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WIMAN'S TYPE INEQUALITY FOR SOME DOUBLE POWER SERIES

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In this paper we prove some analogue of Wiman's inequality for analytic functions $f(z_1, z_2)$ in the domain $\mathbb{T} = \{z \in \mathbb{C}^2: |z_1| < 1, |z_2| < +\infty\}$. The obtained inequality is sharp.

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Доказывается аналог неравенства Вимана для функций $f(z_1, z_2)$, аналитических в области $\mathbb{T} = \{z \in \mathbb{C}^2: |z_1| < 1, |z_2| < +\infty\}$. Полученное неравенство точное.

1. Introduction. Let f be analytic function in the disc $\{z: |z| < R\}$, $0 < R \leq +\infty$, represented by the power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \quad (1)$$

For $r \in (0, R)$ we denote $M_f(r) = \max\{|f(z)|: |z| = r\}$, $\mu_f(r) = \max\{|a_n| r^n: n \geq 0\}$.

It is well known ([1, p. 9], [2, p. 28]) that for each nonconstant entire function $f(z)$ and every $\varepsilon > 0$ there exists a set $E(\varepsilon, f) \subset [1, +\infty)$ such that Wiman's inequality

$$M_f(r) \leq \mu_f(r) (\ln \mu_f(r))^{1/2+\varepsilon}$$

holds for all $r \in [1, +\infty) \setminus E(\varepsilon, f)$, where the set $E(\varepsilon, f)$ has finite logarithmic measure, i.e. $\int_{E(\varepsilon, f)} \frac{dr}{r} < +\infty$.

Let $R = 1$, i.e. $f(z)$ be an analytic function in the unit disc $\mathbb{D} = \{z: |z| < 1\}$. For such a function $f(z)$ and every $\delta > 0$ there exists a set $E_f(\delta) \subset (0, 1)$ of finite logarithmic measure on $(0, 1)$, i.e.

$$\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty,$$

such that for all $r \in (0, 1) \setminus E_f(\delta)$ the inequality

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}$$

holds ([3]). Similar inequality for analytic function in the unit disc one can find in [4].

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Also in [3] it is noted that for the function $g(z) = \sum_{n=1}^{+\infty} \exp\{n^\varepsilon\}z^n$, $\varepsilon \in (0, 1)$ we have

$$\lim_{r \rightarrow 1-0} \frac{M_g(r)}{\frac{\mu_g(r)}{1-r} \ln^{1/2} \frac{\mu_g(r)}{1-r}} \geq C > 0.$$

Some analogues of Wiman's inequality for entire functions of several complex variables one can find in [5]–[11].

The aim of this paper is to prove some analogues of Wiman's inequality for analytic functions represented by the series

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m \tag{2}$$

with the domain of convergence $\mathbb{T} = \{z \in \mathbb{C}^2 : |z_1| < 1, z_2 \in \mathbb{C}\}$.

By \mathcal{A}^2 we denote the class of analytic functions of form (2) with the domain of convergence \mathbb{T} and $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ in \mathbb{T} .

2. Wiman's type inequality for analytic functions in \mathbb{T} . For $r = (r_1, r_2) \in T := [0, 1) \times [0, +\infty)$ and function $f \in \mathcal{A}^2$ we denote

$$\begin{aligned} \Delta_r &= \{(t_1, t_2) \in T : t_1 > r_1, t_2 > r_2\}, \quad M_f(r) = \max\{|f(z)| : |z_1| \leq r_1, |z_2| \leq r_2\}, \\ \mu_f(r) &= \max\{|a_{nm}|r_1^n r_2^m : (n, m) \in \mathbb{Z}_+^2\}, \quad \mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}|r_1^n r_2^m. \end{aligned}$$

Let $D_f(r) = (D_{ij})$ be a 2×2 matrix such that

$$D_{ij} = r_i \frac{\partial}{\partial r_i} \left(r_j \frac{\partial}{\partial r_j} \ln \mathfrak{M}_f(r) \right) = \partial_i \partial_j \ln \mathfrak{M}_f(r), \quad \partial_i = r_i \frac{\partial}{\partial r_i}, \quad i, j \in \{1, 2\}.$$

The following statement can be proved by verbatim repetition of the proof of Theorem 3.1 from [5].

