УДК 517.444

O. A. Ochakovskaya

DESCRIPTION OF POMPEIU SETS IN TERMS OF APPROXIMATIONS OF THEIR INDICATOR FUNCTIONS

O. A. Ochakovskaya. *Description of Pompeiu sets in terms of approximations of their indicator functions*, Mat. Stud. **39** (2013), 142–149.

Let H be an open upper half-space in $\mathbb{R}^n$, $n \geq 2$, and assume that A is a non-empty, open, bounded subset of $\mathbb{R}^n$ such that $\overline{A} \subset H$ and the exterior of A is connected. Let $p \in [2, +\infty)$. It is proved that there is a nonzero function with zero integrals over all sets in $\mathbb{R}^n$ congruent to A if and only if the indicator function of A is the limit in $L^p(H)$ of a sequence of linear combinations of indicator functions of balls in H with radii proportional to positive zeros of the Bessel function $J_{n/2}$. The proportionality coefficient here is the same for all balls and depends only on A.

О. А. Очаковская. *Описание множеств Помпейю в терминах аппроксимации их индикаторов* // Мат. Студiї. – 2013. – Т.39, №2. – C.142–149.

Пусть H — открытое верхнее полупространство в $\mathbb{R}^n$, $n \geq 2$ и пусть A — непустое, открытое, ограниченное подмножество $\mathbb{R}^n$ такое, что $\overline{A} \subset H$ и внешность A связна. Пусть $p \in [2, +\infty)$. Показано, что для существования ненулевой функции с нулевыми интегралами по всем множествам из $\mathbb{R}^n$, конгруэнтным A, необходимо и достаточно, чтобы индикатор A был пределом в $L^p(H)$ последовательности линейных комбинаций индикаторов шаров, лежащих в H, с радиусами, пропорциональными положительным нулям функции Бесселя $J_{n/2}$. При этом коэффициент пропорциональности один и тот же для всех шаров и зависит только от A.

1. Introduction and the central result. Let $\mathbb{R}^n$ be a real Euclidean space of dimension $n \geq 2$ and let $M(n)$ be the group of its rigid motions.

A non-empty open bounded subset A of $\mathbb{R}^n$ is called a *Pompeiu set* if for function $f \in L_{loc}(\mathbb{R}^n)$ the equality

$$\int_{gA} f(x)dx = 0 \tag{1}$$

holding for all $g \in M(n)$ yields $f = 0$. In this case, one says also that A has the *Pompeiu property.*

Next, one says that A fails to have the *weak Pompeiu property* if there is a nonzero solution f of equality (1) such that

$$\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^{-\alpha} dx < +\infty \tag{2}$$

2010 *Mathematics Subject Classification:* 33C80, 44A35, 53C35.
Keywords: Pompeiu property; mean periodicity.

for some $\alpha > 0$ depending on f.

In the sequel we write χ_A for the indicator function of A. Also let $\mathrm{Ext}(A)$ be the exterior of A (i.e., $\mathrm{Ext}(A) = \mathbb{R}^n \setminus \overline{A}$ where $\overline{A}$ is the closure of A). For $\lambda > 0$, let $N_\lambda = \{r > 0\colon J_{n/2}(r\lambda) = 0\}$ where J_k is the kth-order Bessel function of the first kind.

The classical Pompeiu problem about functions satisfying (1) has been studied by many authors (see survey papers [1], [2], that contain an extensive bibliography; see also [3], [4])

The following description of sets with the weak Pompeiu property has been obtained by V. V. Volchkov, see [5].

Theorem 1. *Let A be a non-empty open bounded subset of $\mathbb{R}^n$. Then the following conditions are equivalent.*

- (i) *A fails to have the weak Pompeiu property.*
- (ii) *There exists $\lambda = \lambda(A) > 0$ such that the function χ_A is the limit of a sequence of linear combinations of the indicator functions of balls of radii $r \in N_\lambda$ convergent in $L^1(\mathbb{R}^n)$.*

We note that a similar result for the Pompeiu property holds under the assumption that the set $\mathrm{Ext}(A)$ is connected (see [6]).

