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WIMAN’S TYPE INEQUALITY FOR SOME DOUBLE POWER SERIES


In this paper we prove some analogue of Wiman’s inequality for analytic functions $f(z_1, z_2)$ in the domain $T = \{ z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < +\infty \}$. The obtained inequality is sharp.

1. Introduction. Let $f$ be analytic function in the disc $\{ z : |z| < R \}$, $0 < R \leq +\infty$, represented by the power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n.$$  \hspace{1cm} (1)

For $r \in (0, R)$ we denote $M_f(r) = \max\{|f(z)| : |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n : n \geq 0\}$.

It is well known ([1, p. 9], [2, p. 28]) that for each nonconstant entire function $f(z)$ and every $\varepsilon > 0$ there exists a set $E(\varepsilon, f) \subset [1, +\infty)$ such that Wiman’s inequality

$$M_f(r) \leq \mu_f(r) (\ln \mu_f(r))^{1/2+\varepsilon}$$

holds for all $r \in [1, +\infty) \setminus E(\varepsilon, f)$, where the set $E(\varepsilon, f)$ has finite logarithmic measure, i.e. $\int_{E(\varepsilon, f)} \frac{dr}{r} < +\infty$.

Let $R = 1$, i.e. $f(z)$ be an analytic function in the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$. For such a function $f(z)$ and every $\delta > 0$ there exists a set $E_f(\delta) \subset (0, 1)$ of finite logarithmic measure on $(0, 1)$, i.e.

$$\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty,$$

such that for all $r \in (0, 1) \setminus E_f(\delta)$ the inequality

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}$$

holds ([3]). Similar inequality for analytic function in the unit disc one can find in [4].

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Also in [3] it is noted that for the function \( g(z) = \sum_{n=1}^{+\infty} \exp\{n^\varepsilon\}z^n, \varepsilon \in (0,1) \) we have
\[
\lim_{r \to 1-0} \frac{M_g(r)}{\mu_g(r) \ln^{1/2} \mu_g(r)} \geq C > 0.
\]

Some analogues of Wiman’s inequality for entire functions of several complex variables one can find in [5]–[11].

The aim of this paper is to prove some analogues of Wiman’s inequality for analytic functions represented by the series
\[
f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm}z_1^n z_2^m
\]
with the domain of convergence \( T = \{ z \in \mathbb{C}^2 : |z_1| < 1, z_2 \in \mathbb{C} \} \).

By \( A^2 \) we denote the class of analytic functions of form (2) with the domain of convergence \( T \) and \( \frac{\partial^2}{\partial z_2^2} f(z_1, z_2) \not\equiv 0 \) in \( T \).

2. Wiman’s type inequality for analytic functions in \( T \). For \( r = (r_1, r_2) \in T := [0,1) \times [0, +\infty) \) and function \( f \in A^2 \) we denote
\[
\triangle_r = \{(t_1, t_2) \in T : t_1 > r_1, t_2 > r_2\}, \quad M_f(r) = \max\{|f(z)| : |z_1| \leq r_1, |z_2| \leq r_2\}, \quad \mu_f(r) = \max\{|a_{nm}| r_1^n r_2^m : (n,m) \in \mathbb{Z}_+^2\}, \quad \mathcal{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m.
\]

Let \( D_f(r) = (D_{ij}) \) be a \( 2 \times 2 \) matrix such that
\[
D_{ij} = r_i \frac{\partial}{\partial r_i} \left( r_j \frac{\partial}{\partial r_j} \ln \mathcal{M}_f(r) \right) = \partial_i \partial_j \ln \mathcal{M}_f(r), \quad \partial_i = r_i \frac{\partial}{\partial r_i}, \quad i,j \in \{1,2\}.
\]

The following statement can be proved by verbatim repetition of the proof of Theorem 3.1 from [5].

**Theorem 1.** Let \( f \in A^2 \). There exists an absolute constant \( C_0 \) such that
\[
\mathcal{M}_f(r) \leq C_0 \mu_f(r) (\det(D_f(r) + I))^{1/2},
\]
where \( I \) is the identity \( 2 \times 2 \) matrix.