Theorem 1. *Let $f \in \mathcal{A}^2$. There exists an absolute constant C_0 such that*

$$\mathfrak{M}_f(r) \leq C_0 \mu_f(r) (\det(D_f(r) + I))^{1/2},$$

where I is the identity 2×2 matrix.

We say that $E \subset T$ is set of *asymptotically finite logarithmic measure on T* if there exists $r_0 \in T$ such that

$$\nu_{\ln}(E \cap \Delta_{r_0}) := \iint_{E \cap \Delta_{r_0}} \frac{dr_1 dr_2}{(1-r_1)r_2} < +\infty,$$

i.e. the set $E \cap \Delta_{r_0}$ is a set of *finite logarithmic measure on T* .

Lemma 1. *Let $\delta > 0$ and let $h: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be an increasing function on each variable such that*

$$\int_1^{+\infty} \int_1^{+\infty} \frac{du_1 du_2}{h(u_1, u_2)} < +\infty.$$

Then there exists a set $E \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \setminus E$ the inequalities

$$\det(D_f(r) + I) \leq \frac{1}{1-r_1} \cdot h\left(\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r), r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2\right), \quad (3)$$

$$\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) \leq \frac{1}{1-r_1} \cdot h\left(\ln \mathfrak{M}_f(r), \ln r_2\right), \quad (4)$$

$$\frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) \leq \frac{1}{r_2(1-r_1)^\delta} (\ln \mathfrak{M}_f(r))^{1+\delta} \quad (5)$$

hold.

Proof. Let $E_1 \subset T$ be a set for which inequality (3) does not hold. Now we prove that E_1 is a set of asymptotically finite logarithmic measure. Since $r_j \frac{\partial}{\partial r_j} \ln \mathfrak{M}_f(r)$ is increasing to $+\infty$ for $j \in \{1, 2\}$ as $r_1 \rightarrow 1-0$, $r_2 \rightarrow +\infty$, there exists $r^0 \in T$ such that $r_1^0 > \frac{1}{2}$, $r_2^0 > 1$ and for all $r \in \Delta_{r^0}$ we have

$$r_j \frac{\partial}{\partial r_j} \ln \mathfrak{M}_f(r) + \ln r_j > 1.$$

Then

$$\begin{aligned} \nu_{\ln}(E_1 \cap \Delta_{r^0}) &= \iint_{E_1 \cap \Delta_{r^0}} \frac{dr_1 dr_2}{(1-r_1)r_2} \leq \iint_{E_1 \cap \Delta_{r^0}} \frac{\det(D_f(r) + I)(1-r_1)dr_1 dr_2}{h\left(\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r), r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2\right)(1-r_1)r_2} \leq \\ &\leq \iint_{E_1 \cap \Delta_{r^0}} \frac{1}{r_1 r_2} \cdot \frac{\det(D_f(r) + I)dr_1 dr_2}{h\left(r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) + \ln r_1, r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2\right)}. \end{aligned}$$

Let $U: T \rightarrow \mathbb{R}_+^2$ be a mapping such that $U = (u_1(r), u_2(r))$ and $u_j(r) = r_j \frac{\partial}{\partial r_j} \ln \mathfrak{M}_f(r) + \ln r_j$, $j \in \{1, 2\}$, $r = (r_1, r_2)$. Then for $i, j \in \{1, 2\}$ we obtain

$$\begin{aligned} \frac{\partial u_i}{\partial r_i} &= \frac{\partial}{\partial r_i} \left(r_i \frac{\partial}{\partial r_i} \ln \mathfrak{M}_f(r) + \ln r_i \right) = \frac{1}{r_i} \partial_i \partial_i \ln \mathfrak{M}_f(r) + \frac{1}{r_i}, \\ \frac{\partial u_i}{\partial r_j} &= \frac{\partial}{\partial r_j} \left(r_i \frac{\partial}{\partial r_i} \ln \mathfrak{M}_f(r) + \ln r_i \right) = \frac{1}{r_j} \partial_i \partial_j \ln \mathfrak{M}_f(r), \quad i \neq j. \end{aligned}$$