It is known that Theorem 1 is no longer valid for the space $L^p(\mathbb{R}^n)$, $p \geq 2n/(n+1)$ instead of $L^1(\mathbb{R}^n)$ (see [3, Part 2, Theorem 1.13]). The case $1 < p < 2n/(n+1)$ is still open.

In this paper, we obtain an analog of Theorem 1 in terms of approximation of χ_A in the space $L^p(H)$ where $2 \leq p < +\infty$ and $H = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n\colon x_n > 0\}$. Our main result is as follows.

Theorem 2. *Let A be a non-empty open bounded subset of $\mathbb{R}^n$ such that $\overline{A} \subset H$ and the set $\mathrm{Ext}(A)$ is connected. Let $p \in [2, +\infty)$. Then the following conditions are equivalent.*

- (i) *A does not have the Pompeiu property.*
- (ii) *There exists $\lambda = \lambda(A) > 0$ such that the function χ_A is the limit of a sequence of linear combinations of the indicator functions of balls in H of radii $r \in N_\lambda$ convergent in $L^p(H)$.*

Together with [5], the proof of Theorem 2 shows (see Section 3 below) that the same result for the weak Pompeiu property remains valid. Moreover, in this situation the assumption that $\mathrm{Ext}(A)$ is connected can be removed.

2. Notation and auxiliary statements. Let $r > 0$ and let $B_r = \{x \in \mathbb{R}^n\colon |x| < r\}$ where $|\cdot|$ is the Euclidean norm. Denote by $E_{n/2} = \{\nu_1, \nu_2, \dots\}$ the increasing sequence of all zeros of the function $J_{n/2}$ lying on $(0, +\infty)$.

As usual, $\widehat{f}$ is the Fourier transform of the function f and $f_1 * f_2$ is the convolution of the functions f_1, f_2 (when they are well defined), $\mathrm{supp} f$ is the support of f, and $\mathcal{D}(\mathbb{R}^n)$ is the set of functions in the class $C^\infty(\mathbb{R}^n)$ with compact support. Let Δ be the Laplace operator.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, |x'| = \sqrt{x_1^2 + \dots + x_{n-1}^2}$. Also let $(\cdot, \cdot)$ be the inner product in $\mathbb{R}^{n-1}$ and assume that $dx' = dx_1 \cdots dx_{n-1}$ is the Lebesgue measure in $\mathbb{R}^{n-1}$.

For $\mu > 0$ we set

$$\varphi(t) = \varphi(t, r, \mu) = \begin{cases} (r^2 - t^2)^{\frac{n-1}{4}} J_{\frac{n-1}{2}}(\mu\sqrt{r^2 - t^2}), & |t| < r; \\ 0, & |t| \geq r. \end{cases}$$

Also let $\varphi_{k,\mu}(t) = \varphi(t, \nu_k, \mu), \ k \in \mathbb{N}$.

Lemma 1. *The following assertions are true.*

(*i*) *For* $z \in \mathbb{C}$,

$$\widehat{\varphi}(z) = \sqrt{2\pi} r^{n/2} \mu^{\frac{n-1}{2}} J_{\frac{n}{2}}(r\sqrt{\mu^2 + z^2})(\mu^2 + z^2)^{-\frac{n}{4}}. \tag{3}$$

(*ii*) *All the zeros of* $\widehat{\varphi}_{k,\mu}$ *in* $\mathbb{C} \setminus \{0\}$ *are simple.*

(*iii*) *If* $\zeta \in \mathbb{C}$ *and* $\mu > 0$ *are fixed and* $\widehat{\varphi}_{k,\mu}(\zeta) = 0$ *for all* $k \in \mathbb{N}$ *then* $\zeta^2 + \mu^2 = 1$.

