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WIMAN'S TYPE INEQUALITY FOR SOME DOUBLE POWER SERIES

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In this paper we prove some analogue of Wiman's inequality for analytic functions $f(z_1, z_2)$ in the domain $\mathbb{T} = \{z \in \mathbb{C}^2 \colon |z_1| < 1,\ |z_2| < +\infty\}$. The obtained inequality is sharp.

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Доказывается аналог неравенства Вимана для функций $f(z_1, z_2)$, аналитических в области $\mathbb{T} = \{z \in \mathbb{C}^2 \colon |z_1| < 1,\ |z_2| < +\infty\}$. Полученное неравенство точное.

1. Introduction. Let f be analytic function in the disc $\{z \colon |z| < R\}$, $0 < R \le +\infty$, represented by the power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \tag{1}$$

For $r \in (0, R)$ we denote $M_f(r) = \max\{|f(z)| \colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n \colon n \ge 0\}$.

It is well known ([1, p. 9], [2, p. 28]) that for each nonconstant entire function $f(z)$ and every $\varepsilon > 0$ there exists a set $E(\varepsilon, f) \subset [1, +\infty)$ such that Wiman's inequality

$$M_f(r) \le \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}$$

holds for all $r \in [1, +\infty)\backslash E(\varepsilon, f)$, where the set $E(\varepsilon, f)$ *has finite logarithmic measure*, i.e. $\int_{E(\varepsilon,f)} \frac{dr}{r} < +\infty$.

Let $R = 1$, i.e. $f(z)$ be an analytic function in the unit disc $\mathbb{D} = \{z \colon |z| < 1\}$. For such a function $f(z)$ and every $\delta > 0$ there exists a set $E_f(\delta) \subset (0, 1)$ of finite logarithmic measure on $(0, 1)$, i.e.

$$\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty,$$

such that for all $r \in (0, 1)\backslash E_f(\delta)$ the inequality

$$M_f(r) \le \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}$$

holds ([3]). Similar inequality for analytic function in the unit disc one can find in [4].

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Also in [3] it is noted that for the function $g(z)=\sum_{n=1}^{+\infty}\exp\{n^{\varepsilon}\}z^n$, $\varepsilon\in(0,1)$ we have

$$\varliminf_{r\to 1-0}\frac{M_g(r)}{\frac{\mu_g(r)}{1-r}\ln^{1/2}\frac{\mu_g(r)}{1-r}}\geq C>0.$$

Some analogues of Wiman's inequality for entire functions of several complex variables one can find in [5]–[11].

The aim of this paper is to prove some analogues of Wiman's inequality for analytic functions represented by the series

$$f(z)=f(z_1,z_2)=\sum_{n+m=0}^{+\infty}a_{nm}z_1^nz_2^m \tag{2}$$

with the domain of convergence $\mathbb{T}=\{z\in\mathbb{C}^2\colon |z_1|<1, z_2\in\mathbb{C}\}$.

By $\mathcal{A}^2$ we denote the class of analytic functions of form (2) with the domain of convergence $\mathbb{T}$ and $\frac{\partial}{\partial z_2}f(z_1,z_2)\not\equiv 0$ in $\mathbb{T}$.

2. Wiman's type inequality for analytic functions in $\mathbb{T}$. For $r=(r_1,r_2)\in T{:=}[0,1)\times[0,+\infty)$ and function $f\in\mathcal{A}^2$ we denote

$$\triangle_r=\{(t_1,t_2)\in T\colon t_1>r_1,\ t_2>r_2\},\ M_f(r)=\max\{|f(z)|\colon |z_1|\leq r_1, |z_2|\leq r_2\},$$
$$\mu_f(r)=\max\{|a_{nm}|r_1^nr_2^m\colon (n,m)\in\mathbb{Z}_+^2\},\ \mathfrak{M}_f(r)=\sum_{n+m=0}^{+\infty}|a_{nm}|r_1^nr_2^m.$$

Let $D_f(r)=(D_{ij})$ be a 2×2 matrix such that

$$D_{ij}=r_i\frac{\partial}{\partial r_i}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)\Big)=\partial_i\partial_j\ln\mathfrak{M}_f(r),\quad \partial_i=r_i\frac{\partial}{\partial r_i},\quad i,j\in\{1,2\}.$$

The following statement can be proved by verbatim repetition of the proof of Theorem 3.1 from [5].

