УДК 512.53+515.12

T. Banakh, I. Pastukhova

TOPOLOGICAL AND DITOPOLOGICAL UNOSEMIGROUP

T. Banakh, I. Pastukhova. *Topological and ditopological unosemigroups*, Mat. Stud. **39** (2013), 119–133.

In this paper we introduce and study a new topologo-algebraic structure called a (di)topological unosemigroup. This is a topological semigroup endowed with continuous unary operations of left and right units (which have certain continuous division property called the dicontinuity). We show that the class of ditopological unosemigroups contains all topological groups, all topological semilattices, all uniformizable topological unosemigroups, all compact topological unosemigroups, and is closed under the operations of taking subunosemigroups, Tychonoff product, reduced product, semidirect product, and the Hartman-Mycielski extension.

Т. О. Банах, І. С. Пастухова. *Топологические и дитопологические унополугруппы* // Мат. Студiї. – 2013. – Т.39, №2. – C.119–133.

Вводится и изучается новая топологическая структура — (ди)топологическая унополугруппа. Это топологическая полугруппа с непрерывными операциями левой и правой единицы (имеющими некоторое свойство неперерывной делимости, которая называется динеперерывность). Показано, что класс дитопологичних унополугрупп содержит все топологические группы, топологические полурешетки и равномерные топологические унополугруппы, все компактные унополугруппы, и является замкнутым относительно операций тихоновского произведения, полупрямого и приведенного произведений, а также наследуется подунополугруппами и расширениями Хартмана-Мыцельського.

1. Introduction. In this paper we shall introduce and study a new topologo-algebraic structure called a [di]topological (left, right) unosemigroup. This is a topological semigroup endowed with continuous unary operations of (left, right) units [having certain continuous division property called the dicontinuity]. Introducing topological unosemigroups was motivated by the problem of generalization of Hryniv's Embedding Theorems ([7] on embeddings of Clifford compact topological inverse semigroups into products of cones over compact topological groups) beyond the class of compact topological inverse semigroups.

Topological and ditopological (left, right) unosemigroups will be introduced in Sections 2 and 3. In Sections 4 and 5 we shall study the class of ditopological unosemigroups and shall show that it contains all topological groups, all topological semilattices, and all compact Hausdorff topological unosemigroups. In Section 6 we shall introduce and study some operations over topological (left, right) unosemigroups.

2. Topological unosemigroups. In this section we shall introduce the notion of a topological (left, right) unosemigroup.

Let S be a semigroup (i.e., a non-empty set S endowed with an associative binary operation $\cdot\colon S\times S\to S$). A unary operation $\lambda\colon S\to S$ is called a *left unit operation* on S if

2010 *Mathematics Subject Classification:* 22A15.
Keywords: topological semigroup; unosemigroup; ditopological unosemigroup.

$\lambda(x) \cdot x = x$ for all $x \in S$. A semigroup S endowed with a left unit operation $\lambda\colon S \to S$ is called a *left unosemigroup*.

By analogy we can introduce right unosemigroups. Namely, a unary operation $\rho\colon S \to S$ on a semigroup S is called a *right unit operation* on S if $x \cdot \rho(x) = x$ for all $x \in S$. A *right unosemigroup* is a semigroup S endowed with a right unit operation $\rho\colon S \to S$.

A *unosemigroup* is a semigroup S endowed with a left unit operation $\lambda\colon S \to S$ and a right unit operation $\rho\colon S \to S$.

It should be mentioned that the class of left (resp. right) unosemigroups contains the class of domain (resp. range) semigroups considered in [5].

Now we introduce topological versions of these notions.

A *topological semigroup* is a topological space S endowed with a continuous associative binary operation $\cdot\colon S \times S \to S$. A *topological left unosemigroup* is a topological semigroup S endowed with a continuous left unit operation $\lambda\colon S \to S$. A *topological right unosemigroup* is a topological semigroup S endowed with a continuous right unit operation $\rho\colon S \to S$. A *topological unosemigroup* is a topological semigroup S endowed with a continuous left unit operation $\lambda\colon S \to S$ and a continuous right unit operation $\rho\colon S \to S$.