Proof. Assertions (i) and (ii) were obtained by the author in [7]. Let us prove (iii). First we recall the fact that all the zeros of $J_{n/2}$ are real (see [8, Chapter 23]). Using now (3) we see that for each $k \in \mathbb{N}$ there exists $m = m_k \in \mathbb{N}$ such that $\zeta^2 = \nu_m^2/\nu_k^2 - \mu^2$. This means that the value ν_m/ν_k is independent of k. Taking [5, Lemma 3] into account we conclude that $\nu_m/\nu_k = 1$, which yields the required result. □

Lemma 2. *Let* $k \in \mathbb{N}$ *and* $\mu > 0$ *be fixed and assume that* $J_{\frac{n}{2}}(\mu\nu_k) \neq 0$. *Also let* g *be a function in the class* $C^n[-\nu_k, \nu_k]$ *satisfying the following conditions:*

1) $g^{(s)}(\nu_k) = 0$ *for all* $s \in \{0, \dots, n\}$;
2) $\int_{-\nu_k}^{\nu_k} g^{(s)}(\alpha)\varphi_{k,\mu}(\alpha)\, d\alpha = 0$ *for all* $s \in \{0, \dots, n\}$;
3) $\int_{-\nu_k}^{\nu_k} g(\xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t) e^{i\tau(\xi - t)}\, dt\, d\xi = 0$ *for all* $\tau \in \mathbb{C}$ *such that* $\widehat{\varphi}_{k,\mu}(\tau) = 0$.

Then $g = 0$.

Proof. We can rewrite condition 3) as follows

$$\int_0^{2\nu_k} e^{i\tau\alpha} \int_{-\nu_k}^{\nu_k - \alpha} g(\alpha + \beta)\varphi_{k,\mu}(\beta)\, d\beta\, d\alpha = 0. \tag{4}$$

Consider the entire function

$$w(z) = e^{-\nu_k z} \int_0^{2\nu_k} e^{iz\alpha} \int_{-\nu_k}^{\nu_k - \alpha} g(\alpha + \beta)\varphi_{k,\mu}(\beta)\, d\beta\, d\alpha, \quad z \in \mathbb{C}. \tag{5}$$

Integrating (5) by parts and using conditions 1) and 2) we obtain $|w(z)| \le C_1(1 + |z|)^{-n} e^{\nu_k |\in z|}$, with positive constant C_1 independent of z. It now follows from (4), (5) and Lemma 1(ii) that the function

$$w_1(z) = w(z)/\widehat{\varphi}_{k,\mu}(z) \tag{6}$$

is entire. Using the asymptotic formula for $J_{\frac{n}{2}}(z)$ as $z \to \infty$ (see [8, Chapter 29]) we deduce from (6) and (3) the inequality $|w_1(t \pm it)| \le C_2(1 + |t|)^{\frac{1-n}{2}}$, $t \in \mathbb{R}^1$, with positive C_2 independent of t. Hence it follows from the Phragmén-Lindelöf principle and Liouwille's theorem that $w_1 = 0$. According to (5) this means that $\int_{-\nu_k}^{\nu_k - \alpha} g(\alpha + \beta)\varphi_{k,\mu}(\beta)\, d\beta\, d\alpha = 0$ for all $\alpha \in (0, 2\nu_k)$. Then we have the result of our lemma from Titchmarsh's theorem on convolution (see [9, Appendix VII, Chapter 12]). □

Lemma 3. *Let* $R > r > 0$, $T \in L(\mathbb{R}^1)$, *assume that* $\operatorname{supp} T \subset [-r, r]$ *and let*

$$f(t) = \sum_{m=0}^{M} \sum_{l=1}^{L} a_{m,l} t^m \exp(i b_l t), \quad t \in (-R, R),$$

where $M \in \mathbb{Z}_+$, $L \in \mathbb{N}$, $a_{m,l} \in \mathbb{C}$, *and* $b_l \in \mathbb{C}$ *are pairwise different complex numbers. Assume also that* $f * T = 0$ *and* $\widehat{T}^{(\eta)}(b_k) \neq 0$ *for some* $\eta \in \{0, \dots, M\}$, $k \in \{1, \dots, L\}$. *Then* $a_{m,k} = 0$ *for all* $m \ge \eta$.