Theorem 1. *Let $f\in\mathcal{A}^2$. There exists an absolute constant C_0 such that*

$$\mathfrak{M}_f(r)\leq C_0\mu_f(r)(\det(D_f(r)+I))^{1/2},$$

where I is the identity 2×2 matrix.

We say that $E\subset T$ is set of *asymptotically finite logarithmic measure on T* if there exists $r_0\in T$ such that

$$\nu_{\ln}(E\cap\triangle_{r_0}){:=}\iint\limits_{E\cap\triangle_{r_0}}\frac{dr_1dr_2}{(1-r_1)r_2}<+\infty,$$

i.e. the set $E\cap\triangle_{r_0}$ is a set of *finite logarithmic measure on T.*

Lemma 1. *Let $\delta>0$ and let $h\colon\mathbb{R}_+^2\to\mathbb{R}_+$ be an increasing function on each variable such that*

$$\int\limits_1^{+\infty}\int\limits_1^{+\infty}\frac{du_1du_2}{h(u_1,u_2)}<+\infty.$$

Then there exists a set $E\subset T$ of asymptotically finite logarithmic measure such that for all $r\in T\backslash E$ the inequalities

$$\det(D_f(r)+I)\le \frac{1}{1-r_1}\cdot h\Big(\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r), r_2\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)+\ln r_2\Big), \tag{3}$$

$$\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)\le \frac{1}{1-r_1}\cdot h\Big(\ln\mathfrak{M}_f(r),\ln r_2\Big), \tag{4}$$

$$\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)\le \frac{1}{r_2(1-r_1)^\delta}(\ln\mathfrak{M}_f(r))^{1+\delta} \tag{5}$$

hold.

Proof. Let $E_1\subset T$ be a set for which inequality (3) does not hold. Now we prove that E_1 is a set of asymptotically finite logarithmic measure. Since $r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)$ is increasing to $+\infty$ for $j\in\{1,2\}$ as $r_1\to 1-0,\ r_2\to+\infty$, there exists $r^0\in T$ such that $r_1^0>\frac12,\ r_2^0>1$ and for all $r\in\triangle_{r^0}$ we have

$$r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln r_j>1.$$

Then

$$\nu_{\ln}(E_1\cap\triangle_{r^0})=\iint\limits_{E_1\cap\triangle_{r^0}}\frac{dr_1dr_2}{(1-r_1)r_2}\le\iint\limits_{E_1\cap\triangle_{r^0}}\frac{\det(D_f(r)+I)(1-r_1)dr_1dr_2}{h\Big(\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r), r_2\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)+\ln r_2\Big)(1-r_1)r_2}\le$$
$$\le\iint\limits_{E_1\cap\triangle_{r^0}}\frac{1}{r_1r_2}\cdot\frac{\det(D_f(r)+I)dr_1dr_2}{h\Big(r_1\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)+\ln r_1, r_2\frac{\partial}{\partial r_2}\ln\mathfrak{M}_f(r)+\ln r_2\Big)}.$$

Let $U\colon T\to\mathbb{R}^2_+$ be a mapping such that $U=(u_1(r),u_2(r))$ and $u_j(r)=r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln r_j,\ j\in\{1,2\},\ r=(r_1,r_2)$. Then for $i,j\in\{1,2\}$ we obtain

$$\frac{\partial u_i}{\partial r_i}=\frac{\partial}{\partial r_i}\Big(r_i\frac{\partial}{\partial r_i}\ln\mathfrak{M}_f(r)+\ln r_i\Big)=\frac{1}{r_i}\partial_i\partial_i\ln\mathfrak{M}_f(r)+\frac{1}{r_i},$$
$$\frac{\partial u_i}{\partial r_j}=\frac{\partial}{\partial r_j}\Big(r_i\frac{\partial}{\partial r_i}\ln\mathfrak{M}_f(r)+\ln r_i\Big)=\frac{1}{r_j}\partial_i\partial_j\ln\mathfrak{M}_f(r),\ i\ne j.$$

So, the Jacobian

$$J_1:=\frac{D(u_1,u_2)}{D(r_1,r_2)}=\begin{vmatrix}\frac{\partial u_1}{\partial r_1}&\frac{\partial u_1}{\partial r_2}\\ \frac{\partial u_2}{\partial r_1}&\frac{\partial u_2}{\partial r_2}\end{vmatrix}=\det(D_f(r)+I)\frac{1}{r_1r_2}.$$