Topological (left, right) unosemigroups are objects of the category whose morphisms are continuous (left, right) unohomomorphisms. A function $h\colon X \to Y$ between two topological left unosemigroups (X, λ_X) and (Y, λ_Y) is called a *left unohomomorphism* if h is a semigroup homomorphism and $h(\lambda_X(x)) = \lambda_Y(h(x))$ for all $x \in X$. By analogy we can define *right unohomomorphisms* between topological right unosemigroups and *unohomomorphisms* between topological unosemigroups.

3. Ditopological unosemigroups. To introduce **di**topological (left, right) unosemigroups, we first introduce two kinds of **di**vision operations on each semigroup. Namely, for two points a, b of a semigroup S consider the sets

$$a \leftthreetimes b = \{x \in S\colon ax = b\} \quad \text{and} \quad b \rightthreetimes a = \{x \in S\colon xa = b\}$$

which can be considered as the results of left and right division of b by a. Respectively, for two subsets $A, B \subset S$ the sets

$$A \leftthreetimes B = \bigcup_{(a,b)\in A\times B} a \leftthreetimes b \quad \text{and} \quad B \rightthreetimes A = \bigcup_{(a,b)\in A\times B} b \rightthreetimes a$$

can be considered as the results of left and right division of B by A.

A continuous left unit operation $\lambda\colon S \to S$ on a topological semigroup S is called *dicontinuous* at a point $x \in S$ if for every neighborhood $O_x \subset S$ of x there are neighborhoods $U_x \subset S$ and $W_{\lambda(x)} \subset \lambda(S)$ of the points x and $\lambda(x)$ in S and $\lambda(S)$ respectively, such that $(W_{\lambda(x)} \leftthreetimes U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset O_x$. A left unit operation $\lambda\colon S \to S$ is defined to be *dicontinuous* if it is dicontinuous at each point $x \in X$.

A *ditopological left unosemigroup* is a topological semigroup S endowed with a dicontinuous left unit operation $\lambda\colon S \to S$.

A trivial (but important) example of a ditopological left unosemigroup is an idempotent topological left unosemigroup. A (topological) left unosemigroup (S, λ) is called *idempotent* if $\lambda(x) = x$ for all $x \in S$. In this case $x = \lambda(x) \cdot x = xx$, which means that S is an idempotent semigroup.

Proposition 1. *Each idempotent topological left unosemigroup (S, λ) is ditopological.*

Proof. Given a point $x \in S$ and a neighborhood $O_x \subset S$ of x put $U_x = W_{\lambda(x)} = O_x$ and observe that $(W_{\lambda(x)} \leftthreetimes U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset W_{\lambda(x)} = O_x$, which means that the left unit operation λ is dicontinuous at x and hence the topological left unosemigroup (S, λ) is dicontinuous. □

By analogy we can introduce the right and two-sided versions of these notions.

Namely, a continuous right unit operation $\rho\colon S \to S$ on a topological semigroup S is called *dicontinuous* if for every point $x \in S$ and every neighborhood $O_x \subset S$ of x there are neighborhoods $U_x \subset S$ and $W_{\rho(x)} \subset \rho(S)$ of x and $\rho(x)$ such that $(U_x \rightthreetimes W_{\rho(x)}) \cap \rho^{-1}(W_{\rho(x)}) \subset O_x$. A *ditopological right unosemigroup* is a topological semigroup S endowed with a dicontinuous right unit operation $\rho\colon S \to S$.

A (topological) right unosemigroup (S, ρ) is called *idempotent* if $\rho(x) = x$ for all $x \in X$. By analogy with Proposition 1 we can prove

Proposition 2. *Each idempotent topological right unosemigroup (S, λ) is ditopological.*

A *ditopological unosemigroup* is a topological semigroup S endowed with a dicontinuous left unit operation $\lambda\colon S \to S$ and dicontinuous right unit operation $\rho\colon S \to S$. So, a ditopological unosemigroup carries the structures of a ditopological left and right unosemigroups.