Proof. Simple calculations show that the condition $f * T = 0$ is equivalent to

$$\sum_{m=\nu}^{M} a_{m,l}(-i)^m \binom{m}{\nu} \widehat{T}^{(m-\nu)}(b_l) = 0, \quad \nu \in \{0, \dots, M\},\ l \in \{1, \dots, L\},$$

where $\binom{m}{\nu}$ are the binomial coefficients. The required result is now obvious. □

Lemma 4. *Let $M_r = \{x = (x', x_n) \in \mathbb{R}^n \colon |x_n| \leq r\}$, assume that $u \in L^p(M_r)$ for some $p \in [1, 2]$, and let*

$$v(t) = \int_{B_r} u(x' + t, x_n)dx,\ t \in \mathbb{R}^{n-1}. \tag{7}$$

Then $v \in L^p(\mathbb{R}^{n-1})$ and

$$\widehat{v}(\lambda) = \left(\frac{2\pi}{|\lambda|}\right)^{(n-1)/2} \int_{-r}^{r} \widehat{u}(\lambda, x_n)\varphi(x_n, r, |\lambda|)dx_n \tag{8}$$

for almost all $\lambda \in \mathbb{R}^{n-1}$, where $\widehat{u}$ is the Fourier transform of $u(x', x_n)$ with respect to x'.

Proof. Since $u \in L^p(M_r)$ we see from (7) and Hölder's inequality that $v \in L^p(\mathbb{R}^{n-1})$. Let us prove (8). We claim that

$$\widehat{v}(\lambda) = \int_{B_r} e^{i(\lambda, x')}\widehat{u}(\lambda, x_n)dx. \tag{9}$$

It is enough to consider the cases $p = 1$ and $p = 2$ (see [10, Chapter 1, Section 2]). In the case $p = 1$ relation (9) follows from the definition of the Fourier transform and Fubini's theorem. Suppose now that $p = 2$. For $R > 0$ we set

$$v_R(t) = \int_{B_r} u_R(x' + t, x_n)dx,\ t \in \mathbb{R}^{n-1},$$

where $u_R(x', x_n) = u(x', x_n)$ for $|x'| \leq R$, and $u_R(x', x_n) = 0$ otherwise. Letting $R \to +\infty$ one sees that $u_R \to u$ in $L^2(M_r)$ and hence $v_R \to v$ in $L^2(\mathbb{R}^{n-1})$. In addition,

$$\widehat{v_R}(\lambda) = \int_{B_r} e^{i(\lambda, x')}\widehat{u_R}(\lambda, x_n)dx$$

for almost all $\lambda \in \mathbb{R}^{n-1}$. Passing here to the limit as $R \to \infty$ in the space $\mathcal{D}'(\mathbb{R}^{n-1})$ of the distributions on $\mathbb{R}^{n-1}$ we arrive at (9). Next, passing to repeated integration in (9) one infers that

$$\widehat{v}(\lambda) = \left(\frac{2\pi}{|\lambda|}\right)^{(n-1)/2} \int_{-r}^{r} \widehat{u}(\lambda, x_n) \int e^{i(\lambda, x')}\, dx'\, dx_n,$$

where the inner integral is taken over the ball $\{x' \in \mathbb{R}^{n-1} \colon x_1^2 + \dots + x_{n-1}^2 \leq r^2 - x_n^2\}$. Using the formula for the Fourier transform of the indicator of a ball (see [10, Chapter 4, Theorem 4.15]) we obtain the required result. □

Lemma 5. *Let $f \in L^p(H)$ for some $p \in [1, 2]$ and suppose that $\int_{B_r} f(x + y)dx = 0$ for all $r \in E_{n/2}$ and all $y \in \{x \in \mathbb{R}^n \colon x_n > r\}$. Then there exists $u \in C^\infty(H)$ such that $\triangle u + u = 0$ and $f = u$ in H almost everywhere.*