Therefore,

$$\nu_{\ln}(E_1\cap\triangle_{r^0})\le\iint\limits_{U(E_1\cap\triangle_{r^0})}\frac{du_1du_2}{h(u_1,u_2)}\le\int\limits_1^{+\infty}\int\limits_1^{+\infty}\frac{du_1du_2}{h(u_1,u_2)}<+\infty.$$

Suppose that $E_2\subset T$ is a set for which inequality (4) does not hold. Then we can choose $r^0\in T$ so that $\ln\mathfrak{M}_f(r^0)>1$ and $r_2^0>e$.

$$\nu_{\ln}(E_2\cap\triangle_{r^0})=\iint\limits_{E_2\cap\triangle_{r^0}}\frac{dr_1dr_2}{(1-r_1)r_2}\le\iint\limits_{E_2\cap\triangle_{r^0}}\frac{\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)\cdot(1-r_1)dr_1dr_2}{h(\ln\mathfrak{M}_f(r),\ln r_2)(1-r_1)r_2}.$$

Consider the mapping $V\colon T \to \mathbb{R}_+^2$, where $V = (v_1(r), v_2(r))$ and $v_1 = \ln \mathfrak{M}_f(r),\ v_2 = \ln r_2,\ r = (r_1, r_2)$. So,

$$J_2 := \frac{D(v_1, v_2)}{D(r_1, r_2)} = \begin{vmatrix} \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) & \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) \\ 0 & \frac{1}{r_2} \end{vmatrix} = \frac{1}{r_2} \cdot \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r).$$

Therefore

$$\nu_{\ln}(E_2 \cap \triangle_{r^0}) \le \iint\limits_{E_2 \cap \triangle_{r^0}} \frac{\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) dr_1 dr_2}{h(\ln \mathfrak{M}_f(r), \ln r_2) r_2} = \iint\limits_{V(E_2 \cap \triangle_{r^0})} \frac{du_1 du_2}{h(u_1, u_2)} \le \int\limits_1^{+\infty} \int\limits_1^{+\infty} \frac{du_1 du_2}{h(u_1 u_2)} < +\infty.$$

Let $E_3 \subset T$ be a set for which inequality (5) does not hold. Then

$$\nu_{\ln}(E_3 \cap \triangle_{r^0}) \le \iint\limits_{E_3 \cap \triangle_{r^0}} \frac{\frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) r_2 dr_1 dr_2}{\frac{1}{(1-r_1)^\delta} \ln^{1+\delta} \mathfrak{M}_f(r)(1 - r_1) r_2}.$$

Let r_0 be such that $\ln \mathfrak{M}_f(r^0) > 1$. Define the mapping $W\colon T \to T$, where $W = (w_1(r), w_2(r))$ and $w_1 = r_1,\ w_2 = \ln \mathfrak{M}_f(r),\ r = (r_1, r_2)$. So,

$$J_3 := \frac{D(w_1, w_2)}{D(r_1, r_2)} = \begin{vmatrix} 1 & 0 \\ \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) & \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) \end{vmatrix} = \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r).$$

Therefore,

$$\nu_{\ln}(E_3 \cap \triangle_{r^0}) \le \iint\limits_{W(E_3 \cap \triangle_{r^0})} \frac{du_1 du_2}{(1 - u_1)^{1-\delta} u_2^{1+\delta}} \le \int\limits_0^1 \frac{du_1}{(1 - u_1)^{1-\delta}} \cdot \int\limits_1^{+\infty} \frac{du_2}{u_2^{1+\delta}} < +\infty.$$

It remains to remark that the set $E = \bigcup_{j=1}^3 E_j$ is also a set of asymptotically finite logarithmic measure in T. $\square$

Theorem 2. *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \backslash E$ we obtain*

$$M_f(r) \le \frac{\mu_f(r)}{(1 - r_1)^{1+\delta}} \ln^{1+\delta} \Big(\frac{\mu_f(r)}{1 - r_1} \Big) \cdot \ln^{1/2+\delta} r_2. \tag{6}$$