So, the Jacobian

$$J_1 := \frac{D(u_1, u_2)}{D(r_1, r_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_1}{\partial r_2} \\ \frac{\partial u_2}{\partial r_1} & \frac{\partial u_2}{\partial r_2} \end{vmatrix} = \det(D_f(r) + I) \frac{1}{r_1 r_2}.$$

Therefore,

$$\nu_{\ln}(E_1 \cap \Delta_{r^0}) \leq \iint_{U(E_1 \cap \Delta_{r^0})} \frac{du_1 du_2}{h(u_1, u_2)} \leq \int_1^{+\infty} \int_1^{+\infty} \frac{du_1 du_2}{h(u_1, u_2)} < +\infty.$$

Suppose that $E_2 \subset T$ is a set for which inequality (4) does not hold. Then we can choose $r^0 \in T$ so that $\ln \mathfrak{M}_f(r^0) > 1$ and $r_2^0 > e$.

$$\nu_{\ln}(E_2 \cap \Delta_{r^0}) = \iint_{E_2 \cap \Delta_{r^0}} \frac{dr_1 dr_2}{(1-r_1)r_2} \leq \iint_{E_2 \cap \Delta_{r^0}} \frac{\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) \cdot (1-r_1)dr_1 dr_2}{h(\ln \mathfrak{M}_f(r), \ln r_2)(1-r_1)r_2}.$$

Consider the mapping $V: T \rightarrow \mathbb{R}_+^2$, where $V = (v_1(r), v_2(r))$ and $v_1 = \ln \mathfrak{M}_f(r)$, $v_2 = \ln r_2$, $r = (r_1, r_2)$. So,

$$J_2 := \frac{D(v_1, v_2)}{D(r_1, r_2)} = \begin{vmatrix} \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) & \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) \\ 0 & \frac{1}{r_2} \end{vmatrix} = \frac{1}{r_2} \cdot \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r).$$

Therefore

$$\nu_{\ln}(E_2 \cap \Delta_{r^0}) \leq \iint_{E_2 \cap \Delta_{r^0}} \frac{\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) dr_1 dr_2}{h(\ln \mathfrak{M}_f(r), \ln r_2) r_2} = \iint_{V(E_2 \cap \Delta_{r^0})} \frac{du_1 du_2}{h(u_1, u_2)} \leq \int_1^{+\infty} \int_1^{+\infty} \frac{du_1 du_2}{h(u_1 u_2)} < +\infty.$$

Let $E_3 \subset T$ be a set for which inequality (5) does not hold. Then

$$\nu_{\ln}(E_3 \cap \Delta_{r^0}) \leq \iint_{E_3 \cap \Delta_{r^0}} \frac{\frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) r_2 dr_1 dr_2}{\frac{1}{(1-r_1)^\delta} \ln^{1+\delta} \mathfrak{M}_f(r) (1-r_1) r_2}.$$

Let r_0 be such that $\ln \mathfrak{M}_f(r^0) > 1$. Define the mapping $W: T \rightarrow T$, where $W = (w_1(r), w_2(r))$ and $w_1 = r_1$, $w_2 = \ln \mathfrak{M}_f(r)$, $r = (r_1, r_2)$. So,

$$J_3 := \frac{D(w_1, w_2)}{D(r_1, r_2)} = \begin{vmatrix} 1 & 0 \\ \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) & \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) \end{vmatrix} = \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r).$$

Therefore,

$$\nu_{\ln}(E_3 \cap \Delta_{r^0}) \leq \iint_{W(E_3 \cap \Delta_{r^0})} \frac{du_1 du_2}{(1-u_1)^{1-\delta} u_2^{1+\delta}} \leq \int_0^1 \frac{du_1}{(1-u_1)^{1-\delta}} \cdot \int_1^{+\infty} \frac{du_2}{u_2^{1+\delta}} < +\infty.$$

It remains to remark that the set $E = \bigcup_{j=1}^3 E_j$ is also a set of asymptotically finite logarithmic measure in T . \square

Theorem 2. *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \setminus E$ we obtain*

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \left(\frac{\mu_f(r)}{1-r_1} \right) \cdot \ln^{1/2+\delta} r_2. \quad (6)$$