A (topological) unosemigroup (S, λ, ρ) is called *idempotent* if $\lambda(x) = \rho(x) = x$ for all $x \in X$. Propositions 1 and 2 imply that each idempotent topological unosemigroup is ditopological.

4. Uniformizable topological unosemigroups. In this section we shall prove that the dicontinuity of a continuous left (right) unit operation on a topological semigroup automatically follows from the right (left) uniformizability of the topological semigroup.

We define a topological semigroup S to be *left-uniformizable* (resp. *right-uniformizable*) if the topology of X is generated by a uniformity $\mathcal{U}$ such that for every entourage $U \in \mathcal{U}$ there is an entourage $V \in U$ such that $x \cdot B(y, V) \subset B(xy, U)$ (resp. $B(x, V) \cdot y \subset B(xy, U)$ for all points $x, y \in S$. Here $B(x, V) = \{z \in S\colon (x, z) \in V\}$ stands for the V-ball centered at a point $x \in S$.

Each topological group G is left (resp. right) uniformizable by its left (resp. right) uniformity generated by the base consisting of the entourages $\{(x, y) \in G \times G\colon y \in xU\}$ (resp. $\{(x, y) \in G \times G\colon y \in Ux\}$) where $U = U^{-1}$ runs over symmetric neighborhoods of the idempotent in G.

Theorem 1. *Each continuous left unit operation $\lambda\colon S \to S$ on a right-uniformizable topological semigroup S is dicontinuous. Consequently, each right-uniformizable topological left unosemigroup (S, λ) is a ditopological left unosemigroup.*

Proof. Let $\mathcal{U}$ be a uniformity on S witnessing that the semigroup S is right-uniformizable. To show that the left unit operation $\lambda\colon S \to S$ is dicontinuous, fix a point $x \in S$ and a neighborhood $O_x \subset S$ of x. Since $\mathcal{U}$ generates the topology of S, the neighborhood O_x contains the ball $B(x, U) = \{y \in S\colon (x, y) \in U\}$ for some entourage $U \in \mathcal{U}$. Find an entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$ where

$$V \circ V \circ V = \{(x, y) \in S \times S\colon \exists u, v \in S\ (x, u), (u, v), (v, y) \in V\}.$$

Since S is right-uniformizable by $\mathcal{U}$, for the entourage V there is an entourage $W \in \mathcal{U}$ such that $W \subset V$ and $B(s, W) \cdot t \subset B(st, W)$ for all points $s, t \in S$.

We claim that the neighborhoods $U_x = B(x, W)$ and $W_{\lambda(x)} = B(\lambda(x), W) \cap \lambda(S)$ witness that the left unit operation λ is dicontinuous x. Indeed, take any point $y \in (W_{\lambda(x)} \setminus U_x) \cap \lambda^{-1}(W_{\lambda(x)})$. Then $\lambda(y) \in W_{\lambda(x)}$ and $wy \in U_x$ for some $w \in W_{\lambda(x)}$. It follows from $w, \lambda(y) \in W_{\lambda(y)} = B(\lambda(x), W)$ that

$$\{wy, y\} = \{wy, \lambda(y)y\} \subset B(\lambda(x), W) \cdot y \subset B(\lambda(x)y, V)$$

and hence $(wy, y) \in V \circ V$, which implies

$$y \in B(wy, V \circ V) \subset B(U_x, V \circ V) = B(B(x, W), V \circ V) \subset B(x, V \circ V \circ V) \subset B(x, U) \subset O_x.$$

$\square$

By analogy we can prove

Theorem 2. *Each continuous right unit operation $\lambda\colon S \to S$ on a left-uniformizable topological semigroup S is dicontinuous. Consequently, each left-uniformizable topological right unosemigroup (S, λ) is a ditopological right unosemigroup.*

Theorems 1 and 2 imply

Theorem 3. *A topological unosemigroup (S, λ, ρ) is ditopological if the topological semigroup S is both left-uniformizable and right-uniformizable.*

Each compact Hausdorff space X carries a unique uniformity generating its topology. The uniform continuity of continuous maps defined on a compact Hausdorff space implies the following assertion.