Proof. Let E' and E_0 be the exceptional sets from Theorem 1 and Lemma 1, respectively. Then for $E = E' \cup E_0$ and $h(r_1, r_2) = (r_1 r_2)^{1+\delta}, \delta \in (0, 1)$, we get for all $r \in T \backslash E$

$$\mathfrak{M}_f(r) \le C_0 \mu_f(r) (\det(D_f(r) + I))^{1/2} \le$$
$$\le C_0 \mu_f(r) \bigg(\frac{1}{1 - r_1} \cdot h\Big(\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r), r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2 \Big) \bigg)^{1/2} \le$$
$$\le C_0 \mu_f(r) \bigg(\frac{1}{1 - r_1} \Big(\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) \Big)^{1+\delta} \Big(r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2 \Big)^{1+\delta} \bigg)^{1/2}.$$

Hence by Lemma 1 there exists $r^0 \in T$ such that for all $r \in \triangle_{r^0} \backslash E$ we get

$$\mathfrak{M}_f(r) \le C_0 \mu_f(r) \left(\frac{1}{(1-r_1)^2} \Big(\ln \mathfrak{M}_f(r) \cdot \ln r_2 \Big)^{(1+\delta)^2} \Big(\frac{1}{(1-r_1)^{\delta}} (\ln \mathfrak{M}_f(r))^{1+\delta} + \ln r_2 \Big)^{1+\delta} \right)^{1/2} <$$
$$< \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{(1+\delta)^2} \mathfrak{M}_f(r) \ln^{1/2+2\delta} r_2. \qquad (7)$$

Using inequality (7) we obtain

$$\ln \mathfrak{M}_f(r) \le \ln \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} + (1+\delta)^2 \ln\ln \mathfrak{M}_f(r) + \Big(\frac{1}{2} + 2\delta\Big) \ln\ln r_2 \le$$
$$\le \ln \frac{\mu_f^{1+\delta}(r)}{(1-r_1)^{1+\delta}} + 8 \ln\ln \mathfrak{M}_f(r).$$

Therefore, $\ln \mathfrak{M}_f(r) \le 2 \ln \frac{\mu_f(r)}{1-r_1}$. Finally for all $r \in \triangle_{r^0} \backslash E$ we have

$$M_f(r) \le \mathfrak{M}_f(r) \le \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \left(2 \ln \frac{\mu_f(r)}{1-r_1} \right)^{1+2\delta+\delta^2} \ln^{1/2+2\delta} r_2 <$$
$$< \frac{\mu_f(r)}{(1-r_1)^{1+\delta_1}} \left(\ln \frac{\mu_f(r)}{1-r_1} \right)^{1+\delta_1} \ln^{1/2+\delta_1} r_2,$$

where $\delta_1 > 2(\delta + \delta^2)$. □

3. Some examples. From Theorem 2 we conclude, that for every $\delta > 0$ the set

$$E = \Big\{ r \in T \colon M_f(r) > \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \Big(\frac{\mu_f(r)}{1-r_1} \Big) \ln^{1/2+\delta} r_2 \Big\}$$

has asymptotically finite logarithmic measure.

Remark that exponent 1 by $\frac{1}{1-r_1}$ in inequality (6) cannot be replaced with a smaller number. We suppose that one can replace this exponent with $\varepsilon \in (0,1)$. Consider the function

$$f(z) = f(z_1, z_2) = \frac{\gamma(z_2)}{1-z_1},$$

where $\gamma(z_2)$ is an entire function such that $\ln \mu_\gamma(r_2) < r_2$, $r_2 \to +\infty$. Then

$$M_f(r) = \frac{M_\gamma(r_2)}{1-r_1}, \quad \mu_f(r) = \mu_\gamma(r_2).$$

Denote $r'(\varepsilon) \in (0,1)$ such that for all $r_1 > r'(\varepsilon)$ we have

$$\frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} > e.$$

Thus,

$$B' = \Big\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) \colon \frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} > r_2^2 \Big\} \subset$$
$$\subset \Big\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) \colon \frac{1}{1-r_1} > \frac{1}{(1-r_1)^{\varepsilon}} \Big(\ln \frac{1}{1-r_1} + r_2 \Big) \sqrt{\ln r_2} \Big\} \subset$$

$$\subset\Big\{r\in[r'(\varepsilon),1)\times[e,+\infty)\colon \frac{\mu_\gamma(r_2)}{1-r_1}>\frac{\mu_\gamma(r_2)}{(1-r_1)^\varepsilon}\ln\frac{\mu_\gamma(r_2)}{1-r_1}\sqrt{\ln r_2}\Big\}\subset$$

$$\subset\Big\{r\in[r'(\varepsilon),1)\times[e,+\infty)\colon M_f(r_1,r_2)>\frac{\mu_f(r_1,r_2)}{(1-r_1)^\varepsilon}\ln\frac{\mu_f(r_1,r_2)}{1-r_1}\sqrt{\ln r_2}\Big\}=B$$

