1D NONNEGATIVE SCHRÖDINGER OPERATORS WITH POINT INTERACTIONS


Let $Y$ be an infinite discrete set of points in $\mathbb{R}$, satisfying the condition $\inf\{|y-y'|, \, y, y' \in Y, \, y' \neq y\} > 0$. In the paper we prove that the systems $\{\delta(x-y)\}_{y \in Y}$, $\{\delta'(x-y)\}_{y \in Y}$, $\{\delta(x-y), \, \delta'(x-y)\}_{y \in Y}$ form Riesz bases in the corresponding closed linear spans in the Sobolev spaces $W^{-1}_2(\mathbb{R})$ and $W^{-2}_2(\mathbb{R})$. As an application, we prove the transversality of the Friedrichs and Krein nonnegative selfadjoint extensions of the nonnegative symmetric operators $A_0$, $A'$, and $H_0$ defined as restrictions of the operator $A = -\frac{d^2}{dx^2}$, dom($A$) = $W^{-2}_2(\mathbb{R})$ to the linear manifolds dom($A_0$) = $\{f \in W^{-2}_2(\mathbb{R}) : f(y) = 0, \, y \in Y\}$, dom($A'$) = $\{g \in W^{-2}_2(\mathbb{R}) : g'(y) = 0, \, y \in Y\}$, and dom($H_0$) = $\{f \in W^{-2}_2(\mathbb{R}) : f(y) = 0, \, f'(y) = 0, \, y \in Y\}$, respectively. Using the divergence forms, the basic nonnegative boundary triplets for $A_0^*$, $A'^*$, and $H_0^*$ are constructed.


Пусть $Y$ бесконечное дискретное множество точек в $\mathbb{R}$, удовлетворяющее условию $\inf\{|y-y'|, \, y, y' \in Y, \, y' \neq y\} > 0$. Мы показываем, что системы $\{\delta(x-y)\}_{y \in Y}$, $\{\delta'(x-y)\}_{y \in Y}$, $\{\delta(x-y), \, \delta'(x-y)\}_{y \in Y}$ образуют базисы Рисса в соответствующих замкнутых линейных оболочках в пространствах Соболева $W^{-1}_2(\mathbb{R})$ и $W^{-2}_2(\mathbb{R})$. В приложении мы показываем трансверсальность неотрицательных самосопряженных расширений Фридрихса и Крейна неотрицательных симметрических операторов $A_0$, $A'$ и $H_0$, определенных как сужение оператора $A = -\frac{d^2}{dx^2}$, dom($A$) = $W^{-2}_2(\mathbb{R})$ на линейные многообразия dom($A_0$) = $\{f \in W^{-2}_2(\mathbb{R}) : f(y) = 0, \, y \in Y\}$, dom($A'$) = $\{g \in W^{-2}_2(\mathbb{R}) : g'(y) = 0, \, y \in Y\}$ и dom($H_0$) = $\{f \in W^{-2}_2(\mathbb{R}) : f(y) = 0, \, f'(y) = 0, \, y \in Y\}$ соответственно. Используя дивергентную форму, построены базисные неотрицательные граничные тройки для $A_0^*$, $A'^*$ и $H_0^*$.

1. Introduction. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_- = \{j \in \mathbb{Z}, \, j \leq -1\}$, $\mathbb{Z}_+ = \{j \in \mathbb{Z}, \, j \geq 1\}$. By $\mathbb{J}$ we will denote one of the sets $\mathbb{Z}$, $\mathbb{Z}_-$, $\mathbb{Z}_+$. Let $Y$ be a finite or infinite monotone sequence of points in $\mathbb{R}$. When $Y$ is infinite we will suppose that

$$\inf\{|y_j - y_k|, \, j \neq k\} = d > 0.$$  \hspace{1cm} (1)

For an infinite $Y$, the following three cases are possible

$Y = \{y_j, \, j \in \mathbb{Z}\}$, if $\inf\{Y\} = -\infty$ and $\sup\{Y\} = +\infty$,

$Y = \{y_j, \, j \in \mathbb{Z}_-\}$, if $y_{-1} = \sup\{Y\} < +\infty$, $Y = \{y_j, \, j \in \mathbb{Z}_+\}$, if $y_1 = \inf\{Y\} > -\infty$.

Clearly, the notation $Y = \{y_j, \, : j \in \mathbb{J}\}$ serves all these cases.

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Let $W_{2}^{±1}(\mathbb{R}), W_{2}^{±2}(\mathbb{R})$ be Sobolev spaces. Define in the Hilbert space $L_{2}(\mathbb{R})$ the linear operators

$$
\text{dom}(A_0) = \{ f \in W_{2}^{2}(\mathbb{R}) : f(y) = 0, \; y \in Y \}, \quad A_0 := -\frac{d^2}{dx^2},
$$

$$
\text{dom}(A') = \{ g \in W_{2}^{2}(\mathbb{R}) : g'(y) = 0, \; y \in Y \}, \quad A' := -\frac{d^2}{dx^2},
$$

$$
\text{dom}(H_0) = \{ f \in W_{2}^{2}(\mathbb{R}) : f(y) = 0, \; f'(y) = 0, \; y \in Y \}, \quad H_0 := -\frac{d^2}{dx^2}.
$$

The operators $A_0, A'$, and $H_0$ are basic for investigations of Hamiltonians on the real line corresponding to the $\delta, \delta'$ and $\delta - \delta'$ interactions, respectively ([1]). They are symmetric, densely defined, closed, and nonnegative ([1]), and are restrictions of the selfadjoint and nonnegative operator $A$ defined by

$$
\text{dom}(A) = W_{2}^{2}(\mathbb{R}), \quad A = -\frac{d^2}{dx^2},
$$

In addition, the operators $A_0$ and $A'$ are symmetric extensions of the operator $H_0$. The adjoint operators are given by

$$
\text{dom}(A_0^*) = W_{2}^{1}(\mathbb{R}) \cap W_{2}^{2}(\mathbb{R} \setminus Y), \quad A_0^* = -\frac{d^2}{dx^2},
$$

$$
\text{dom}(A'^*) = \{ g \in W_{2}^{2}(\mathbb{R}) : g'(y+) = g'(y-), \; y \in Y \}, \quad A'^* = -\frac{d^2}{dx^2},
$$

$$
\text{dom}(H_0^*) = W_{2}^{2}(\mathbb{R} \setminus Y), \quad H_0^* = -\frac{d^2}{dx^2}.
$$

It is well known ([1]) that

$$
\delta_y = \delta(x - y) \in W_{2}^{-1}(\mathbb{R}) \setminus L_{2}(\mathbb{R}), \quad (\delta_y)' = \delta'(x - y) \in W_{2}^{-2}(\mathbb{R}) \setminus W_{2}^{-1}(\mathbb{R}),
$$

where $\delta(x - y)$ and $\delta'(x - y)$ are the delta-function and its derivative.

We have the following chain of Hilbert spaces $W_{2}^{2}(\mathbb{R}) \subset W_{2}^{1}(\mathbb{R}) \subset L_{2}(\mathbb{R}) \subset W_{2}^{-1}(\mathbb{R}) \subset W_{2}^{-2}(\mathbb{R})$. The triplets $W_{2}^{2}(\mathbb{R}) \subset L_{2}(\mathbb{R}) \subset W_{2}^{-2}(\mathbb{R})$ and $W_{2}^{1}(\mathbb{R}) \subset L_{2}(\mathbb{R}) \subset W_{2}^{-1}(\mathbb{R})$ are rigged Hilbert spaces, i.e., the Hilbert space $W_{2}^{-2}(\mathbb{R})$ ($W_{2}^{-1}(\mathbb{R})$, respectively) is the set of all continuous anti-linear functionals on $W_{2}^{2}(\mathbb{R})$ (on $W_{2}^{1}(\mathbb{R})$, respectively, [6]).