Corollary 1. *Each compact Hausdorff topological (left, right) unosemigroup is ditopological.*

Since discrete topological semigroups are (left and right) uniformizable by the discrete uniformity, Theorem 3 implies

Corollary 2. *Each discrete topological (left, right) unosemigroup is ditopological.*

5. Ditopological inverse semigroups. Continuous left and right unit operations appear naturally in topological inverse semigroups. Let us recall that a semigroup S is called *inverse* if for each element $x \in S$ there is a unique element $x^{-1} \in S$ (called *the inverse to* x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. By a *topological inverse semigroup* we understand an inverse semigroup S endowed with a topology such that the semigroup operation $\cdot\colon S \times S \to S$ and the operation of inversion $(\,)^{-1}\colon S \to S$ are continuous.

Each topological inverse semigroup S possesses the canonical continuous left unit operation $\lambda\colon S \to S$, $\lambda\colon x \mapsto xx^{-1}$, and the canonical continuous right unit operation $\rho\colon S \to S$, $\rho\colon x \mapsto x^{-1}x$, which turn S into a topological unosemigroup.

Observe that the sets $\lambda(S) = \{xx^{-1}\colon x \in S\}$ and $\rho(S) = \{x^{-1}x\colon x \in S\}$ coincide with the set $E(S) = \{x \in S\colon xx = x\}$ of idempotents of S. It is well-known that the set $E(S)$ is a commutative subsemigroup of the inverse semigroup S.

A topological inverse semigroup S is called a *ditopological inverse semigroup* if the topological unosemigroup (S, λ, ρ) is ditopological.

The following proposition shows that the dicontinuity of the left unit operation on a topological inverse semigroup is equivalent to the dicontinuity of the right unit operation.

Proposition 3. *Let S be a topological inverse semigroup. The left unit operation $\lambda\colon S \to S$, $\lambda\colon x \mapsto xx^{-1}$, is dicontinuous at a point $x \in S$ if and only if the right unit operation $\rho\colon S \to S$, $\rho\colon x \mapsto x^{-1}x$, is dicontinuous at the point $x^{-1} \in S$.*

Proof. Assume that the left unit operation λ is dicontinuous at a point $x \in X$. To show that the right unit operation ρ is dicontinuous at the point x^{-1}, fix a neighborhood $O_{x^{-1}} \subset S$ of x^{-1}. Then $O^{-1}_{x^{-1}} = \{y^{-1}\colon y \in O_{x^{-1}}\}$ is a neighborhood of the point x in S. By the dicontinuity of the left unit operation λ on S, the points x and $\lambda(x) = xx^{-1}$ have neighborhoods $U_x \subset S$ and $W_{xx^{-1}} \subset E(S)$ such that $(W_{xx^{-1}} \searrow U_x) \cap \lambda^{-1}(W_{xx^{-1}}) \subset O^{-1}_{x^{-1}}$. After the inversion the latter inclusion turns into $(U_x^{-1} \swarrow W^{-1}_{xx^{-1}}) \cap \rho^{-1}(W_{xx^{-1}}) \subset O_{x^{-1}}$. So, $U_{x^{-1}} = U_x^{-1}$ and $W_{\rho(x^{-1})} = W^{-1}_{xx^{-1}} \cap W_{xx^{-1}}$ are neighborhoods of the points x^{-1} and $\rho(x^{-1}) = xx^{-1}$ witnessing the dicontinuity of the right unit operation ρ at x^{-1}.

By analogy we can prove that the dicontinuity of the right unit operation ρ at $x^{-1} \in S$ implies the dicontinuity of the left unit operation λ at the point x. □

By a *topological semilattice* we understand a commutative idempotent topological semigroup S. Each topological semilattice is a topological inverse semigroup with $x^{-1} = x$ for all $x \in S$.