Proof. Let φ be an arbitrary function in the class $C^\infty(\mathbb{R}^n)$ with support inside the ball B_1. We set

$$F(x) = \int_H f(y)\varphi(x+y)\,dy, \quad x \in H_1 = \{x \in \mathbb{R}^1,\ x_n > 1\}. \tag{10}$$

Let $r \in E_{\frac{n}{2}}$. In view of (10), one has

$$\int_{B_r(0)} F(x_1+t_1, \ldots, x_{n-1}+t_{n-1}, x_n+y)\,dx = 0 \tag{11}$$

for all $t \in (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ and $y > r+1$. Assume that $\mu \geq 0$ and $\lambda \in \mathbb{R}^{n-1}$ are related by $\mu = |\lambda|$. For brevity we shall write $g_\lambda(x_n)$ for the Fourier transform of $F(x', x_n)$ with respect to x'. Using Lemma 4 we see from (11) that

$$\int_{-\nu_k}^{\nu_k} g_\lambda(x_n+y)\varphi_{k,\mu}(x_n)\,dx_n = 0, \tag{12}$$

for all $k \in \mathbb{N},\ y > \nu_k + 1$ and almost all $\lambda \in \mathbb{R}^{n-1}$. Now define $g_{\lambda,k}(u) = g_\lambda(u + \nu_k + 2),\ u > \nu_k - 1$. Thus $g_{\lambda,k} \in C^\infty(-\nu_k - 1, +\infty)$ and

$$\int_{-\nu_k}^{\nu_k} g_{\lambda,k}(\xi+\eta)\varphi_{k,\mu}(\xi)\,d\xi = 0, \quad \eta > -1, \tag{13}$$

for each $k \in \mathbb{N}$ (see (10)).

Assume that $\eta > -1$, $z \in \mathbb{C}$, and let

$$W_{k,\mu}(\eta, z) = \int_{-\nu_k}^{\nu_k} g_{\lambda,k}(\eta+\xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t)e^{iz(\xi-t)}\,dt\,d\xi. \tag{14}$$

Then one has

$$\frac{\partial W_{k,\mu}}{\partial \eta} = \int_{-\nu_k}^{\nu_k} g'_{\lambda,k}(\eta+\xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t)e^{iz(\xi-t)}\,dt\,d\xi.$$

Integrating by parts we find that

$$\frac{\partial W_{k,\mu}}{\partial \eta} = \widehat{\varphi}_{k,\mu}(z)e^{i\nu_k z}g_{\lambda,k}(\eta+\nu_k) - \int_{-\nu_k}^{\nu_k} g_{\lambda,k}(\eta+\xi)\varphi_{k,\mu}(\xi)\,d\xi -$$
$$-iz\int_{-\nu_k}^{\nu_k} g_{\lambda,k}(\eta+\xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t)e^{iz(\xi-t)}\,dt\,d\xi.$$

This together with (13) yields

$$\frac{\partial W_{k,\mu}}{\partial \eta} = -izW_{k,\mu} + \widehat{\varphi}_{k,\mu}(z)e^{i\nu_k z}g_{\lambda,k}(\eta+\nu_k). \tag{15}$$

Let $N_{k,\mu} = \{z \in \mathbb{C}\colon \widehat{\varphi}_{k,\mu}(z) = 0\}$ and assume that $\tau \in N_{k,\mu}$. Equality (15) shows that the value $W_{k,\mu}(\eta, \tau)e^{iz\eta}$ is independent of η. Because of (14) this means that

$$\int_{-\nu_k}^{\nu_k} g_{\lambda,k}(\eta+\xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t)e^{i\tau(\xi-t)}\,dt\,d\xi = C_{k,\mu}(\tau)e^{-i\tau\eta}, \quad \eta > -1, \tag{16}$$