Let $Y = \{ y_j \in \mathbb{R}, \; j \in J \}$ be a discrete set in $\mathbb{R}$ satisfying (1). Define the following subspaces

$$
\Phi = \overline{\text{span}}_{W_{2}^{-2}(\mathbb{R})} \{ \delta'(x - y), \; y \in Y \} \quad \text{(the closure in} \; W_{2}^{-2}(\mathbb{R})),
$$

$$
\Psi_{-1} = \overline{\text{span}}_{W_{2}^{-1}(\mathbb{R})} \{ \delta(x - y), \; y \in Y \} \quad \text{(the closure in} \; W_{2}^{-1}(\mathbb{R})),
$$

$$
\Psi_{-2} = \overline{\text{span}}_{W_{2}^{-2}(\mathbb{R})} \{ \delta(x - y), \; y \in Y \} \quad \text{(the closure in} \; W_{2}^{-2}(\mathbb{R})),
$$

$$
\Omega = \overline{\text{span}}_{W_{2}^{-2}(\mathbb{R})} \{ \delta(x - y), \; \delta'(x - y), \; y \in Y \} \quad \text{(the closure in} \; W_{2}^{-2}(\mathbb{R})).
$$
Clearly, $\Psi_{-1} \subseteq \Psi_{-2}$. It is known ([1]) that $\Phi \cap L_2(\mathbb{R}) = \{0\}$, $\Psi_{-2} \cap L_2(\mathbb{R}) = \{0\}$, $\Omega \cap L_2(\mathbb{R}) = \{0\}$. Therefore, the operators $A', A_0$, and $H_0$ are densely defined and

$$\text{dom}(A') = \{f \in W_2^2(\mathbb{R}) : (f, \varphi) = 0, \varphi \in \Phi\},$$

$$\text{dom}(A_0) = \{f \in W_2^2(\mathbb{R}) : (f, \psi) = 0, \psi \in \Psi_{-2}\},$$

$$\text{dom}(H_0) = \{f \in W_2^2(\mathbb{R}) : (f, \omega) = 0, \omega \in \Omega\}.$$  (8, 9, 10)

In this paper we establish some new connections between the Sobolev spaces $W_2^1(\mathbb{R})$, $W_2^2(\mathbb{R})$ and the Hilbert space $\ell_2$. Using these connections we prove that

- $\Psi_{-1} = \Psi_{-2}$;
- the systems $\{\delta(x - y_j)\}_{j \in \mathbb{J}}$, $\{\delta'(x - y_j)\}_{j \in \mathbb{J}}$, $\{\delta(x - y_j), \delta'(x - y_j)\}_{j \in \mathbb{J}}$ form the Riesz bases of the subspaces $\Psi_{-2}$, $\Phi$, and $\Omega$, respectively;
- the Friedrichs and Krein extensions of $A'$, $A_0$, and $H_0$ are mutually transversal.

Finally, we construct basic positive boundary triplets ([2], [3]) for $A^*$, $A^*_0$, and $H^*_0$ and give descriptions of all nonnegative selfadjoint extensions.

2. The Sobolev spaces $W_2^1(\mathbb{R})$, $W_2^2(\mathbb{R})$ and the Hilbert space $\ell_2$. In this Section we establish some connections between the Hilbert spaces $W_2^1(\mathbb{R})$, $W_2^2(\mathbb{R})$ and the Hilbert space $\ell_2(\mathbb{J})$.

**Proposition 1.** Suppose $Y$ is infinite and (1) holds. Then

1) If $g \in W_2^2(\mathbb{R})$ then the sequences $\{g(y_j), y_j \in Y\}$ and $\{g'(y_j), y_j \in Y\}$ belong to $\ell_2(\mathbb{J})$.

Moreover, there exists a positive constant $c$ such that

$$\|\{g(y_j)\}\|_{\ell_2(\mathbb{J})} \leq c\|g\|_{W_2^2(\mathbb{R})}, \|\{g'(y_j)\}\|_{\ell_2(\mathbb{J})} \leq c\|g\|_{W_2^2(\mathbb{R})}, \forall g \in W_2^2(\mathbb{R}).$$

2) If $\{a_j, j \in \mathbb{J}\}$, $\{b_j, j \in \mathbb{J}\} \in \ell_2(\mathbb{J})$ then there exists a function $g \in W_2^2(\mathbb{R})$ such that $g(y_j) = a_j$, $g'(y_j) = b_j$, $\forall j \in \mathbb{J}$.

**Proof.** 1) Let $g \in W_2^2(\mathbb{R})$. One can verify that the equalities

$$g(y_j) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y_j|} (g(x) - \text{sgn}(x-y_j)g'(x)) \, dx, \quad g'(y_j) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y_j|} (g'(x) - \text{sgn}(x-y_j)g''(x)) \, dx$$

hold. Further

$$|g(y_j)| \leq \frac{1}{2} \sum_{n \in \mathbb{J}} \left( \int_{y_{n-1}}^{y_n} e^{-2|x-y_j|} \, dx \right)^{1/2} \left( \int_{y_{n-1}}^{y_n} |g(x) - \text{sgn}(x-y_j)g'(x)|^2 \, dx \right)^{1/2} = \frac{1}{2} \sum_{n \in \mathbb{J}} M_{jn} h_n,$$

where $\{h_n, n \in \mathbb{J}\} \in \ell_2(\mathbb{J})$ because

$$\sum_{n \in \mathbb{J}} h_n^2 = \sum_{n \in \mathbb{J}} \int_{y_{n-1}}^{y_n} |g(x) - \text{sgn}(x-y_j)g'(x)|^2 \, dx \leq$$

$$\leq 2 \sum_{n \in \mathbb{J}} \int_{y_{n-1}}^{y_n} (|g(x)|^2 + |g'(x)|^2) \, dx \leq 2\|g\|_{W_2^2(\mathbb{R})}^2 < \infty,$$

$$\sum_{n \in \mathbb{J}} M_{jn} = \sum_{n \in \mathbb{J}} \left( \int_{y_{n-1}}^{y_n} e^{-2|x-y_j|} \, dx \right)^{1/2} \leq \sum_{n \in \mathbb{J}} \frac{1}{\sqrt{2}} \left\{ \begin{array}{ll} e^{-|y_n-y_j|}, & n \leq j, \\ e^{-|y_{n-1}-y_j|}, & n \geq j+1 \end{array} \right\} \leq$$

$$\leq \sqrt{2} \sum_{m \in \mathbb{Z}} e^{-|m|d} = \sqrt{2} \frac{e^d + 1}{e^d - 1}.$$
Let $M$ be the linear operator in $\ell_2(\mathbb{J})$ given by the matrix $\|M_{jn}\|_{j,n \in \mathbb{J}}$. Then the Holmgren bound of $M$ ([1, Appendix C]) satisfies

$$
\|M\|_H = \left( \sup_{j} \sum_{n} |M_{jn}| \right)^{1/2} \left( \sup_{n} \sum_{j} |M_{jn}| \right)^{1/2} \leq \sqrt{2} \frac{e^d + 1}{e^d - 1} < \infty.
$$

It follows that $M$ is bounded in $\ell_2(\mathbb{J})$. Hence

$$
\sum_{j \in \mathbb{J}} |g(y_j)|^2 \leq \frac{1}{4} \sum_{j \in \mathbb{J}} \left( \sum_{n \in \mathbb{J}} M_{jn} h_n \right)^2 = \frac{1}{4} \|M h\|_{\ell_2(\mathbb{J})}^2 \leq \frac{1}{4} \|M\|_H^2 \|g\|_{W_2^2(\mathbb{R})}^2 \leq \left( \frac{1}{\sqrt{2} e^d - 1} \right)^2 \|g\|_{W_2^2(\mathbb{R})}^2 = c_2^2 \|g\|_{W_2^2(\mathbb{R})}^2 < \infty. \quad (11)
$$

Similarly $\sum_{j \in \mathbb{J}} |g'(y_j)|^2 \leq c_2^2 \|g\|_{W_2^2(\mathbb{R})}^2 < \infty$. So, $\{g(y_j), y_j \in Y\}, \{g'(y_j), y_j \in Y\} \in \ell_2(\mathbb{J})$.