Denote $R=\sqrt{\frac{1}{(1-r_1)^{1-\varepsilon}}-\ln^2\frac{1}{1-r_1}}$. Then for $\nu_{\ln}$-measure of the set B is estimated as follows

$$\nu_{\ln}(B)=\iint\limits_{B}\frac{dr_1dr_2}{(1-r_1)r_2}\geq\iint\limits_{B'}\frac{dr_1dr_2}{(1-r_1)r_2}\geq\int\limits_{r'(\varepsilon)}^{1}\Bigg(\int\limits_{1}^{R}\frac{dr_2}{r_2}\Bigg)\frac{dr_1}{1-r_1}\geq$$

$$=\frac{1}{2}\int\limits_{r'(\varepsilon)}^{1}\ln\Bigg(\frac{1}{(1-r_1)^{1-\varepsilon}}-\ln^2\frac{1}{1-r_1}\Bigg)\frac{dr_1}{1-r_1}\geq\frac{1}{2}\int\limits_{r'(\varepsilon)}^{1}\frac{dr_1}{1-r_1}=+\infty.$$

We remark, that none of the exponents 1 in (6) can be replaced with a smaller number. Let us consider the function

$$f(z_1,z_2)=\sum_{n=0}^{+\infty}\frac{z_1^n}{n!}\sum_{m=1}^{+\infty}e^{m^\varepsilon}z_2^m=\varphi(z_1)\cdot\psi(z_2),\ \varepsilon\in(0,1).$$

Remark that for this function there exists $r_0\in T$ such that for some $\delta>0$ we have

$$\Big\{r\in\triangle_{r^0}\colon M_f(r)>\frac{\mu_f(r)}{(1-r_1)^{1+\delta}}\ln^{1+\delta}\Big(\frac{\mu_f(r)}{1-r_1}\Big)\ln^{1/2+\delta}r_2\Big\}\subset$$

$$\subset\Big\{r\in\triangle_{r^0}\colon M_f(r)>\frac{\mu_f(r)}{(1-r_1)^{1+\delta}}\ln^{1+2\delta}\Big(\frac{\mu_f(r)}{1-r_1}\Big)\Big\}.$$

For the analytic function $\varphi(z_1)$ there exist constants $C_1'(\varepsilon)\in(0,1)$ and $r_1\in(r_1^0,1)$ such that for all $r_1\geq r_1^0$ we get

$$C_1'(\varepsilon)\frac{\mu_\varphi(r_1)}{1-r_1}\leq\frac{M_\varphi(r_1)}{\sqrt{\ln M_\varphi(r_1)}}\leq\frac{1}{C_1'(\varepsilon)}\frac{\mu_\varphi(r_1)}{1-r_1}.\tag{8}$$

It follows from inequality (8), that for $r_1\geq r_1^0$ and some constant $C_1(\varepsilon)<C_1'(\varepsilon)$ we obtain

$$M_\varphi(r_1)\geq C_1(\varepsilon)\frac{\mu_\varphi(r_1)}{1-r_1}\ln^{1/2}\frac{\mu_\varphi(r_1)}{1-r_1},\tag{9}$$

and for the entire function $\psi(z_2)$ we obtain for every $\varepsilon>0$ and $r_2\in(r_2^0,+\infty)$

$$M_\psi(r_2)\geq(\sqrt{2\pi}-\varepsilon)\mu_\psi(r_2)\ln^{\frac{1}{2}}\mu_\psi(r_2).\tag{10}$$

So, $M_f(r_1,r_2)=M_\varphi(r_1)M_\psi(r_2)$ and for $r\in(r_1^0;1)\times(r_2^0,+\infty)$ we get

$$M_f(r)\geq(\sqrt{2\pi}-\varepsilon)C_1(\varepsilon)\frac{\mu_f(r)}{1-r_1}\Bigg(\ln\frac{\mu_\varphi(r_1)}{1-r_1}\cdot\ln\mu_\psi(r_2)\Bigg)^{\frac{1}{2}}.\tag{11}$$