where the constant $C_{k,\mu}(\tau)$ is independent of η. Next, let $s \in \mathbb{N}$ and $h > \nu_s$. It follows from (16) and (13) that

$$C_{k,\mu}(\tau) \int_{-\nu_s+h}^{\nu_s+h} e^{-i\tau\eta} \varphi_{s,\mu}(\eta - h)\, d\eta = 0,$$

whence $C_{k,\mu}(\tau)e^{-i\tau h}\widehat{\varphi}_{s,\mu}(\tau) = 0$. This together with (16) implies that if for some $k \in \mathbb{N}$ and $\tau \in N_{k,\mu}$ there exists $s \in \mathbb{N}$ such that $\widehat{\varphi}_{s,\mu}(\tau)$ is nonzero, then

$$\int_{-\nu_k}^{\nu_k} g_{\lambda,k}(\eta + \xi) \int_{-\nu_k}^{\nu_k} \varphi_{k,\mu}(t) e^{i\tau(\xi - t)}\, dt\, d\xi = 0, \quad \eta > -1. \tag{17}$$

Assertion (iii) of Lemma 1 shows that (17) holds for each $\tau \in N_{k,\mu}$ such that $\tau^2 + \mu^2 \neq 1$. If the last condition is valid then one has

$$\int_{-\nu_k}^{\nu_k} \Phi_{\lambda,k}(\eta + \xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t) e^{i\tau(\xi - t)}\, dt\, d\xi = 0, \quad \eta > -1, \tag{18}$$

where

$$\Phi_{\lambda,k}(t) = g''_{\lambda,k}(t) + (1 - \mu^2) g_{\lambda,k}(t), \quad t > -\nu_k - 1. \tag{19}$$

For the case where $\tau^2 + \mu^2 = 1$ relation (18) remains valid because of (16). By differentiating (18) with respect to η and by setting $\eta = 0$ we infer that

$$\int_{-\nu_k}^{\nu_k} \Phi^{(m)}_{\lambda,k}(\xi) \int_{-\nu_k}^{\xi} \varphi_{k,\mu}(t) e^{i\tau(\xi - t)}\, dt\, d\xi = 0, \tag{20}$$

for all $\tau \in N_{k,\mu}$ and $m \in \mathbb{Z}_+$. We now put $g(u) = \sum_{j=1}^{2n+3} \alpha_{j,k} \Phi^{(j)}_{\lambda,k}(u)$, where the constants $\alpha_{j,k} \in \mathbb{C}$ are selected so that $\sum_{j=1}^{2n+3} |\alpha_{j,k}| \neq 0$ and the function g satisfies assumptions 1) and 2) of Lemma 2. This is possible since $2n+3$ is grater than the total number of equations in assumptions 1) and 2) of Lemma 2. Choosing λ so that $J_{\frac{n}{2}}(|\lambda|\nu_k) \neq 0$ for each $k \in \mathbb{N}$, we conclude from (20) and Lemma 2 that $g = 0$ on $[-\nu_k, \nu_k]$. Using now (21) and (19), one sees from (17) and Titchmarsh's theorem on convolution (see [9, Appendix VII, Chapter 12]) that $g = 0$ on $(-\nu_k, +\infty)$. By the definition of g_λ and (19) it follows that g_λ is a solution of some linear differential equation with nonzero constant coefficients depending on λ. This means that g_λ has the form

$$g_\lambda(t) = \sum_{m=0}^{M} \sum_{l=1}^{L} a_{\lambda,m,l} t^m \exp(i b_l t), \quad t > 1,$$

where the constants $a_{\lambda,m,l} \in \mathbb{C}$, $\nu_l \in \mathbb{C}$, $M \in \mathbb{Z}_+$, $L \in \mathbb{N}$ depend, in general, on λ. In the sequel, without loss of generality, we assume that the numbers b_l are pairwise distinct. Bearing (12) in mind one concludes from Lemma 3 that