Theorem 4. *The class of ditopological inverse semigroups contains all topological groups, all topological semilattices, all compact Hausdorff topological inverse semigroups, and all discrete topological inverse semigroups.*

Proof. Let G be a topological group. In this case λ and ρ are constant operations assigning to each $x \in G$ the unique idempotent e of the group G, so $\lambda(G) = \rho(G) = \{e\}$ is a singleton. By Proposition 3, to prove that G is a ditopological inverse semigroup, it suffices to check that the left unit operation λ is dicontinuous at each point $x \in G$. Given any neighborhood $O_x \subset G$ of x, put $U_x = O_x$ and $W_{\lambda(x)} = \{e\}$ and observe that

$$(W_{\lambda(x)} \searrow U_x) \cap \lambda^{-1}(W_{\lambda(x)}) = (\{e\} \searrow U_x) \cap G = U_x = O_x,$$

which means that the left unit operation λ is discontinuous at x. So, the topological group G is ditopological. The same conclusion can be also derived from the right-uniformizabity of topological groups and Theorem 1.

Each topological semilattice, being an idempotent topological unosemigroup, is a ditopological unosemigroup according to Propositions 1 and 2. Corollaries 1 and 2 imply that the class of ditopological inverse semigroups contains all compact Hausdorff topological inverse semigroups and all discrete topological inverse semigroups. □

Now we present a simple example of a locally compact commutative topological inverse semigroup S which is not ditopological.

Example 1. There exists a commutative topological inverse semigroup S such that

1) S is countable, metrizable, and locally compact;
2) the idempotent semilattice $E(S)$ of S is compact and each subgroup of S has cardinality ≤ 2;
3) the left unit operation $\lambda\colon S \to S$, $\lambda\colon x \mapsto xx^{-1}$, is not dicontinuous, which implies that the topological inverse semigroup S is not ditopological.

Proof. Let H be a two-element group and $T = \{0\} \cup \{\frac{1}{n}\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a convergent sequence endowed with the semilattice operation

$$xy = \begin{cases} x, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Then the product $T \times H$ is a commutative inverse semigroup whose idempotent semilattice coincides with the set $E = T \times \{e\}$ where e is the idempotent of the group H. Let h be the non-identity element of the two-element group H.

Endow the inverse semigroup $S = T \times H$ with the topology τ which induces the original (compact metrizable) topology on the set $T \times \{e\}$ and the discrete topology on $T \times \{h\}$. It is easy to see that the topology τ is metrizable, locally compact, and turns S into a topological inverse semigroup.

We claim that the left unit operation $\lambda\colon S \to S$, $\lambda\colon x \mapsto xx^{-1}$, is not dicontinuous at the point $x = (0, h) \in T \times H$. Assuming the opposite, for the neighborhood $O_x = \{x\}$ we would find neighborhoods $U_x \subset S$ of x and $W_{\lambda(x)} \subset \lambda(S)$ of the idempotent $\lambda(x) = (0, e)$ such that $(W_{\lambda(x)} \diagdown U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset O_x = \{x\}$. By the definition of the topology on S, the neighborhood $W_{\lambda(x)}$ contains a point $(\frac{1}{n}, e)$ for some $n \in \mathbb{N}$. It follows from $(0, e) \cdot (\frac{1}{n}, h) = (0, h) = x \in U_x$ and $\lambda\big((\frac{1}{n}, h)\big)) = (\frac{1}{n}, e) \in W_{\lambda(x)}$ that

$$(\tfrac{1}{n}, h) \in (W_{\lambda(x)} \diagdown U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset \{x\} = \{(0, h)\}$$

and hence $\frac{1}{n} = 0$, which is a desired contradiction. □

6. Operations over ditopological unosemigroups. As we know from Theorem 4, the class of ditopological unosemigroups contains all topological groups, all topological semilattices, and all compact Hausdorff topological inverse semigroups. In this section we shall show that this class is stable under many natural operations over topological unosemigroups.