2) Let

$$
f_{\alpha}(t) = \begin{cases} 
e \exp \left( \frac{\alpha^2}{t^2} \right) \frac{-a^2(a+bt)}{t^2-a^2}, & |t| \leq \alpha; \\ 0, & \text{otherwise}, \end{cases}
$$

Clearly $f_{\alpha}(t) \in W_2^2(\mathbb{R})$ and $f_{\alpha}(0) = a$. Further

$$
f'_{\alpha}(t) = \begin{cases} e \exp \left( \frac{\alpha^2}{t^2} \right) \frac{\alpha^2}{(t^2-a^2)^2} (b t^4 + 2 a t^2 + 2 b \alpha^2 t + 2 \alpha^4), & |t| \leq \alpha; \\ 0, & \text{otherwise}, \end{cases}
$$

and $f'_{\alpha}(0) = b$.

$$
f''_{\alpha}(t) = \begin{cases} e \exp \left( \frac{\alpha^2}{t^2} \right) \frac{\alpha^2}{(t^2-a^2)^2} \frac{\alpha^2(2-2bt^2-6b^2t^2+12b^2t^4-4a^2t^4-2a^2t^2+6a^2t^2+8a^2t^2)}{(t^2-a^2)^2}, & |t| \leq \alpha; \\ 0, & \text{otherwise}. \end{cases}
$$

Let $\{a_k, k \in \mathbb{J}\}$, $\{b_k, k \in \mathbb{J}\} \in \ell_2(\mathbb{J})$,

$$
g_k(x) = f_{d/2}(x - y_k) = \begin{cases} e \exp \left( \frac{(d/2)^2}{(x-y_k)^2-(d/2)^2} \right) \frac{-(d/2)(a_k+b_k(x-y_k))}{((x-y_k)^2-(d/2)^2)}, & |x-y_k| \leq d/2; \\ 0, & \text{otherwise}, \end{cases}
$$

and $g(x) = \sum_{k \in \mathbb{J}} g_k(x)$, then $g(y_k) = a_k$, $g'(y_k) = b_k$. Now we show that the function $g(x)$ belongs to $W_2^2(\mathbb{R})$.

$$
\int_{\mathbb{R}} |g(x)|^2 \, dx = \int_{\mathbb{R}} \sum_{k \in \mathbb{J}} |g_k(x)|^2 \, dx \leq \sum_{k \in \mathbb{J}} \int_{y_k-d/2}^{y_k+d/2} e^2 \exp \left( \frac{2(d/2)^2}{(x-y_k)^2-(d/2)^2} \right) \frac{(d/2)^4 |a_k+b_k(x-y_k)|^2}{((x-y_k)^2-(d/2)^2)^2} \, dx = \sum_{k \in \mathbb{J}} e^2 (d/2)^4 \int_{-d/2}^{d/2} \exp \left( \frac{2(d/2)^2}{t^2-(d/2)^2} \right) \frac{|a_k+b_k|}{(t^2-(d/2)^2)^2} \frac{|a_k+b_k|^2}{(t^2-(d/2)^2)^2} \, dt \leq 2e^2 (d/2)^4 \sum_{k \in \mathbb{J}} \int_{-d/2}^{d/2} \exp \left( \frac{2(d/2)^2}{t^2-(d/2)^2} \right) \frac{dt}{(t^2-(d/2)^2)^2} +
$$
Set
\[ I_1 = \int_{-d/2}^{d/2} \exp\left( \frac{2(d/2)^2}{t^2 - (d/2)^2} \right) \frac{dt}{(t^2 - (d/2)^2)^2}, \quad I_2 = \int_{-d/2}^{d/2} \exp\left( \frac{2(d/2)^2}{t^2 - (d/2)^2} \right) \frac{t^2 dt}{(t^2 - (d/2)^2)^2}, \]
then we obtain \( \int \|g(x)\|^2 dx \leq 2e^2(d/2)^4 \left( \|a\|_{\ell_2(J)}^2 I_1 + \|b\|_{\ell_2(J)}^2 I_2 \right) < \infty \). Similarly
\[ \int \|g'(x)\|^2 dx \leq e^2(d/2)^4 \left( \|a\|_{\ell_2(J)}^2 P_1 + \|b\|_{\ell_2(J)}^2 P_2 \right) < \infty, \]
\[ \int \|g''(x)\|^2 dx \leq e^2(d/2)^4 \left( \|a\|_{\ell_2(J)}^2 S_1 + \|b\|_{\ell_2(J)}^2 S_2 \right) < \infty. \]
So, \( g(x) \in W_2^2(\mathbb{R}) \).

**Corollary 1.** If \( f \in W_2^1(\mathbb{R}) \) then the sequence \( \{f(y_j), \quad y_j \in Y\} \) belongs to \( \ell_2(\mathbb{J}) \).

**Proof.** Due to inequality (11) we have
\[ \|\{f(y_j), \quad y_j \in Y\}\|_{\ell_2(\mathbb{J})}^2 \leq \left( \frac{1}{\sqrt{2}e^d - 1} \right)^2 \|f\|_{W_2^1(\mathbb{R})}^2 < \infty. \]

**Proposition 2.** If \( f \in W_2^1(\mathbb{R} \setminus Y) \) then the sequence \( \{f(y_j+) - f(y_j-), \quad y_j \in Y\} \) belongs to \( \ell_2(\mathbb{J}) \).

**Proof.** Let \( g(x) \) from \( W_2^1(\mathbb{R} \setminus Y) \) be real, then the equalities
\[ g^2(y_j-) - g^2(y_j-1+)e^{-(y_j-y_{j-1})} = \int_{y_{j-1}+}^{y_j-} e^{-|x-y_j|} (g^2(x) + 2g(x)g'(x)) dx, \]
\[ g^2(y_{j-1}+) - g^2(y_j-)e^{-(y_{j-1}-y_j)} = \int_{y_{j-1}+}^{y_j} e^{-|x-y_{j-1}|} (g^2(x) - 2g(x)g'(x)) dx \]
hold. From (12) we have
\[ (g^2(y_j-) + g^2(y_{j-1}+))(1 - e^{-(y_j-y_{j-1})}) = \]
\[ = \int_{y_{j-1}+}^{y_j} \left[ g^2(x)(e^{-|x-y_j|} + e^{-|x-y_{j-1}|}) + 2g(x)g'(x)(e^{-|x-y_j|} - e^{-|x-y_{j-1}|}) \right] dx \leq \]
\[ \leq \int_{y_{j-1}+}^{y_j} \left[ 2g^2(x) + 4|g(x)g'(x)| \right] dx \leq \int_{y_{j-1}+}^{y_j} \left[ 4g^2(x) + 2g^2(x) \right] dx. \]
Since \( 1 - e^{-(y_j-y_{j-1})} \geq 1 - e^{-d} \), we obtain
\[ \sum_{j \in \mathbb{J}} (g^2(y_j-) + g^2(y_{j-1}+))(1 - e^{-d}) \leq \int_{\mathbb{R} \setminus Y} \left[ 4g^2(x) + 2g^2(x) \right] dx, \]
and hence
\[ \sum_{j \in \mathbb{J}} (g^2(y_j-) + g^2(y_j+)) < \infty. \]  

Consider \( f(x) = f_R(x) + i f_I(x) \) from \( W_2^1(\mathbb{R} \setminus Y) \), then for \( f_R(x) \) and \( f_I(x) \) inequality (13) holds and hence \( \sum_{j \in \mathbb{J}} (|f(y_j-)|^2 + |f(y_j+)|^2) < \infty. \)