$$g_\lambda(x_n) = C_1(\lambda) e^{-i\sqrt{1-|\lambda|^2}x_n} + C_2(\lambda) e^{i\sqrt{1-|\lambda|^2}x_n}, \tag{21}$$

where $C_1(\lambda)$ and $C_2(\lambda)$ are complex constants depending on λ. Next, let $y > r > 0$ and $t \in \mathbb{R}^{n-1}$. Applying (21), by Lemmas 4 and 1(i) we obtain

$$\int_{B_r} F(x' + t, x_n + y) dx = (2\pi r)^{(n/2)} J_{n/2}(r) F(t, y).$$

By the theorem on ball means for solutions of Helmholtz's equation (see[5]) this means that $\Delta F + F = 0$ in H_1. Since the function φ above was arbitrary, one deduces from (10) that $\Delta f + f = 0$ in H in the distribution sense. This equality and the ellipticity of the operator Δ ensure us that f almost everywhere equal to a function $u \in C^{\infty}(H)$ such that $\triangle u + u = 0$. Hence the lemma is proved. □

3. Proof of the main result. We now proceed to the proof of Theorem 2.
(i)→ (ii). It follows from the assumptions and ([3, Part 4, Theorem 1.2]) that there exists $\lambda = \lambda(A) > 0$ such that the equation

$$\Delta u + \lambda^2 u = \chi_A \tag{22}$$

has a solution $u \in C^1(\mathbb{R}^n)$ with compact support (here equality (22) is understood in the sense of distributions). We select a non-trivial function $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\operatorname{supp} v \subset H$ where $v = \chi_A * \phi$. Setting $w = u * \phi$ one sees from (22) that

$$\Delta w + \lambda^2 w = v. \tag{23}$$

We claim that each continuous linear functional Ψ on $L^p(H)$ annihilating the indicator functions of all balls in H of radii $r \in N_\lambda$ also annihilates v. By Riesz's theorem and Lemma 5 such a functional has the following form

$$\Psi(g) = \int_H g(x) f(x) dx, \ g \in L^p(H), \tag{24}$$

where $f \in L^q(H)$, $q = p/(p-1)$, and

$$\Delta f + \lambda^2 f = 0 \ \text{ in } H. \tag{25}$$

The last equality and the ellipticity of the operator Δ mean that f is almost everywhere equal to a real analytic function on H. Using (23), (24), and (25), we have $\Psi(v) = \int_H w(x)$ $(\Delta f + \lambda^2 f)(x) dx = 0$ proving the claiming. Thus the convolution $\chi_A * \phi$ is the limit of a sequence of linear combinations of the indicator functions of balls in H of radii $r \in N_\lambda$ convergent in $L^p(H)$. Now, from the arbitrariness of ϕ and ([10, Chapter 1, Theorem 1.18]) we obtain the required result.
(ii)→ (i). For $\lambda = \lambda(A)$ we consider a non-trivial non-negative function $\varphi \in \mathcal{D}(\mathbb{R}^1)$ with support on $[a, b] \subset (\lambda/2, +\infty)$. Now define

$$f(x) = \int_a^b \frac{J_{(n-3)/2}(\sqrt{t^2+\lambda^2}|x'|)}{(\sqrt{t^2+\lambda^2}|x'|)^{(n-3)/2}} e^{-tx_n} \varphi(t) dt, \ x \in \mathbb{R}^n. \tag{26}$$

Repeating the argument in the proof of Theorem 2 in [7] we see that $f \in (C^{\infty} \cap L^q)(H)$ for each $q \geq 1$. In addition $f(x) > 0$ for $x = (0, \ldots, 0, x_n)$ and relation (25) is satisfied. Using the mean theorem for Helmholtz's equation we conclude that $\int_B f(x) dx = 0$ for each ball $B \subset H$ with radius $r \in N_\lambda$. Hence by the assumption in (ii) and the Hölder inequality one obtains $\int_{A+h} f(x) dx = 0$ for each $h \in H$. Take $h = h_1 + h_2$ where $h_1, h_2 \in H$. It follows from (26) and the arbitrariness of h_2 and φ that