6.1. Subunosemigroups of topological (left, right) unosemigroups. Let (S, λ) be a topological left unosemigroup and $X \subset S$ be a subsemigroup such that $\lambda(X) \subset X$. Then $\lambda|X\colon X \to X$ is a continuous left unit operation on X, which implies that $(X, \lambda|X)$ is a topological left unosemigroup. Such a left unosemigroup will be called a *left subunosemigroup* of (S, λ).

By analogy we can define *right subunosemigroups* of topological right unosemigroups and *subunosemigroups* of topological unosemigroups.

Since the dicontinuity of a left or right unit operation $u\colon S \to S$ on a topological semigroup S implies the dicontinuity of the restriction $u|X$ to any subsemigroup $X \subset S$ with $u(X) \subset X$, we obtain the following result.

Proposition 4. *If a topological (left,right) unosemigroup S is ditopological, then so is any its subunosemigroup $X \subset S$.*

6.2. Tychonoff products of topological (left, right) unosemigroups. For any family of topological left unosemigroups $(S_\alpha, \lambda_\alpha)$, $\alpha \in A$, the Tychonoff product $S = \prod_{\alpha\in A} S_\alpha$ carries the natural structure of a topological left unosemigroup endowed with the left unit operation $\lambda\colon S \to S$, $\lambda\colon (x_\alpha)_{\alpha\in A} \mapsto \big(\lambda_\alpha(x_\alpha)\big)_{\alpha\in A}$. By analogy we can define the operation of the Tychonoff product of (right) unosemigroups.

The dicontinuity of left (right) unit operations on the topological semigroups S_α, $\alpha \in A$, implies the dicontinuity of the left (right) unit operation on their Tychonoff product. This proves our next proposition.

Proposition 5. *The Tychonoff product of ditopological (left, right) unosemigroups is a ditopological (left, right) unosemigroup.*

6.3. The reduced product of topological (left, right) unosemigroups. Let X, Y be topological semigroups and $I \subset X$ be a closed two-sided ideal in X. By the *reduced product* $X \times_I Y$ of the semigroups X and Y over the ideal I we mean the set $I \cup ((X \setminus I) \times Y)$ endowed with the smallest topology such that

- the map $(X \setminus I) \times Y \hookrightarrow X \times_I Y$ is a topological embedding;
- the projection $\pi\colon X \times_I Y \to X$ is continuous.

Here

$$\pi(z) = \begin{cases} z, & \text{if } z \in I; \\ x, & \text{if } z = (x, y) \in (X \setminus I) \times Y, \end{cases}$$

for any $z \in X \times_I Y$.

The semigroup operation on $X \times_I Y$ is defined as a unique binary operation on $X \times_I Y$ such that the projection $q\colon X \times Y \to X \times_I Y$ defined by

$$q(x, y) = \begin{cases} x, & \text{if } x \in I; \\ (x, y), & \text{otherwise} \end{cases}$$

is a semigroup homomorphism.

If the semigroups X and Y carry continuous left unit operations $\lambda_X\colon X \to X$ and $\lambda_Y\colon Y \to Y$ such that $\lambda_X(I) \subset I$, then the reduced product $X \times_I Y$ carries a natural left unit operation $\lambda\colon X \times_I Y \to X \times_I Y$ defined by the formula

$$\lambda(z) = \begin{cases} \lambda_X(z), & \text{if } z \in I; \\ (\lambda_X(x), \lambda_Y(y)), & \text{if } z = (x, y) \in (X \setminus I) \times Y. \end{cases}$$

This formula is well-defined since for every $x \in X \setminus I$ the equality $x = \lambda_X(x) \cdot x \notin I$ implies that $\lambda_X(x) \notin I$.

The semigroup $X \times_I Y$ endowed with the left unit operation λ is a topological left unosemigroup called the *reduced product* of the topological left unosemigroups (X, λ_X) and (Y, λ_Y).