We say that \( E \subset T \) is set of asymptotically finite logarithmic measure on \( T \) if there exists \( r_0 \in T \) such that
\[
\nu_{ln}(E \cap \triangle_{r_0}) := \int \int_{E \cap \triangle_{r_0}} \frac{dr_1dr_2}{(1-r_1)r_2} < +\infty,
\]
i.e. the set \( E \cap \triangle_{r_0} \) is a set of finite logarithmic measure on \( T \).

**Lemma 1.** Let \( \delta > 0 \) and let \( h: \mathbb{R}_+^2 \to \mathbb{R}_+ \) be an increasing function on each variable such that
\[
\int_1^{+\infty} \int_1^{+\infty} \frac{du_1du_2}{h(u_1, u_2)} < +\infty.
\]
Then there exists a set $E \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \setminus E$ the inequalities

$$\det(D_f(r) + I) \leq \frac{1}{1 - r_1} \cdot h\left(\frac{\partial}{\partial r_1} \ln M_f(r), r_2 \frac{\partial}{\partial r_2} \ln M_f(r) + \ln r_2\right),$$

$$\frac{\partial}{\partial r_1} \ln M_f(r) \leq \frac{1}{1 - r_1} \cdot h\left(\ln M_f(r), r_2\right),$$

$$\frac{\partial}{\partial r_2} \ln M_f(r) \leq \frac{1}{r_2 (1 - r_1)^{1+\delta}} (\ln M_f(r))^{1+\delta}$$

hold.

Proof. Let $E_1 \subset T$ be a set for which inequality (3) does not hold. Now we prove that $E_1$ is a set of asymptotically finite logarithmic measure. Since $r_j \frac{\partial}{\partial r_j} \ln M_f(r)$ is increasing to $+\infty$ for $j \in \{1, 2\}$ as $r_1 \to 1 - 0$, $r_2 \to +\infty$, there exists $r^0 \in T$ such that $r_1^0 > \frac{1}{2}$, $r_2^0 > 1$ and for all $r \in \Delta_{r^0}$ we have

$$r_j \frac{\partial}{\partial r_j} \ln M_f(r) + \ln r_j > 1.$$

Then

$$\nu_{in}(E_1 \cap \Delta_{r^0}) = \int_{E_1 \cap \Delta_{r^0}} \frac{dr_1 dr_2}{(1 - r_1) r_2} \leq \int_{E_1 \cap \Delta_{r^0}} \frac{\det(D_f(r) + I)(1 - r_1) dr_1 dr_2}{h\left(\frac{\partial}{\partial r_1} \ln M_f(r), r_2 \frac{\partial}{\partial r_2} \ln M_f(r) + \ln r_2\right)(1 - r_1) r_2} \leq \int_{E_1 \cap \Delta_{r^0}} \frac{1}{r_1 r_2} \cdot h\left(\frac{\partial}{\partial r_1} \ln M_f(r) + \ln r_1, r_2 \frac{\partial}{\partial r_2} \ln M_f(r) + \ln r_2\right).$$

Let $U: T \to \mathbb{R}^2_+$ be a mapping such that $U = (u_1(r), u_2(r))$ and $u_j(r) = r_j \frac{\partial}{\partial r_j} \ln M_f(r) + \ln r_j$, $j \in \{1, 2\}$, $r = (r_1, r_2)$. Then for $i, j \in \{1, 2\}$ we obtain

$$\frac{\partial u_i}{\partial r_i} = \frac{\partial}{\partial r_i} \left(r_i \frac{\partial}{\partial r_i} \ln M_f(r) + \ln r_i\right) = \frac{1}{r_i} \frac{\partial}{\partial r_i} \ln M_f(r) + \frac{1}{r_i},$$

$$\frac{\partial u_i}{\partial r_j} = \frac{\partial}{\partial r_j} \left(r_i \frac{\partial}{\partial r_i} \ln M_f(r) + \ln r_i\right) = \frac{1}{r_j} \frac{\partial}{\partial r_j} \ln M_f(r), \ i \neq j.$$