Consider the positive increasing functions $g_1(r_1)=\ln(\mu_\varphi(r_1)/(1-r_1))$ and $g_2(r_2)=\ln\mu_\psi(r_2)$. We define

$$A=\Big\{r\in T\colon g_1(r_1)\cdot g_2(r_2)>\frac{1}{18}(g_1(r_1)+g_2(r_2))^2\Big\}\supset\Big\{r\in T\colon \frac{1}{2}<\frac{g_1(r_1)}{g_2(r_2)}<2\Big\}=E^*.$$

Then, for $r\in A$ we get

$$g_1(r_1)g_2(r_2)\geq\frac{g_2^2(r_2)}{2}=\frac{1}{18}(g_2(r_2)+2g_2(r_2))^2\geq\frac{1}{18}(g_1(r_1)+g_2(r_2))^2.$$

Moreover, there exists the inverse function $g_2^{-1}\colon\mathbb{R}_+\to(r_0,1)$, which is also increasing.

For $r_1\in(r_1^0,1)$ we define r_1^* and r_2^* such that

$$r_1^*=g_2^{-1}\Big(\frac{g_1(r_1)}{2}\Big),\ r_2^*=g_2^{-1}(2g_1(r_1)).$$

Therefore, from inequality (11) we have for all $r\in E^*$

$$M_f(r)\geq\frac{(\sqrt{2\pi}-\varepsilon)C_1(\varepsilon)}{18}\frac{\mu_f(r)}{1-r_1}\left(\ln\frac{\mu_\varphi(r_1)}{1-r_1}+\ln\mu_\psi(r_2)\right)=\frac{(\sqrt{2\pi}-\varepsilon)C_1(\varepsilon)}{18}\frac{\mu_f(r)}{1-r_1}\ln\frac{\mu_f(r)}{1-r_1}.$$

It remains to prove that set E^* is a set of infinite asymptotically logarithmic measure.

$$\mu(E^*)=\iint\limits_{\triangle_{r_1^0}\cap E^*}\frac{dr_1dr_2}{(1-r_1)r_2}=\int\limits_{r_0^1}^{1}\int\limits_{r_1^*}^{r_2^*}\frac{dr_1dr_2}{(1-r_1)r_2}=$$

$$=\int\limits_{r_0^1}^{1}(\ln r_2^*-\ln r_1^*)\frac{dr_1}{1-r_1}=\int\limits_{r_0^1}^{1}\left(\ln g_2^{-1}(2g_1(r_1))-\ln g_2^{-1}\Big(\frac{g_1(r_1)}{2}\Big)\right)\frac{dr_1}{1-r_1}.$$

Since $\lim\limits_{r\to+\infty}\frac{M_\psi(r)}{\mu_\psi(r)\ln^{1/2}\mu_\psi(r)}=\sqrt{2\pi}$ and $g_2(r)=\ln\mu_\psi(r)$, we get

$$r-\ln(\sqrt{2\pi}+\varepsilon)-\frac{1}{2}\ln g_2(r)<g_2(r)<r-\ln(\sqrt{2\pi}-\varepsilon)-\frac{1}{2}\ln g_2(r),\ r\to+\infty. \tag{12}$$

From (12) we have

$$g_2^{-1}(r)>r, \tag{13}$$

$$g_2(r)>r-\ln(\sqrt{2\pi}+\varepsilon)-\frac{1}{2}\ln g_2(r)>r-\ln(\sqrt{2\pi}+\varepsilon)-\frac{1}{2}\ln r>\frac{2}{3}r,$$

$$g_2^{-1}(r)<\frac{3}{2}r,\ r\to+\infty. \tag{14}$$

Using (13) and (14) we finally obtain

$$\nu_{\ln}(E^*)\geq\int\limits_{r_0^1}^{1}\left(\ln(2g_1(r_1))-\ln\Big(\frac{1}{2}\cdot\frac{3}{2}g_1(r_1)\Big)\right)\frac{dr_1}{1-r_1}=\int\limits_{r_0^1}^{1}\ln\frac{8}{3}\cdot\frac{dr_1}{1-r_1}=+\infty.$$

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