Theorem 5. *If the topological left unosemigroups* $\mathbf{X} = (X, \lambda_X)$ *and* $\mathbf{Y} = (Y, \lambda_Y)$ *are ditopological, then so is their reduced product* $\mathbf{X} \times_I \mathbf{Y} = (X \times_I Y, \lambda)$.

Proof. Assume that the topological left unosemigroups $\mathbf{X} = (X, \lambda_X)$ and $\mathbf{Y} = (Y, \lambda_Y)$ are ditopological. To show that $X \times_I Y$ is a ditopological left unosemigroup, fix a point $z \in X \times_I Y$ and a neighborhood $O_z \subset X \times_I Y$ of z. We divide the proof into two parts.

First assume that $z = (x, y) \in (X \setminus I) \times Y$. In this case we can assume that $O_z = O_x \times O_y$ for some open neighborhoods $O_x \subset X \setminus I$ and $O_y \subset Y$ of the points $x \in X \setminus I$ and $y \in Y$, respectively. Since X and Y are left ditopological unosemigroups, there are open neighborhoods $U_x \subset X$, $W_{\lambda_X(x)} \subset \lambda_X(X)$ of the points x, $\lambda_X(x)$ and $U_y \subset Y$, $W_{\lambda_Y(y)} \subset \lambda_Y(Y)$ of y, $\lambda_Y(y)$ such that

$$(W_{\lambda_X(x)} \leftthreetimes U_x) \cap \lambda_X^{-1}(W_{\lambda_X(x)}) \subset O_x \quad \text{and} \quad (W_{\lambda_Y(y)} \leftthreetimes U_y) \cap \lambda_Y^{-1}(W_{\lambda_Y(y)}) \subset O_y.$$

It follows from $\lambda_X(x) \cdot x = x \notin I$ that $\lambda_X(x) \notin I$. So, we can assume that the sets U_x and $W_{\lambda_X(x)}$ are contained in $X \setminus I$.

We claim that the open sets $U_z = U_x \times U_y$ and $W_{\lambda(z)} = W_{\lambda_X(x)} \times W_{\lambda_Y(y)}$ witness that the left unit operation λ on $X \times_I Y$ is dicontinuous at the point z. Given any point $u \in X \times_I Y$ such that $u \in (W_{\lambda(z)} \setminus U_z) \cap \lambda^{-1}(W_{\lambda(z)})$, we need to show that $u \in O_z$. It follows that $\lambda(u) \in W_{\lambda(z)}$ and $wu \in U_z$ for some $w \in W_{\lambda(z)}$, which implies that $w, u \notin I$ and hence $w = (x_w, y_w)$ and $u = (x_u, y_u)$ for some points $x_w, x_u \in X \setminus I$ and $y_w, y_u \in Y$. Since $wu = (x_w x_u, y_w y_u) \in U_z = U_x \times U_y$ and $\lambda(u) = (\lambda_X(x_u), \lambda_Y(y_u)) \in W_{\lambda(z)} = W_{\lambda_X(x)} \times W_{\lambda_Y(y)}$, we obtain $x_w x_u \in U_x, y_w y_u \in U_y$ and $\lambda_X(x_u) \in W_{\lambda_X(x)}, \lambda_Y(y_u) \in W_{\lambda_Y(y)}$. Hence

$$x_u \in (W_{\lambda_X(x)} \setminus U_x) \cap \lambda_X^{-1}(W_{\lambda_X(x)}) \subset O_x \quad \text{and} \quad y_u \in (W_{\lambda_Y(y)} \setminus U_y) \cap \lambda_Y^{-1}(W_{\lambda_Y(y)}) \subset O_y$$

and thus $u \in O_x \times O_y = O_z$.