So, the Jacobian

$$J_1 := \frac{D(u_1, u_2)}{D(r_1, r_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_2}{\partial r_1} \\ \frac{\partial u_1}{\partial r_2} & \frac{\partial u_2}{\partial r_2} \end{vmatrix} = \det(D_f(r) + I) \frac{1}{r_1 r_2}.$$ 

Therefore,

$$\nu_{in}(E_1 \cap \Delta_{r^0}) \leq \int_{U(E_1 \cap \Delta_{r^0})} \frac{du_1 du_2}{h(u_1, u_2)} \leq \int_{1}^{+\infty} \int_{1}^{+\infty} \frac{du_1 du_2}{h(u_1, u_2)} < +\infty.$$

Suppose that $E_2 \subset T$ is a set for which inequality (4) does not hold. Then we can choose $r^0 \in T$ so that $\ln M_f(r^0) > 1$ and $r_2^0 > e$.

$$\nu_{in}(E_2 \cap \Delta_{r^0}) = \int_{E_2 \cap \Delta_{r^0}} \frac{dr_1 dr_2}{(1 - r_1) r_2} \leq \int_{E_2 \cap \Delta_{r^0}} \frac{\partial}{\partial r_1} \ln M_f(r) \cdot (1 - r_1) dr_1 dr_2,$$
Consider the mapping \( V : T \rightarrow \mathbb{R}^2_+ \), where \( V = (v_1(r), v_2(r)) \) and \( v_1 = \ln M_f(r) \), \( v_2 = \ln r_2 \), \( r = (r_1, r_2) \). So,

\[
J_2 := \frac{D(v_1, v_2)}{D(r_1, r_2)} = \begin{bmatrix} \frac{\partial}{\partial r_1} \ln M_f(r) & \frac{\partial}{\partial r_2} \ln M_f(r) \\ 0 & \frac{1}{r_2} \end{bmatrix} = \frac{1}{r_2} \cdot \frac{\partial}{\partial r_1} \ln M_f(r).
\]

Therefore

\[
\nu_{ln}(E_2 \cap \Delta_{r_0}) \leq \int \int_{E_2 \cap \Delta_{r_0}} \frac{\partial}{\partial r_1} \ln M_f(r) r_2 dr_1 dr_2 = \int \int_{V(E_2 \cap \Delta_{r_0})} \frac{du_1 du_2}{h(u_1, u_2)} \leq \int \int_{1}^{+\infty} \frac{du_1 du_2}{h(u_1 u_2)} < +\infty.
\]

Let \( E_3 \subset T \) be a set for which inequality (5) does not hold. Then

\[
\nu_{ln}(E_3 \cap \Delta_{r_0}) \leq \int \int_{E_3 \cap \Delta_{r_0}} \frac{\partial}{\partial r_1} \ln M_f(r) r_2 dr_1 dr_2 = \int \int_{V(E_3 \cap \Delta_{r_0})} \frac{du_1 du_2}{h(u_1, u_2)} \leq \int \int_{1}^{+\infty} \frac{du_1 du_2}{h(u_1 u_2)} < +\infty.
\]

Let \( r_0 \) be such that \( \ln M_f(r_0) > 1 \). Define the mapping \( W : T \rightarrow T \), where \( W = (w_1(r), w_2(r)) \) and \( w_1 = r_1 \), \( w_2 = \ln M_f(r) \), \( r = (r_1, r_2) \). So,

\[
J_3 := \frac{D(w_1, w_2)}{D(r_1, r_2)} = \begin{bmatrix} \frac{\partial}{\partial r_1} \ln M_f(r) & \frac{\partial}{\partial r_2} \ln M_f(r) \\ 0 & \frac{\partial}{\partial r_2} \ln M_f(r) \end{bmatrix} = \frac{\partial}{\partial r_1} \ln M_f(r).
\]

Therefore,

\[
\nu_{ln}(E_3 \cap \Delta_{r_0}) \leq \int \int_{W(E_3 \cap \Delta_{r_0})} \frac{du_1 du_2}{(1 - u_1)^{1-\delta} u_2^{1+\delta}} \leq \int_{0}^{1} \frac{du_1}{(1 - u_1)^{1-\delta}} \cdot \int_{1}^{+\infty} \frac{du_2}{u_2^{1+\delta}} < +\infty.
\]

It remains to remark that the set \( E = \bigcup_{j=1}^{3} E_j \) is also a set of asymptotically finite logarithmic measure in \( T \).