Proof. Let E' and E_0 be the exceptional sets from Theorem 1 and Lemma 1, respectively. Then for $E = E' \cup E_0$ and $h(r_1, r_2) = (r_1 r_2)^{1+\delta}$, $\delta \in (0, 1)$, we get for all $r \in T \setminus E$

$$\begin{aligned} \mathfrak{M}_f(r) &\leq C_0 \mu_f(r) (\det(D_f(r) + I))^{1/2} \leq \\ &\leq C_0 \mu_f(r) \left(\frac{1}{1-r_1} \cdot h \left(\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r), r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2 \right) \right)^{1/2} \leq \\ &\leq C_0 \mu_f(r) \left(\frac{1}{1-r_1} \left(\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) \right)^{1+\delta} \left(r_2 \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) + \ln r_2 \right)^{1+\delta} \right)^{1/2}. \end{aligned}$$

Hence by Lemma 1 there exists $r^0 \in T$ such that for all $r \in \Delta_{r^0} \setminus E$ we get

$$\begin{aligned} \mathfrak{M}_f(r) &\leq C_0 \mu_f(r) \left(\frac{1}{(1-r_1)^2} \left(\ln \mathfrak{M}_f(r) \cdot \ln r_2 \right)^{(1+\delta)^2} \left(\frac{1}{(1-r_1)^\delta} (\ln \mathfrak{M}_f(r))^{1+\delta} + \ln r_2 \right)^{1+\delta} \right)^{1/2} < \\ &< \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{(1+\delta)^2} \mathfrak{M}_f(r) \ln^{1/2+2\delta} r_2. \end{aligned} \quad (7)$$

Using inequality (7) we obtain

$$\begin{aligned} \ln \mathfrak{M}_f(r) &\leq \ln \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} + (1+\delta)^2 \ln \ln \mathfrak{M}_f(r) + \left(\frac{1}{2} + 2\delta \right) \ln \ln r_2 \leq \\ &\leq \ln \frac{\mu_f^{1+\delta}(r)}{(1-r_1)^{1+\delta}} + 8 \ln \ln \mathfrak{M}_f(r). \end{aligned}$$

Therefore, $\ln \mathfrak{M}_f(r) \leq 2 \ln \frac{\mu_f(r)}{1-r_1}$. Finally for all $r \in \Delta_{r^0} \setminus E$ we have

$$\begin{aligned} M_f(r) &\leq \mathfrak{M}_f(r) \leq \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \left(2 \ln \frac{\mu_f(r)}{1-r_1} \right)^{1+2\delta+\delta^2} \ln^{1/2+2\delta} r_2 < \\ &< \frac{\mu_f(r)}{(1-r_1)^{1+\delta_1}} \left(\ln \frac{\mu_f(r)}{1-r_1} \right)^{1+\delta_1} \ln^{1/2+\delta_1} r_2, \end{aligned}$$

where $\delta_1 > 2(\delta + \delta^2)$. □

3. Some examples. From Theorem 2 we conclude, that for every $\delta > 0$ the set

$$E = \left\{ r \in T : M_f(r) > \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \left(\frac{\mu_f(r)}{1-r_1} \right) \ln^{1/2+\delta} r_2 \right\}$$

has asymptotically finite logarithmic measure.

Remark that exponent 1 by $\frac{1}{1-r_1}$ in inequality (6) cannot be replaced with a smaller number. We suppose that one can replace this exponent with $\varepsilon \in (0, 1)$. Consider the function

$$f(z) = f(z_1, z_2) = \frac{\gamma(z_2)}{1-z_1},$$

where $\gamma(z_2)$ is an entire function such that $\ln \mu_\gamma(r_2) < r_2$, $r_2 \rightarrow +\infty$. Then

$$M_f(r) = \frac{M_\gamma(r_2)}{1-r_1}, \quad \mu_f(r) = \mu_\gamma(r_2).$$

Denote $r'(\varepsilon) \in (0, 1)$ such that for all $r_1 > r'(\varepsilon)$ we have

$$\frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} > e.$$

Thus,

$$\begin{aligned} B' &= \left\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) : \frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} > r_2^2 \right\} \subset \\ &\subset \left\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) : \frac{1}{1-r_1} > \frac{1}{(1-r_1)^\varepsilon} \left(\ln \frac{1}{1-r_1} + r_2 \right) \sqrt{\ln r_2} \right\} \subset \end{aligned}$$