Since \( |f(y_j+) - f(y_j-)|^2 \leq 2(|f(y_j-)|^2 + |f(y_j+)|^2) \), we obtain that \( \{ f(y_j+) - f(y_j-) \}, j \in \mathbb{J} \} \subset \ell_2(\mathbb{J}). \) \[ \square \]

3.1. Applications. Let \( A \) be an unbounded self-adjoint operator in a Hilbert space \( H \) and let \( H_{+2} \subset H_{+1} \subset H \subset H_{-1} \subset H_{-2} \) be the chain of rigged Hilbert spaces (6) constructed by means of \( A: H_{+2} = \text{dom}(A), H_{+1} = \text{dom}(|A|^{1/2}) \) with norms \( \| f \|_k = (|A|^{k/2} f)^2 + \| f \|^2, k \in \{1, 2\} \). The “negative” Hilbert spaces \( H_{-k} (k \in \{1, 2\}) \) are the completion of \( H \) with respect to the norms
\[ \| f \|_{-k} = \sup_{g \in H_k, \| g \|_k = 1} |(f, g)|. \]

The operator \( A \) has an extension \( A \in \mathcal{L}(H_k, H_{k-1}), k \in \{0, 1\} \) (\( H_0 := H \) and \( |A|^{1/2} \in \mathcal{L}(H_k, H_{k-1}), k \in \{-1, 0\} \) is an extension of \( |A|^{1/2} \). The resolvent \( R_z = (A - zI)^{-1}, z \in \rho(A) \) has an extension \( R_z = (A - zI)^{-1} \in \mathcal{L}(H_{-k}, H_{-k+2}), k \in \{0, 1, 2\} \). Let \( \Phi \) be a subspace in \( H_{-2} \) such that
\[ \Phi \cap H = \{0\}, \]
then the operator \( A' \) defined by
\[ \text{dom}(A') = \left\{ f \in H_{+2}: (f, \varphi) = 0 \text{ for all } \varphi \in \Phi \right\}, \quad A' = A| \text{dom}(A') \]
is a closed, densely defined symmetric operator with the defect numbers equal to \( \dim \Phi \). For the defect subspace \( \mathfrak{N}_2(A') = \ker(A'' - zI) \) the formula \( \mathfrak{N}_2(A') = R_z^* \Phi \) holds.

Suppose that \( A \) is a nonnegative operator. Then as it is well known, \( A \) is the Friedrichs extension of \( A' \) if and only if \( \Phi \cap H_{-1} = \{0\} \).

The operator \( A \) given by (5) is nonnegative and self-adjoint in \( H = L_2(\mathbb{R}) \). Set for convenience
\[ H_{+2} = \text{dom}(A) = W_2^2(\mathbb{R}), \quad H_{+1} = \text{dom}(A^{1/2}) = W_2^1(\mathbb{R}), \quad H_{-1} = W_2^{-1}(\mathbb{R}), \quad H_{-2} = W_2^{-2}(\mathbb{R}). \]

As mentioned above, (see (7)) one has \( \delta_y = \delta(x-y) \in H_{-1} \setminus H, (\delta_y)' = \delta'(x-y) \in H_{-2} \setminus H_{-1} \). Let \( Y = \{y_j \in \mathbb{R}, j \in \mathbb{J}\} \) be a discrete set in \( \mathbb{R} \) satisfying (1).

The defect subspaces of \( A', A_0 \), and \( H_0 \) are given by (see [1])
\[ \mathfrak{N}_\lambda(A') = \overline{\text{span}} \left\{ \text{sgn}(x - y_j) \exp(i\sqrt{\lambda}|x - y_j|), j \in \mathbb{J} \right\}, \]
\[ \mathfrak{N}_\lambda(A_0) = \overline{\text{span}} \{ \exp(i\sqrt{\lambda}|x - y_j|), j \in \mathbb{J} \}, \]
\[ \mathfrak{N}_\lambda(H_0) = \overline{\text{span}} \{ \exp(i\sqrt{\lambda}|x - y_j|), \text{sgn}(x - y_j) \exp(i\sqrt{\lambda}|x - y_j|), j \in \mathbb{J} \}, \]
respectively.

3.2. Riesz bases. Recall [8] that a countable set of vectors \( \{g_j\} \) forms a Riesz basis in a separable Hilbert space \( \mathfrak{H} \) if \( \overline{\text{span}} \{g_j\} = \mathfrak{H} \) and there exist two positive numbers \( a_1 \) and \( a_2 \)
Therefore, the system

\[ a_2 \sum_{j=1}^{n} |c_j|^2 \leq \left\| \sum_{j=1}^{n} c_j g_j \right\|_\delta^2 \leq a_1 \sum_{j=1}^{n} |c_j|^2. \]

Since \( \{e_j\}_{j \in \mathbb{J}} \) forms a Riesz basis \( \mathcal{H} \), every \( f \in \mathcal{H} \) has an expansion \( f = \sum_{j \in \mathbb{J}} c_j e_j \) with \( \sum_{j \in \mathbb{J}} |c_j|^2 < \infty \), and conversely, if \( \sum_{j \in \mathbb{J}} |c_j|^2 < \infty \) then the series \( \sum_{j \in \mathbb{J}} c_j e_j \) converges in \( \mathcal{H} \).

**Proposition 3.** The systems \( \{\delta(x - y_j)\}_{j \in \mathbb{J}}, \{\delta'(x - y_j)\}_{j \in \mathbb{J}} \) and \( \{\delta(x - y_j), \delta'(x - y_j)\}_{j \in \mathbb{J}} \) form Riesz bases of the subspaces \( \Psi_{\mathbb{J}}, \Phi \) and \( \Omega \), respectively.

**Proof.** We will show that \( \{\delta(x - y_j), \delta'(x - y_j)\}_{j \in \mathbb{J}} \) is a Riesz basis of the subspace \( \Omega \).

Let \( f = \sum_{j} a_j \delta(x - y_j) + b_j \delta'(x - y_j) \in \Omega \), where \( \{a_j\}_{j \in \mathbb{J}}, \{b_j\}_{j \in \mathbb{J}} \in l_2(\mathbb{J}) \), then using the first statement of Propositions (1) we get

\[
\left\| \sum_{j} a_j \delta(x - y_j) + b_j \delta'(x - y_j) \right\|_{\mathcal{H}}^2 = \sup_{g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1} |(f, g)|^2 = \sup_{g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1} \left| \sum_{j} a_j g(y_j) + b_j g'(y_j) \right|^2 \leq 2 \left( \sup_{g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1} \sum_{j} |a_j|^2 \|g(y_j)\|^2 + \sup_{g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1} \sum_{j} |b_j|^2 \|g'(y_j)\|^2 \right) = C_1 \|a\|_{l_2(\mathbb{J})}^2 + C_2 \|b\|_{l_2(\mathbb{J})}^2 < \infty.
\]

On the other hand, using the second statement of Proposition (1) we have

\[
\sup_{g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1} \left| \sum_{j} a_j g(y_j) + b_j g'(y_j) \right|^2 \geq \left( \sum_{j} a_j \bar{a}_j \|a\|^2 + b_j \|b\|^2 \right)^2 = (\|a\|_{l_2(\mathbb{J})}^2 + \|b\|_{l_2(\mathbb{J})}^2)^2.
\]

Therefore, the system \( \{\delta(x - y_j), \delta'(x - y_j)\}_{j \in \mathbb{J}} \) forms a Riesz basis of the subspace \( \Omega \).

