukova Dumka", Kiev, 1965 (in Russian). English translation in Translations of Mathematical Monographs. Amer. Math. Soc. Providence, 1968, V.17.
7. Bruk V.M. *On one class of boundary value problems with a spectral parameter in the boundary condition*// Mat. Sbornik. – 1976. – V.100, №2. – P. 210–216. (in Russian)
8. Gohberg I.C., Kreĭn M.G., Introduction to the theory of linear nonselfadjoint operators. – M: Nauka, 1965 (in Russian). English translation in Translations of Mathematical Monographs, V.18, Amer. Math. Soc., Providence, R.I., 1969.

9. Gorbachuk M.L., Gorbachuk V.I., Boundary value problems for differential-operator equations. – K.: Naukova Dumka, 1984. (in Russian)
10. Kato T., Perturbation theory for linear operators. – Springer-Verlag, 1966.
11. Kochubeĭ A.N. *On extensions of symmetric operators and symmetric binary relations*// Math. Zametki. – 1975. – V.17, №1. – P. 41–48. (in Russian)
12. Kochubeĭ A.N. *One dimensional point interactions*// Ukr. Math. J. – 1989. – V.41, №10. – P. 90–95.
13. Kostenko A.S., Malamud M.M. *1-D Schrődinger operators with local point interactions on a discrete set*// J. Differ. Equations. – 2010. – V.249, №2. – P. 253–304.
14. Kreĭn M.G. *The theory of selfadjoint extensions of semibounded Hermitian transformations and its applications, I*// Mat. Sbornik. – 1947. – V.20, №3. – P. 431–495. (in Russian)
15. Lyantse V.E., Storozh O.G., Methods of the theory of unbounded operators. – K.: Naukova Dumka, 1983. (in Russian)
16. Malamud M.M. *On some classes of Hermitian operators with gaps*// Ukrainian Mat. J. – 1992. – V.44, №2. – P. 215–234. (in Russian)
17. Pipa H.M., Storozh O.G. *Accretive perturbations of proper extensions for positively definite operator*// Mat. Stud. – 2006. – V.25, №2. – P. 181–190. (in Ukrainian)

Department of Mathematical Analysis
East Ukrainian National University
yury_kovalev@ukr.net
yury.kovalev.lugansk@gmail.com

Received 20.03.2012