$$\begin{aligned} & \subset \left\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) : \frac{\mu_\gamma(r_2)}{1-r_1} > \frac{\mu_\gamma(r_2)}{(1-r_1)^\varepsilon} \ln \frac{\mu_\gamma(r_2)}{1-r_1} \sqrt{\ln r_2} \right\} \subset \\ & \subset \left\{ r \in [r'(\varepsilon), 1) \times [e, +\infty) : M_f(r_1, r_2) > \frac{\mu_f(r_1, r_2)}{(1-r_1)^\varepsilon} \ln \frac{\mu_f(r_1, r_2)}{1-r_1} \sqrt{\ln r_2} \right\} = B \end{aligned}$$

Denote $R = \sqrt{\frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1}}$. Then for ν_{\ln} -measure of the set B is estimated as follows

$$\begin{aligned} \nu_{\ln}(B) &= \iint_B \frac{dr_1 dr_2}{(1-r_1)r_2} \geq \iint_{B'} \frac{dr_1 dr_2}{(1-r_1)r_2} \geq \int_{r'(\varepsilon)}^1 \left(\int_1^R \frac{dr_2}{r_2} \right) \frac{dr_1}{1-r_1} \geq \\ &= \frac{1}{2} \int_{r'(\varepsilon)}^1 \ln \left(\frac{1}{(1-r_1)^{1-\varepsilon}} - \ln^2 \frac{1}{1-r_1} \right) \frac{dr_1}{1-r_1} \geq \frac{1}{2} \int_{r'(\varepsilon)}^1 \frac{dr_1}{1-r_1} = +\infty. \end{aligned}$$

We remark, that none of the exponents 1 in (6) can be replaced with a smaller number. Let us consider the function

$$f(z_1, z_2) = \sum_{n=0}^{+\infty} \frac{z_1^n}{n!} \sum_{m=1}^{+\infty} e^{m\varepsilon} z_2^m = \varphi(z_1) \cdot \psi(z_2), \quad \varepsilon \in (0, 1).$$

Remark that for this function there exists $r_0 \in T$ such that for some $\delta > 0$ we have

$$\begin{aligned} & \left\{ r \in \Delta_{r^0} : M_f(r) > \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \left(\frac{\mu_f(r)}{1-r_1} \right) \ln^{1/2+\delta} r_2 \right\} \subset \\ & \subset \left\{ r \in \Delta_{r^0} : M_f(r) > \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+2\delta} \left(\frac{\mu_f(r)}{1-r_1} \right) \right\}. \end{aligned}$$

For the analytic function $\varphi(z_1)$ there exist constants $C'_1(\varepsilon) \in (0, 1)$ and $r_1 \in (r_1^0, 1)$ such that for all $r_1 \geq r_1^0$ we get

$$C'_1(\varepsilon) \frac{\mu_\varphi(r_1)}{1-r_1} \leq \frac{M_\varphi(r_1)}{\sqrt{\ln M_\varphi(r_1)}} \leq \frac{1}{C'_1(\varepsilon)} \frac{\mu_\varphi(r_1)}{1-r_1}. \quad (8)$$

It follows from inequality (8), that for $r_1 \geq r_1^0$ and some constant $C_1(\varepsilon) < C'_1(\varepsilon)$ we obtain

$$M_\varphi(r_1) \geq C_1(\varepsilon) \frac{\mu_\varphi(r_1)}{1-r_1} \ln^{1/2} \frac{\mu_\varphi(r_1)}{1-r_1}, \quad (9)$$

and for the entire function $\psi(z_2)$ we obtain for every $\varepsilon > 0$ and $r_2 \in (r_2^0, +\infty)$

$$M_\psi(r_2) \geq (\sqrt{2\pi} - \varepsilon) \mu_\psi(r_2) \ln^{\frac{1}{2}} \mu_\psi(r_2). \quad (10)$$

So, $M_f(r_1, r_2) = M_\varphi(r_1)M_\psi(r_2)$ and for $r \in (r_1^0, 1) \times (r_2^0, +\infty)$ we get

$$M_f(r) \geq (\sqrt{2\pi} - \varepsilon) C_1(\varepsilon) \frac{\mu_f(r)}{1-r_1} \left(\ln \frac{\mu_\varphi(r_1)}{1-r_1} \cdot \ln \mu_\psi(r_2) \right)^{\frac{1}{2}}. \quad (11)$$