The other statements can be proved similarly. \( \square \)

### 3.3. Transversality of the Friedrichs and Krein extensions.

Let \( H \) be a separable Hilbert space and let \( \mathcal{A} \) be a densely defined closed symmetric and nonnegative operator. Denote by \( \mathcal{A}^* \) the adjoint to \( \mathcal{A} \), by \( \widetilde{\mathcal{A}} \) a nonnegative selfadjoint extension of \( \mathcal{A} \). It is well known ([11]) that the operator \( \mathcal{A} \) admits at least one nonnegative self-adjoint extension \( \mathcal{A}_F \) called the **Friedrichs extension**, which is defined as follows. Denote by \( \mathcal{A}[-, -] \) the closure of the sesquilinear form (see [10])

\[
\mathcal{A}[f, g] = (\mathcal{A}f, g), \ f, g \in \text{dom}(\mathcal{A}),
\]

and let \( \mathcal{D}[\mathcal{A}] \) be the domain of this closure. According to the first representation theorem ([10]) there exists a nonnegative self-adjoint operator \( \mathcal{A}_F \) associated with \( \mathcal{A}[-, -] \), i.e., \( (\mathcal{A}_F h, \psi) = \mathcal{A}[h, \psi], \ h \in \mathcal{D}[\mathcal{A}], \ \psi \in \text{dom}(\mathcal{A}_F). \) Clearly \( \mathcal{A} \subset \mathcal{A}_F \subset \mathcal{A}^* \), where \( \mathcal{A}^* \) is the adjoint operator to \( \mathcal{A} \). It follows that \( \text{dom}(\mathcal{A}_F) = \mathcal{D}[\mathcal{A}] \cap \text{dom}(\mathcal{A}^*). \) By the second representation theorem, the equalities \( \mathcal{D}[\mathcal{A}] = \text{dom}(\mathcal{A}^{1/2}_F) \) and \( \mathcal{A}[\phi, \psi] = (\mathcal{A}_F^{1/2} \phi, \mathcal{A}_F^{1/2} \psi), \ \phi, \psi \in \mathcal{D}[\mathcal{A}] \) hold.
M. G. Kreĭn in [14] discovered one more nonnegative self-adjoint extension of $\mathcal{A}$ having extremal property to be minimal (in the sense of the corresponding quadratic forms) among others nonnegative self-adjoint extensions of $\mathcal{A}$. This extension we will denote by $\mathcal{A}_K$ and call it the Kreĭn extension of $\mathcal{A}$.

Recall that two selfadjoint extensions $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ of a symmetric operator $\mathcal{A}$ are called disjoint if $\text{dom}(\tilde{\mathcal{A}}_1) \cap \text{dom}(\tilde{\mathcal{A}}_2) = \text{dom}(\mathcal{A})$ and transversal if $\text{dom}(\tilde{\mathcal{A}}_1) + \text{dom}(\tilde{\mathcal{A}}_2) = \text{dom}(\mathcal{A}^*)$. We need the following statement ([4], [16]).

**Proposition 4.** The Friedrichs and Kreĭn extensions $\mathcal{A}_F$ and $\mathcal{A}_K$ are transversal if $\mathcal{N}_z \subset \text{dom}(\mathcal{A}_{1/2})$ at least for one (hence for all) $z \in \mathbb{C} \setminus [0, \infty)$.

In what follows we will consider our operators (2)–(4) in the $p$-representation by means of the Fourier transform

$$\hat{f}(p) = (\mathcal{F} f)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ipx} \, dx.$$

Note that

$$(\mathcal{F}\delta_y)(p) = \hat{\delta}_y(p) = \frac{1}{\sqrt{2\pi}} e^{-ipy}, \quad (\mathcal{F}\delta_y')(p) = \hat{\delta}'_y(p) = \frac{ipe^{-ipy}}{\sqrt{2\pi}},$$

and the Fourier transformation $\mathcal{F}$ is a unitary operator from $L_2(\mathbb{R}, dx)$ onto $L_2(\mathbb{R}, dp)$. In addition

$$\text{dom}(\tilde{\mathcal{A}}) = \tilde{\mathcal{H}}_+ = \left\{ \hat{f} \in L_2(\mathbb{R}, dp): \int_{\mathbb{R}} |\hat{f}(p)|^2 (p^4 + 1) \, dp < \infty \right\}, \quad (\tilde{\mathcal{A}} \hat{f})(p) = p^2 \hat{f}(p),$$

$$\text{dom}(\tilde{\mathcal{A}}^{1/2}) = \tilde{\mathcal{H}}_+ = \left\{ \hat{f} \in L_2(\mathbb{R}, dp): \int_{\mathbb{R}} |\hat{f}(p)|^2 (p^2 + 1) \, dp < \infty \right\}, \quad (\tilde{\mathcal{A}}^{1/2} \hat{f})(p) = |p| \hat{f}(p).$$

$$\text{dom}(\tilde{\mathcal{A}}_0) = \tilde{\mathcal{H}}_+ = \left\{ \hat{f} \in L_2(\mathbb{R}, dp): \int_{\mathbb{R}} e^{ip\psi_j} \hat{f}(p) \, dp = 0, \quad j \in \mathbb{J} \right\}, \quad (\tilde{\mathcal{A}}_0 \hat{f})(p) = p^2 \hat{f}(p),$$

$$\text{dom}(\tilde{\mathcal{H}}_0) = \left\{ \hat{f} \in \tilde{\mathcal{H}}_+ : \int_{\mathbb{R}} e^{ip\psi_j} \hat{f}(p) \, dp = 0, \quad \int_{\mathbb{R}} e^{ip\psi_j} \hat{f}(p) \, dp = 0 \right\}, \quad (\tilde{\mathcal{H}}_0 \hat{f})(p) = p^2 \hat{f}(p).$$

The pairs of operators $\langle \tilde{\mathcal{A}}, \mathcal{A} \rangle$, $\langle \tilde{\mathcal{A}}', \mathcal{A}' \rangle$, $\langle \tilde{\mathcal{A}}_0, \mathcal{A}_0 \rangle$, and $\langle \tilde{\mathcal{H}}_0, \mathcal{H}_0 \rangle$ are unitary equivalent since $\mathcal{F}A = \tilde{\mathcal{A}} \mathcal{F}$. Clearly, $\tilde{\mathcal{H}}_+ = \mathcal{F} \mathcal{H}_+, \tilde{\mathcal{H}}_+ = \mathcal{F} \mathcal{H}_+$,

$$\tilde{\mathcal{H}}_+ = \mathcal{F} \mathcal{H}_+ = \left\{ \hat{f} \in \tilde{\mathcal{H}}_+ : \frac{\hat{f}(p)}{p^4 + 1} \in \mathcal{H}_+ \right\}, \quad \|\hat{f}(p)\|^2_{\tilde{\mathcal{H}}_+} = \int_{\mathbb{R}} \frac{|\hat{f}(p)|^2}{p^4 + 1} \, dp,$$

$$\tilde{\mathcal{H}}_+ = \mathcal{F} \mathcal{H}_+ = \left\{ \hat{f} \in \tilde{\mathcal{H}}_+ : \frac{\hat{f}(p)}{p^2 + 1} \in \mathcal{H}_+ \right\}, \quad \|\hat{f}(p)\|^2_{\tilde{\mathcal{H}}_+} = \int_{\mathbb{R}} \frac{|\hat{f}(p)|^2}{p^2 + 1} \, dp,$$

$$\tilde{\mathcal{A}} \hat{f} = p^2 \hat{f}(p), \quad \tilde{\mathcal{A}}: \tilde{\mathcal{H}}_+ \to \mathcal{H}_+, \quad \mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R}).$$

Let $\Phi = \mathcal{F} \Phi$, $\Psi_- = \mathcal{F} \Psi_-, \hat{\Psi}_- = \mathcal{H}_- \Phi$, $\hat{\Omega} = \mathcal{F} \Omega$. Then

$$\Phi = \text{span}_{\tilde{\mathcal{H}}_+} \{pe^{-ip\xi_j}, j \in \mathbb{J} \}, \quad \hat{\Psi}_- = \text{span}_{\tilde{\mathcal{H}}_+} \{e^{-ip\xi_j}, j \in \mathbb{J} \},$$

$$\hat{\Psi}_- = \text{span}_{\tilde{\mathcal{H}}_+} \{e^{-ip\xi_j}, j \in \mathbb{J} \}, \quad \hat{\Omega} = \text{span}_{\tilde{\mathcal{H}}_+} \{e^{-ip\xi_j}, pe^{-ip\xi_j}, j \in \mathbb{J} \}. $$
Theorem 1. The equality $\Psi_{-2} = \Psi_{-1}$ holds.