**Theorem 2.** Let \( f \in \mathcal{A}^2 \). For every \( \delta > 0 \) there exists a set \( E = E(\delta, f) \subset T \) of asymptotically finite logarithmic measure such that for all \( r \in T \setminus E \) we obtain

\[
M_f(r) \leq \frac{\mu_f(r)}{(1 - r_1)^{1+\delta}} \ln^{1+\delta} \left( \frac{\mu_f(r)}{1 - r_1} \right) \cdot \ln^{1/2+\delta} r_2.
\]

**Proof.** Let \( E' \) and \( E_0 \) be the exceptional sets from Theorem 1 and Lemma 1, respectively. Then for \( E = E' \cup E_0 \) and \( h(r_1, r_2) = (r_1 r_2)^{1+\delta} \), \( \delta \in (0, 1) \), we get for all \( r \in T \setminus E \)

\[
M_f(r) \leq C_0 \mu_f(r) (\det(D_f(r) + I))^{1/2} \leq C_0 \mu_f(r) \left( \frac{1}{1 - r_1} \cdot h \left( \frac{\partial}{\partial r_1} \ln M_f(r), r_2 \frac{\partial}{\partial r_2} \ln M_f(r) + \ln r_2 \right) \right)^{1/2} \leq C_0 \mu_f(r) \left( \frac{1}{1 - r_1} \left( \frac{\partial}{\partial r_1} \ln M_f(r) \right)^{1+\delta} \left( r_2 \frac{\partial}{\partial r_2} \ln M_f(r) + \ln r_2 \right)^{1+\delta} \right)^{1/2}.
\]
Hence by Lemma 1 there exists $r^0 \in T$ such that for all $r \in \Delta_{r^0} \setminus E$ we get
\[
\mathcal{M}_f(r) \leq C_0 \mu_f(r) \left( \frac{1}{(1-r_1)^2} \left( \ln \mathcal{M}_f(r) \cdot \ln r_2 \right)^{(1+\delta)^2} \left( \frac{1}{(1-r_1)^\delta} (\ln \mathcal{M}_f(r))^{1+\delta} + \ln r_2 \right)^{1+\delta} \right)^{1/2} < \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{(1+\delta)^2} \mathcal{M}_f(r) \ln^{1/2+2\delta} r_2.
\]

Using inequality (7) we obtain
\[
\ln \mathcal{M}_f(r) \leq \ln \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} + (1+\delta)^2 \ln \ln \mathcal{M}_f(r) + \left( \frac{1}{2} + 2\delta \right) \ln \ln r_2 \leq \ln \frac{\mu_f(1+\delta)}{(1-r_1)^{1+\delta}} + 8 \ln \ln \mathcal{M}_f(r).
\]

Therefore, $\ln \mathcal{M}_f(r) \leq 2 \ln \frac{\mu_f(r)}{1-r_1}$. Finally for all $r \in \Delta_{r^0} \setminus E$ we have
\[
M_f(r) \leq \mathcal{M}_f(r) \leq \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \left( 2 \ln \frac{\mu_f(r)}{1-r_1} \right)^{1+2\delta+\delta^2} \ln^{1/2+2\delta} r_2 < \frac{\mu_f(r)}{(1-r_1)^{1+\delta_i}} \left( \ln \frac{\mu_f(r)}{1-r_1} \right)^{1+\delta_i} \ln^{1/2+\delta_i} r_2,
\]
where $\delta_i > 2(\delta + \delta^2)$. \hfill \Box

3. Some examples. From Theorem 2 we conclude, that for every $\delta > 0$ the set
\[
E = \left\{ r \in T : M_f(r) > \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \left( \frac{\mu_f(r)}{1-r_1} \right) \ln^{1/2+\delta} r_2 \right\}
\]
has asymptotically finite logarithmic measure.