Consider the positive increasing functions $g_1(r_1) = \ln(\mu_\varphi(r_1)/(1-r_1))$ and $g_2(r_2) = \ln \mu_\psi(r_2)$. We define

$$A = \left\{ r \in T : g_1(r_1) \cdot g_2(r_2) > \frac{1}{18}(g_1(r_1) + g_2(r_2))^2 \right\} \supset \left\{ r \in T : \frac{1}{2} < \frac{g_1(r_1)}{g_2(r_2)} < 2 \right\} = E^*.$$

Then, for $r \in A$ we get

$$g_1(r_1)g_2(r_2) \geq \frac{g_2^2(r_2)}{2} = \frac{1}{18}(g_2(r_2) + 2g_2(r_2))^2 \geq \frac{1}{18}(g_1(r_1) + g_2(r_2))^2.$$

Moreover, there exists the inverse function $g_2^{-1} : \mathbb{R}_+ \rightarrow (r_0, 1)$, which is also increasing.

For $r_1 \in (r_1^0, 1)$ we define r_1^* and r_2^* such that

$$r_1^* = g_2^{-1}\left(\frac{g_1(r_1)}{2}\right), \quad r_2^* = g_2^{-1}(2g_1(r_1)).$$

Therefore, from inequality (11) we have for all $r \in E^*$

$$M_f(r) \geq \frac{(\sqrt{2\pi} - \varepsilon)C_1(\varepsilon)}{18} \frac{\mu_f(r)}{1-r_1} \left(\ln \frac{\mu_\varphi(r_1)}{1-r_1} + \ln \mu_\psi(r_2) \right) = \frac{(\sqrt{2\pi} - \varepsilon)C_1(\varepsilon)}{18} \frac{\mu_f(r)}{1-r_1} \ln \frac{\mu_f(r)}{1-r_1}.$$

It remains to prove that set E^* is a set of infinite asymptotically logarithmic measure.

$$\begin{aligned} \mu(E^*) &= \iint_{\Delta_{r_1^0} \cap E^*} \frac{dr_1 dr_2}{(1-r_1)r_2} = \int_{r_0^1}^1 \int_{r_1^*}^{r_2^*} \frac{dr_1 dr_2}{(1-r_1)r_2} = \\ &= \int_{r_0^1}^1 (\ln r_2^* - \ln r_1^*) \frac{dr_1}{1-r_1} = \int_{r_0^1}^1 \left(\ln g_2^{-1}(2g_1(r_1)) - \ln g_2^{-1}\left(\frac{g_1(r_1)}{2}\right) \right) \frac{dr_1}{1-r_1}. \end{aligned}$$

Since $\lim_{r \rightarrow +\infty} \frac{M_\psi(r)}{\mu_\psi(r) \ln^{1/2} \mu_\psi(r)} = \sqrt{2\pi}$ and $g_2(r) = \ln \mu_\psi(r)$, we get

$$r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln g_2(r) < g_2(r) < r - \ln(\sqrt{2\pi} - \varepsilon) - \frac{1}{2} \ln g_2(r), \quad r \rightarrow +\infty. \quad (12)$$

From (12) we have

$$g_2^{-1}(r) > r, \quad (13)$$

$$g_2(r) > r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln g_2(r) > r - \ln(\sqrt{2\pi} + \varepsilon) - \frac{1}{2} \ln r > \frac{2}{3}r,$$

$$g_2^{-1}(r) < \frac{3}{2}r, \quad r \rightarrow +\infty. \quad (14)$$

Using (13) and (14) we finally obtain

$$\nu_{\ln}(E^*) \geq \int_{r_0^1}^1 \left(\ln(2g_1(r_1)) - \ln\left(\frac{1}{2} \cdot \frac{3}{2}g_1(r_1)\right) \right) \frac{dr_1}{1-r_1} = \int_{r_0^1}^1 \ln \frac{8}{3} \cdot \frac{dr_1}{1-r_1} = +\infty.$$

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