Proof. Let $f \in \Psi_{-2}$, then $f = \sum_k c_k \delta(x - y_k) \leq \infty$. Using Corollary (1) we have

$$\|f\|^2_{H_{-1}} = \sup_{g \in H_1, \|g\|_1 = 1} \|f, g\|^2 = \sup_{g \in H_1, \|g\|_1 = 1} \left| \sum_k c_k g(y_k) \right|^2 \leq \sum_k |c_k|^2 \sup_{g \in H_1, \|g\|_1 = 1} \sum_k |g(y_k)|^2 < \infty.$$ 

Therefore, $\Psi_{-2} \subset H_{-1}$ and $\Psi_{-2} = \Psi_{-1}$.

Corollary 2. The systems $\{\{e^{-ipy}\}_{j \in J}, \{pe^{-ipy}\}_{j \in J}\}$ form Riesz bases of the subspaces $\widehat{\Psi}_{-1}$, $\hat{\Phi}$ and $\widehat{\mathcal{N}}_{-1}(\hat{A}_0)$, $\widehat{\mathcal{N}}_{-1}(\hat{A}')$, respectively.

Proof. Since the operator $\mathcal{F}$ unitarily maps $H_{-2}$ onto $\widehat{H}_{-2}$, by Proposition 3, the systems $\{e^{-ipy}\}_{j \in J}$ and $\{pe^{-ipy}\}_{j \in J}$ form Riesz bases of $\widehat{\Psi}_{-1}$ and $\hat{\Phi}$, respectively. Let $\widehat{\mathcal{N}}_{-1}(\hat{A}') = \ker(\hat{A}' + I)$, $\widehat{\mathcal{N}}_{-1}(\hat{A}_0) = \ker(\hat{A}_0 + I)$. Then $\widehat{\mathcal{N}}_{-1}(\hat{A}') = (\hat{A}' + I)^{-1} \hat{\Phi}$, $\widehat{\mathcal{N}}_{-1}(\hat{A}_0) = (\hat{A}_0 + I)^{-1} \widehat{\mathcal{N}}_{-1}(\hat{A}_0)$, and $\{pe^{-ipy}\}_{j \in J}$ is a Riesz basis of $\widehat{\mathcal{N}}_{-1}(\hat{A}') \subset H$, $\{\frac{pe^{-ipy}}{p^2 + 1}\}_{j \in J}$ is a Riesz basis of $\widehat{\mathcal{N}}_{-1}(\hat{A}_0) \subset \widehat{H}_{+1}$.

Theorem 2. The equality $\Phi \cap H_{-1} = \{0\}$ holds.

Proof. Let $g \in \hat{\Phi}$, then $g(p) = \sum_k c_k p e^{-ipy_k}$, but by Corollary (2) $\int_{\mathbb{R}} \frac{1}{p^2 + 1} \left| \sum_k c_k p e^{-ipy_k} \right|^2 dp = \infty$, hence $g$ does not belong to $\widehat{H}_{-1}$, i.e. $\Phi \cap H_{-1} = \{0\}$ and $\Phi \cap H_{-1} = \{0\}$.

Corollary 3. The Friedrichs and Krein extensions of the operators $H_0, A', A_0$ are transversal.

Proof. Let $u \in \widehat{\mathcal{N}}_{-1}(\widehat{H}_0)$, then $u(p) = \sum_k \frac{a_k e^{-ipy_k}}{p^2 + 1} + \sum_k \frac{b_k pe^{-ipy_k}}{p^2 + 1}$. Using Corollary (2) we have

$$\sup_{f \in \text{dom} (\hat{H}_0)} \frac{|(\hat{H}_0f, u)|^2}{(\hat{H}_0f, f)} = \sup_{f \in \text{dom} (\hat{H}_0)} \frac{\left| \int_{\mathbb{R}} p^2 f(p) \overline{u(p)} dp \right|^2}{\int_{\mathbb{R}} p^2 |f(p)|^2 dp} \leq \sup_{f \in \text{dom} (\hat{H}_0)} \frac{\int_{\mathbb{R}} p^4 |f(p)|^2 dp \int_{\mathbb{R}} |u(p)|^2 dp}{\int_{\mathbb{R}} p^2 |f(p)|^2 dp} \leq \int_{\mathbb{R}} \left( \sum_k \frac{a_k e^{-ipy_k}}{p^2 + 1} + \sum_k \frac{b_k pe^{-ipy_k}}{p^2 + 1} \right)^2 dp.$$

So, $\widehat{\mathcal{N}}_{-1}(\hat{H}_0) \subset \text{dom}(\hat{H}_0^{1/2})$. Therefore, due to Proposition 4 the extensions $\hat{H}_0F$ and $\hat{H}_0K$ as well as $H_0F$ and $H_0K$ are transversal.

Transversalness of the Friedrichs and Krein extensions of the operators $A'$ and $A_0$ can be proved similarly.

Corollary 4. The operator $A$ is the Friedrichs extension of the operator $A'$.

Proof. Since $\Phi \cap H_{-1} = \{0\}$, we get that $A'|F = A$.

3.4. Basic boundary triplets for operators $A_0^*$, $A'^*$ and $H_0^*$. Let $S$ be a closed densely defined symmetric operator with equal defect numbers in $\mathfrak{H}$. Let $\mathcal{H}$ be some Hilbert space, $\Gamma_1$ and $\Gamma_2$ be linear mappings of $\text{dom}(S'^*)$ into $\mathcal{H}$. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for adjoint operator $S'^*$ ([7], [11], [9]), if

$$\langle S'^* x, y \rangle - \langle x, S'^* y \rangle = \langle \Gamma_1 x, \Gamma_0 y \rangle_\mathcal{H} - \langle \Gamma_0 x, \Gamma_1 y \rangle_\mathcal{H} \quad \text{for all} \quad x, y \in \text{dom}(S'^*), \quad (16)$$

\[ \]
and a mapping \( \Gamma: x \mapsto \{ \Gamma_0 x, \Gamma_1 x \} \), \( x \in \text{dom}(S^*) \) is a surjection of \( \text{dom}(S^*) \) onto \( \mathcal{H} \oplus \mathcal{H} \).

Let \( S \) be a densely defined and nonnegative operator. Suppose that the Friedrichs and Kreín extensions of \( S \) are transversal. The boundary triplet \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) for \( S^* \) is called basic ([2], [3] rigid for positive definite \( S \) [17]) if \( \ker(\Gamma_0) = \text{dom}(S_F) \), \( \ker(\Gamma_1) = \text{dom}(S_K) \). A basic boundary triplet is positive [2] and (see [3]) \( S_K[x, y] = (S^*x, y) - (\Gamma_1 x, \Gamma_0 y)_{\mathcal{H}}, \ x, y \in \text{dom}(S^*) \).

**Proposition 5** ([2]). Let \( S \) be a densely defined and nonnegative operator with transversal Friedrichs and Kreín extensions and let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a basic boundary triplet for \( S^* \). Then the mapping

\[
\Theta \mapsto S_\Theta := S^* \upharpoonright \Gamma^{-1} \Theta = S^* \upharpoonright \{ f \in \text{dom}(S^*) : (\Gamma_0 f, \Gamma_1 f) \in \Theta \}
\]  

establishes a bijective correspondence between the set of all selfadjoint nonnegative linear relations \( \Theta \) in \( \mathcal{H} \) and the set of all nonnegative selfadjoint extensions of \( S_\Theta \subseteq S^* \) of \( S \).