Remark that exponent 1 by $\frac{1}{1-r_1}$ in inequality (6) cannot be replaced with a smaller number. We suppose that one can replace this exponent with $\varepsilon \in (0, 1)$. Consider the function
\[
f(z) = f(z_1, z_2) = \frac{\gamma(z_2)}{1-z_1},
\]
where $\gamma(z_2)$ is an entire function such that $\ln \mu_\gamma(r_2) < r_2$, $r_2 \to +\infty$. Then
\[
M_f(r) = \frac{M_\gamma(r_2)}{1-r_1}, \quad \mu_f(r) = \mu_\gamma(r_2).
\]

Denote $r'(\varepsilon) \in (0, 1)$ such that for all $r_1 > r'(\varepsilon)$ we have
\[
\frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} > \varepsilon.
\]

Thus,
\[
B' = \left\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) : \frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} > \frac{r_2^2}{1-r_1} \right\} \subset \left\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) : \frac{1}{1-r_1} > \frac{1}{(1-r_1)^{\varepsilon}} \left( \ln \frac{1}{1-r_1} + r_2 \right) \sqrt{\ln r_2} \right\} \subset
\]
Remark that for this function there exists

\[\frac{\mu_f(r_1, r_2)}{1 - r_1} > \frac{\mu_f(r_2)}{1 - r_1} \ln \frac{\mu_f(r_2)}{1 - r_1} \sqrt{\ln r_2}\] 

\[\subset \{ r \in [r', 1) \times [e, +\infty) : M_f(r_1, r_2) > \frac{\mu_f(r_1, r_2)}{1 - r_1} \ln \frac{\mu_f(r_1, r_2)}{1 - r_1} \sqrt{\ln r_2} \} = B\]

Denote \( R = \sqrt{\frac{1}{1 - r_1} - \ln^2 \frac{1}{1 - r_1}} \). Then for \( \nu_\mu \)-measure of the set \( B \) is estimated as follows

\[
\nu_\mu(B) = \frac{1}{2} \int_{r'(e)}^{1} \ln \left( \frac{1}{(1 - r_1)^{1 - e} \ln^2 (1 - r_1)} \right) dr_1 \geq \frac{1}{2} \int_{r'(e)}^{1} \frac{dr_1}{1 - r_1} = +\infty.
\]

We remark, that none of the exponents \( 1 \) in (6) can be replaced with a smaller number. Let us consider the function

\[f(z_1, z_2) = \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} e^{n^2} z_2^m = \varphi(z_1) \cdot \psi(z_2), \quad \varepsilon \in (0, 1).\]

Remark that for this function there exists \( r_0 \in T \) such that for some \( \delta > 0 \) we have

\[
\{ r \in \Delta_{\varphi} : M_f(r) > \frac{\mu_f(r)}{1 - r_1} \ln^{1+\delta} \left( \frac{\mu_f(r)}{1 - r_1} \right) \ln^{1/2+\delta} r_2 \} \subset
\]

\[
\{ r \in \Delta_{\varphi} : M_f(r) > \frac{\mu_f(r)}{1 - r_1} \ln^{1+\delta} \left( \frac{\mu_f(r)}{1 - r_1} \right) \}.
\]

For the analytic function \( \varphi(z_1) \) there exist constants \( C_1'(\varepsilon) \in (0, 1) \) and \( r_1 \in (r_1^0, 1) \) such that for all \( r_1 \geq r_1^0 \) we get

\[
C_1'(\varepsilon) \frac{\mu_\varphi(r_1)}{1 - r_1} \leq \frac{M_\varphi(r_1)}{\sqrt{\ln M_\varphi(r_1)}} \leq \frac{1}{C_1'(\varepsilon)} \frac{\mu_\varphi(r_1)}{1 - r_1}.
\] (8)

It follows from inequality (8), that for \( r_1 \geq r_1^0 \) and some constant \( C_1(\varepsilon) < C_1'(\varepsilon) \) we obtain

\[
M_\varphi(r_1) \geq C_1(\varepsilon) \frac{\mu_\varphi(r_1)}{1 - r_1} \ln^{1/2} \frac{\mu_\varphi(r_1)}{1 - r_1}.
\] (9)

and for the entire function \( \psi(z_2) \) we obtain for every \( \varepsilon > 0 \) and \( r_2 \in (r_2^0, +\infty) \)

\[
M_\psi(r_2) \geq (\sqrt{2\pi} - \varepsilon) \mu_\psi(r_2) \ln^2 \mu_\psi(r_2).
\] (10)