$$\int_{A+h_1} \frac{J_{(n-3)/2}(\sqrt{t^2+\lambda^2}|x'|)}{(\sqrt{t^2+\lambda^2}|x'|)^{(n-3)/2}} e^{-tx_n} dx = 0 \tag{27}$$

for each $t > \lambda/2$. For the case where $n = 2$ this yields $\int_{A+h_1} \cos(\sqrt{t^2+\lambda^2}x_1)e^{-tx_2}dx_1dx_2 = 0$ (see [8, formula (7.3)]). Since $h_1 \in H$ could be arbitrary the last relation shows that

$$\int_A \exp(-tx_2 + i\sqrt{t^2+\lambda^2}x_1)dx_1dx_2 = 0. \tag{28}$$

Assume now that $n > 2$. It follows from (27) and [3, Part 1, formula (5.29)] that

$$\int_{\mathbb{S}^{n-2}} \int_{A+h_1} \exp\left(-tx_n + i\sqrt{t^2+\lambda^2}\sum_{j=1}^{n-1} x_j\sigma_j\right) dxd\omega(\sigma) = 0,$$

where $d\omega$ is area measure on $\mathbb{S}^{n-2}$. As before, this yields

$$\int_A \exp\left(-tx_n + i\sqrt{t^2+\lambda^2}\sum_{j=1}^{n-1} x_j\sigma_j\right) dx = 0 \tag{29}$$

for all $\sigma = (\sigma_1, \ldots, \sigma_{n-1}) \in \mathbb{S}^{n-2}$. Since the left-hand parts in (28) and (29) are holomorphic functions of variable t in the disk $\{t \in \mathbb{C}\colon |t| < \lambda\}$ one sees from (28) and (29) that $\int_A \exp(i\sum_{j=1}^n \zeta_j x_j)dx = 0$ for all $(\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$ such that $|\zeta| = \lambda$. Thus A fails to have the weak Pompeiu property (see, for instance, [5]). Hence A is not a Pompeiu set and the proof of Theorem 2 is complete.

REFERENCES

1. Zalcman L. *A bibliographic survey of the Pompeiu problem; in: B. Fuglede et al. (eds.)*// Approximation by Solutions of Partial Differential Equations. – Dordrecht: Kluwer Academic Publishers, 1992. – P. 185–194.
2. Zalcman L. *Supplementary bibliography to 'A bibliographic survey of the Pompeiu problem'; in Radon Transforms and Tomography*// Contemp. Math. – 2001. – V.278. – P. 69–74.
3. Volchkov V.V. Integral Geometry and Convolution Equations. – Dordrecht: Kluwer Academic Publishers, 2003. – 454 p.
4. Volchkov V.V., Volchkov Vit.V. Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group. – London: Springer, 2009. – 671 p.
5. Volchkov V.V. *New theorems on the mean for solutions of the Helmholtz equation*// Russian Acad. Sci. Sb. Math. – 1994. – V.79. – P. 281–286.
6. Volchkov V.V. *Theorems on ball means values in symmetric spaces*// Sbornik: Math. – 2001. – V.192. – P. 1275–1296.
7. Ochakovskaya O.A. *Precise characterizations of admissible rate of decrease of a non-trivial function with zero ball means*// Sbornik: Math. – 2008. – V.199. – P. 45–65.
8. Korenev B.G. Introduction to the theory of Bessel functions. – Moscow: Nauka, 1971. – 288 p.
9. Levin B. Ya. Distribution of zeros of entire functions. – Providence, RI: Amer. Math. Soc., 1964. – 632 p.
10. Stein E., Weiss G. Introduction to Fourier Analysis on Euclidean Spaces. – New Jersey: Princeton University Press, 1971. – 332 p.

Institute of Applied Mathematics and Mechanics
NAS of Ukraine, Donetsk
ochakovskaja@yandex.ua

Received 07.02.2013