Assume that

(A) \( L_1 \) and \( L_2 \) are two closed densely defined operators in the Hilbert space \( \mathfrak{H} \) taking values in a Hilbert space \( \mathcal{H} \) and such that \( L_1 \subset L_2 \).

**Theorem 3** ([5]). Let condition (A) be fulfilled. If the operator \( \mathcal{A} = L_2^* L_1 \) is densely defined and \( \mathcal{A}^* = L_1^* L_2 \), then

1) the operator \( \mathcal{A}_F = L_1^* L_1 \) is the Friedrichs extension of \( \mathcal{A} \);

2) the Friedrichs and Kreín extensions of \( \mathcal{A} \) are transversal;

3) the operator

\[
\text{dom}\mathcal{A}_K = \{ f \in \text{dom}(L_2) : P_{\text{ran}(L_1)} L_2 f \in \text{dom}(L_2^*) \},
\]

\[
\mathcal{A}_K f = L_2^* P_{\text{ran}(L_1)} L_2 f = L_1^* L_2 f, \ f \in \text{dom}(\mathcal{A}_K)
\]

is the Kreín extension of \( \mathcal{A} \) and

\[
\mathcal{D}[\mathcal{A}_K] = \text{dom}(L_2), \quad \mathcal{A}_K[u, v] = (P_{\text{ran}(L_1)} L_2 u, P_{\text{ran}(L_1)} L_2 v), \quad u, v \in \text{dom}(L_2).
\]

The operator \( \mathcal{A} = L_2^* L_1 \) called an operator in the divergence form.

According to V. E. Lyantse and O. G. Storozh ([15]) a pair \( \{ \mathcal{H}, \Gamma \} \) is called a boundary pair for \( L_1 \subset L_2 \), if \( \mathcal{H} \) is a Hilbert space, \( \Gamma \in \mathcal{L}(\text{dom}(L_2), \mathcal{H}) \) and \( \ker(\Gamma) = \text{dom}(L_1) \), \( \text{ran}(\Gamma) = \mathcal{H} \). Let \( \{ \mathcal{H}, \Gamma \} \) be a boundary pair for \( L_1 \subset L_2 \). Then there exists a linear operator \( G \in \mathcal{L}(\text{dom}(L_1^*), \mathcal{H}) \) such that \( \{ \mathcal{H}, G \} \) is a boundary pair for \( L_2^* \subset L_1^* \) and the Green identity

\[
(L_1^* f, u)_H - (f, L_2 u)_\mathfrak{H} = (G f, \Gamma u)_\mathcal{H}, \ f \in \text{dom}(L_1^*), \ u \in \text{dom}(L_2)
\]  

holds. The set \( \{ \mathcal{H}, G, \Gamma \} \) is called the boundary triplet for a pair of the operators \( L_1 \subset L_2 \).

**Theorem 4.** Let condition (A) be fulfilled and let \( \{ \mathcal{H}, G, \Gamma \} \) be a boundary triplet for \( L_1 \subset L_2 \). If the operator \( \mathcal{A} = L_2^* L_1 \) is densely defined and \( \mathcal{A}^* = L_1^* L_2 \), then

1. the triplet \( \Pi = \{ \mathcal{H}, \Gamma, GP_{\text{ran}(L_1)} L_2 \} \) is a basic for \( \mathcal{A}^* \);

2. the mapping

\[
\Theta \mapsto \mathcal{A}_\Theta := \mathcal{A}^* \upharpoonright \Gamma^{-1} \Theta = \mathcal{A}^* \upharpoonright \{ f \in \text{dom}(\mathcal{A}^*) : (\Gamma f, GP_{\text{ran}(L_1)} L_2 f) \in \Theta \}
\]  

establishes a bijective correspondence between all nonnegative selfadjoint extensions of the operator \( \mathcal{A} \) and all nonnegative selfadjoint linear relations \( \Theta \) in \( \mathcal{H} \).
Proof. By Theorem 3, the Friedrichs and Krein extensions of $A$ are transversal, $\mathcal{D}[A_K] = \text{dom}(L_2), \mathcal{D}[A_F] = \text{dom}(L_1)$. Hence, $\{H, \Gamma\}$ is a boundary pair for $A$. Let $x \in \text{dom}(A^*) = \text{dom}(L_1L_2)$ and $y \in \text{dom}(L_2)$. Then $P_{\text{ran}(L_1)}L_2x = L_2x - P_{\ker(L_2^\ast)}L_2x \in \text{dom}(L_1^\ast)$. Using Theorem 3 and (18) we get

$$A_K[x, y] = (P_{\text{ran}(L_1)}L_2x, L_2u)_H = (L_1^\ast P_{\text{ran}(L_1)}L_2x, y)_{\delta} - (GP_{\text{ran}(L_1)}L_2x, \Gamma y)_H = (L_1^\ast L_2x, y)_{\delta} - (GP_{\text{ran}(L_1)}L_2x, \Gamma y)_H.$$ 

In particular, for $x, y \in \text{dom}(A^*)$ taking into account that the form $A_K[x, y]$ is Hermitian, we have $(A^\ast x, y) - (x, A^\ast y) = (GP_{\text{ran}(L_1)}L_2x, \Gamma y)_H - (\Gamma x, GP_{\text{ran}(L_1)}L_2y)_H$. Thus, the triplet $\Pi = \{H, \Gamma, GP_{\text{ran}(L_1)}L_2\}$ is basic for $S^\ast$. From Proposition 5 we get that statement (2) holds true. \hfill \Box

Consider in $L_2(\mathbb{R})$ the following operators

$$\text{dom}(L_0) = \{f \in W_2^1(\mathbb{R}) : f(y) = 0, y \in Y\}, \quad L_0 = i \frac{d}{dx}, \quad (20)$$

$$\text{dom}(L) = W_2^1(\mathbb{R}), \quad L = i \frac{d}{dx}. \quad (21)$$

From (20) it follows that $L_0$ is a densely defined symmetric operator and its adjoint $L_0^\ast$ is given by

$$\text{dom}(L_0^\ast) = W_2^1(\mathbb{R} \setminus Y), \quad L_0^\ast = i \frac{d}{dx}. \quad (22)$$

The operator $L$ is a selfadjoint extension of $L_0$. So, we have $L_0 \subset L \subset L_0^\ast$. From (3)–(22) it follows that

$$A_0 = LL_0, \quad A' = L_0L, \quad H_0 = L_0^2, \quad A = L^2, \quad A_0^\ast = L_0^*L, \quad A^\ast = LL_0^\ast, \quad H_0^* = L_0^2. \quad (23)$$

Using representation (23) and Theorem 3 the explicit expressions for the Friedrichs and Krein extensions of $A_0$, $A'$ and $H_0$ and their transversality have been obtained in [5]. In the next statements for the operators $A^\ast$, $A_0^\ast$ and $H_0^\ast$ explicit expressions for the basic boundary triplets and abstract boundary conditions for all nonnegative selfadjoint extensions are obtained.

**Proposition 6.** Set

$$\mathcal{H} = \begin{cases} \mathbb{C}^m, & Y \text{ consists of } m \text{ points}; \\ \ell_2(J), & Y \text{ is infinite}, \end{cases} \quad \text{dom}(\Gamma) = W_2^1(\mathbb{R} \setminus Y),$$

$$\Gamma u = \{i(u(y_j^+) - u(y_j^-)), j \in J\}, \quad \text{dom}(G) = W_2^1(\mathbb{R}), \quad Gf = \{f(y_j), j \in J\}.$$ 

Then

(i) $\{\mathcal{H}, \Gamma, G\}$ is the boundary triplet for pair $L \subset L_0^\ast$;

(ii) the triplet $\Pi = \{\mathcal{H}, \Gamma, G\mathcal{L}_0^\ast\}$ is basic for $A^\ast$, where $G\mathcal{L}_0^\ast$ is given by the relation $G\mathcal{L}_0^\ast f = \{if'(y_j), j \in J\}, \quad f \in \text{dom}(A^\ast)$;

(iii) the mapping

$$\Theta \mapsto A_0^\ast = A^\ast \upharpoonright \{f \in \text{dom}(A^\ast) : (\{i(f(y_j^+) - f(y_j^-)), j \in J\}, \{if'(y_j), j \in J\}) \in \Theta\}$$

establishes a one-to-one correspondence between all nonnegative selfadjoint extensions of the operator $A'$ and all nonnegative selfadjoint linear relation $\Theta$ in $\mathcal{H}$. 