So, \( M_f(r_1, r_2) = M_\varphi(r_1) M_\psi(r_2) \) and for \( r \in (r_1^0, 1) \times (r_2^0, +\infty) \) we get

\[
M_f(r) \geq (\sqrt{2\pi} - \varepsilon) C_1(\varepsilon) \frac{\mu_f(r)}{1 - r_1} \left( \ln \frac{\mu_\varphi(r_1)}{1 - r_1}, \ln \mu_\psi(r_2) \right)^{1/2}.
\] (11)
Consider the positive increasing functions $g_1(r_1) = \ln(\mu_r(r_1)/(1-r_1))$ and $g_2(r_2) = \ln \mu_\psi(r_2)$. We define

$$A = \left\{ r \in T: g_1(r_1) \cdot g_2(r_2) > \frac{1}{18}(g_1(r_1) + g_2(r_2))^2 \right\} \supset \left\{ r \in T: \frac{1}{2} < \frac{g_1(r_1)}{g_2(r_2)} < 2 \right\} = E^*.$$

Then, for $r \in A$ we get

$$g_1(r_1)g_2(r_2) \geq \frac{g_2^2(r_2)}{2} = \frac{1}{18}(g_2(r_2) + 2g_2(r_2))^2 \geq \frac{1}{18}(g_1(r_1) + g_2(r_2))^2.$$

Moreover, there exists the inverse function $g_2^{-1}: \mathbb{R}^+ \to (r_0, 1)$, which is also increasing.

For $r_1 \in (r_0, 1)$ we define $r_1^*$ and $r_2^*$ such that

$$r_1^* = g_2^{-1}\left(\frac{g_1(r_1)}{2}\right), \quad r_2^* = g_2^{-1}(2g_1(r_1)).$$

Therefore, from inequality (11) we have for all $r \in E^*$

$$M_f(r) \geq \frac{(\sqrt{2\pi} - \varepsilon)C_1(\varepsilon) \mu_f(r)}{18} \left(\ln \frac{\mu_r(r_1)}{1-r_1} + \ln \mu_\psi(r_2)\right) = \frac{(\sqrt{2\pi} - \varepsilon)C_1(\varepsilon) \mu_f(r)}{18} \ln \frac{\mu_f(r)}{1-r_1}.$$

It remains to prove that set $E^*$ is a set of infinite asymptotically logarithmic measure.

$$\mu(E^*) = \int_{\Delta_{r_1\cap E^*}} dr_1dr_2 = \int_{r_1^*}^{r_2^*} \int_{r_1}^{r_2} dr_1dr_2 = \int_{r_1^*}^{r_2^*} \frac{dr_1}{1-r_1} = \int_{r_1^*}^{r_2^*} \ln g_2^{-1}(2g_1(r_1)) - \ln g_2^{-1}\left(\frac{g_1(r_1)}{2}\right) \frac{dr_1}{1-r_1}.$$

Since $\lim_{r \to +\infty} \frac{M_f(r)}{\mu_\psi(r)} = \sqrt{2\pi}$ and $g_2(r) = \ln \mu_\psi(r)$, we get

$$r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln g_2(r) < g_2(r) < r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln g_2(r), \quad r \to +\infty. \quad (12)$$

From (12) we have

$$g_2^{-1}(r) > r, \quad g_2(r) > r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln g_2(r) > r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln r > \frac{2}{3}r, \quad g_2^{-1}(r) < \frac{3}{2}r, \quad r \to +\infty. \quad (14)$$

Using (13) and (14) we finally obtain

$$\nu_\infty(E^*) \geq \int_{r_1^*}^{r_2^*} \ln(2g_1(r_1)) - \ln\left(\frac{1}{2} \cdot \frac{3}{2}g_1(r_1)\right) \frac{dr_1}{1-r_1} = \int_{r_1^*}^{r_2^*} \ln \frac{8}{3} \cdot \frac{dr_1}{1-r_1} = +\infty.$$
REFERENCES


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