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Proof. By the definition of a boundary triplet for the pair \( L_1 \subseteq L_2 \), where \( L_1 = \mathcal{L}, \ L_2 = \mathcal{L}_0^* \) we get \( \text{dom}(\Gamma) = \text{dom}(L_2) = \text{dom}(\mathcal{L}_0^*) = W_2^1(\mathbb{R}) \setminus \mathcal{Y} \) and \( \ker(\Gamma) = \text{dom}(\mathcal{L}) = W_2^1(\mathbb{R}) \). Similarly, \( \text{dom}(\Phi) = \text{dom}(L_1^*) = \text{dom}(\mathcal{L}) = W_2^1(\mathbb{R}), \ \ker(\Phi) = \text{dom}(L_2^*) = \text{dom}(\mathcal{L}_0) = \{ u \in W_2^1(\mathbb{R}) : u(y) = 0, \ y \in \mathcal{Y} \} \). Further, the Green identity

\[
(L_1 f, u)_H - (f, L_2 u)_H = \int_R i f'(x) u(x) \, dx - \int_R f(x) i u'(x) \, dx =
\]

\[
i \sum_{j \in \mathbb{J}} \left( \int_{I_j} f'(x) u(x) \, dx + \int_{I_j} f(x) u'(x) \, dx \right) = i \sum_{j \in \mathbb{J}} f(x) u(x) \bigg|_{y_{j+1}}^{y_j} =
\]

\[
i \sum_{j \in \mathbb{J}} f(y_j) \left( u(y_{j+1}) - u(y_j) \right) = \sum_{j \in \mathbb{J}} f(y_j) i (u(y_{j+1}) - u(y_j)) = (Gf, \Gamma u)_H
\]

holds. Due to Propositions 1 and 2 the operators \( \Gamma \) and \( \mathcal{G} \) are bounded. Hence the triplet \( \{ \mathcal{H}, \mathcal{G}, \Gamma \} \) is the boundary triplet for the pair \( \mathcal{L} \subseteq \mathcal{L}_0^* \).

Further, since \( \ker(\mathcal{L}) = \{ 0 \} \) and applying Theorem 4 we get (ii) and (iii). \( \square \)

Recall [5], that

\[
P_{\text{ran}(\mathcal{L}_0)} \mathcal{L}_0^* f = i f' - i \sum_k \frac{1}{d_k} (f(y_{k+1} - 0) - f(y_k + 0)) \chi_k, \ f \in \text{dom}(\mathcal{L}_0^*),
\]

where the functions \( \{ \chi_k \}_{k \in \mathbb{J}} \) \( \chi_k \) is the characteristic function of the interval \( [y_k, y_{k+1}] \), \( d_k = |y_k - y_{k+1}| \), \( k \in \mathbb{J} \).

Proposition 7. Set

\[
\mathcal{H} = \begin{cases} 
\mathbb{C}^m, & \text{\textit{Y} consists of m points;} \\
\ell_2(\mathbb{J}), & \text{\textit{Y} is infinite,}
\end{cases} \quad \text{dom}(\Gamma) = W_2^1(\mathbb{R}), \ \Gamma u = \{ iu(y_j), \ j \in \mathbb{J} \},
\]

\[
\text{dom}(\mathcal{G}) = W_2^1(\mathbb{R} \setminus \mathcal{Y}), \ \mathcal{G} f = \{ (f(y_j^+) - f(y_j^-)), \ j \in \mathbb{J} \},
\]

then

(i) \( \{ \mathcal{H}, \Gamma, \mathcal{G} \} \) is a boundary triplet for the pair \( \mathcal{L}_0 \subseteq \mathcal{L} \);

(ii) the triplet \( \Pi = \{ \mathcal{H}, \Gamma, \mathcal{G}P_{\text{ran}(\mathcal{L}_0)} \mathcal{L} \} \) is a basic for \( \mathcal{A}_0^* \), where

\[
\mathcal{G}P_{\text{ran}(\mathcal{L}_0)} \mathcal{L} f = \frac{i f'(y_j^+) - i f'(y_j^-)}{y_{j+1} - y_j} \frac{f(y_{j+1}^-) - f(y_j^+)}{y_{j+1} - y_j} + i \frac{f(y_j^-) - f(y_{j-1}^+)}{y_j - y_{j-1}}, \ j \in \mathbb{J},
\]

\( f \in \text{dom}(\mathcal{A}_0^*) \);

(iii) the mapping

\[
\Theta \mapsto \mathcal{A}_0 \Theta = \mathcal{A}_0^* \upharpoonright \text{dom}(\mathcal{A}_0^*): \left( \{ iu(f_j), \ j \in \mathbb{J} \}, \left\{ i f'(y_j^+) - i f'(y_j^-) \frac{f(y_{j+1}^-) - f(y_j^+)}{y_{j+1} - y_j} + i \frac{f(y_j^-) - f(y_{j-1}^+)}{y_j - y_{j-1}}, \ j \in \mathbb{J} \right\} \right) \in \Theta
\]

establishes a one-to-one correspondence between all nonnegative selfadjoint extensions of the operator \( \mathcal{A}_0 \) and all nonnegative selfadjoint linear relation \( \Theta \) in \( \mathcal{H} \).
Proposition 8. Set
\[
H = \begin{cases} \mathbb{C}^{2m}, & Y \text{ consists of } m \text{ points;} \\ \ell_2(J) \otimes \mathbb{C}^2, & Y \text{ is infinite}, \end{cases}
\]
\[
\Gamma u = \{(iu(y_j^-), iu(y_j^+)), \ j \in J\}, \ \text{dom}(G) = W^1_2(\mathbb{R} \setminus Y), Gf = \{(f(y_j^-), f(y_j^+)), \ j \in J\}.
\]
Then
(i) \(\{H, \Gamma, G\}\) is a boundary triplet for pair \(L_0 \subset L^*_0\);
(ii) the triplet \(\Pi = \{H, \Gamma, GP_{\text{ran}(L_0)} L^*_0\}\) is basic for \(H^*_0\), where
\[
GP_{\text{ran}(L_0)} L^*_0 f = \left\{ \left( i f'(y_j^-) - i \frac{f(y_j^-) - f(y_{j-1}^+)}{y_j - y_{j-1}} , i f'(y_j^+) - i \frac{f(y_{j+1}^-) - f(y_j^+)}{y_{j+1} - y_j} \right), \ j \in J \right\},
\]
f \(\in\) \(\text{dom}(H^*_0)\);
(iii) the mapping
\[
\Theta \mapsto H_{0\Theta} = H^*_0 \upharpoonright \left\{ f \in \text{dom}(H^*_0) : \left\{ (-i f(y_j^-), i f(y_j^+)) , \ j \in J \right\} \right\},
\]
\[
\left\{ \left( i f'(y_j^-) - i \frac{f(y_j^-) - f(y_{j-1}^+)}{y_j - y_{j-1}} , i f'(y_j^+) - i \frac{f(y_{j+1}^-) - f(y_j^+)}{y_{j+1} - y_j} \right), \ j \in J \right\} \in \Theta
\]
establishes a one-to-one correspondence between all nonnegative selfadjoint extensions of the operator \(H_0\) and all nonnegative selfadjoint linear relations \(\Theta\) in \(\mathcal{H}\).

Other boundary triplets for \(H^*_0\) have been constructed in [12] and in [13].

REFERENCES


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