In the case where $z \in I$, we can assume that O_z is of the form $O_z = \pi^{-1}(O'_z)$, where O'_z is an open neighborhood of the point $z \in I$ in the left unosemigroup X. Since λ_X is left dicontinuous at z, there are open neighborhoods $U'_z \subset X$ and $W'_{\lambda_X(z)} \subset \lambda_X(X)$ of the points $z \in X$ and $\lambda_X(z) \in \lambda_X(X)$ such that $(W'_{\lambda_X(z)} \setminus U'_z) \cap \lambda_X^{-1}(W'_{\lambda_X(z)}) \subset O'_z$. The latter inclusion implies that for the open neighborhoods $U_z = \pi^{-1}(U'_z) \subset X \times_I Y$ of z and $W_{\lambda(z)} = \lambda(X \times_I Y) \cap \pi^{-1}(W'_{\lambda_X(z)}) \subset \lambda(X \times_I Y)$ of $\lambda(z)$ we get

$$(W_{\lambda(x)} \setminus U_z) \cap \lambda^{-1}(W_{\lambda(z)}) \subset \pi^{-1}\big((W'_{\lambda_X(z)} \setminus U'_z) \cap \lambda_X^{-1}(W'_{\lambda_X(z)})\big) \subset \pi^{-1}(O'_z) = O_z,$$

which witnesses that the left unit operation λ on $X \times_I Y$ is left dicontinuous at z. Thus, the reduced product $\mathbf{X} \times_I \mathbf{Y} = (X \times_I Y, \lambda)$ is a ditopological left unosemigroup. □

By analogy we can introduce the reduced product of topological right unosemigroups. Namely, if $\mathbf{X} = (X, \rho_X)$ and $\mathbf{Y} = (Y, \rho_Y)$ are topological right unosemigroups and $I \subset X$ is a closed two-sided ideal with $\rho_X(I) \subset I$, then the reduced product $X \times_I Y$ carries an induced right unit operation $\rho \colon X \times_I Y \to X \times_I Y$ defined by

$$\rho(z) = \begin{cases} \rho_X(z), & \text{if } z \in I; \\ (\rho_X(x), \rho_Y(y)), & \text{if } z = (x, y) \in (X \setminus I) \times Y. \end{cases}$$

The reduced product $X \times_I Y$ endowed with the right unit operation ρ is a topological right unosemigroup called the *reduced product* of the topological right unosemigroups (X, λ_X) and (Y, λ_Y).

By analogy with Theorem 5, we can prove

Theorem 6. *If the topological right unosemigroups* $\mathbf{X} = (X, \rho_X)$ *and* $\mathbf{Y} = (Y, \rho_Y)$ *are ditopological, then so is their reduced product* $\mathbf{X} \times_I \mathbf{Y} = (X \times_I Y, \rho)$.

The above discussion implies that for topological unosemigroups $\mathbf{X} = (X, \lambda_X, \rho_X)$ and $\mathbf{Y} = (Y, \lambda_Y, \rho_Y)$ and a closed two-sided ideal $I \subset X$ with $\lambda_X(I) \cup \rho_X(I) \subset I$, the triple $\mathbf{X} \times_I \mathbf{Y} = (X \times_I Y, \lambda, \rho)$ is a topological unosemigroup. This topological unosemigroup will be called *reduced product* of the topological unosemigroups $\mathbf{X}$ and $\mathbf{Y}$. Theorems 5 and 6 imply

Corollary 3. *If topological unosemigroups* $\mathbf{X}$ *and* $\mathbf{Y}$ *are ditopological, then so is their reduced product* $\mathbf{X} \times_I \mathbf{Y}$.

Now we present some important examples of reduced products.

Example 2. Let G be a topological group and let $\mathbf{2} = (\{0,1\}, \min)$ be a two-element semilattice endowed with the discrete topology. By Theorem 4, the semigroups G and $\mathbf{2}$ endowed with the canonical left and right unit operations are ditopological inverse semigroups and by Corollary 3, so is their reduced product $\dot{G} = \mathbf{2} \times_{\{0\}} G$ called the 0-*extension* of G.

Example 3. Let G be a topological group and let $\mathbb{I}$ be the unit interval $[0,1]$ endowed with the semilattice operation of minimum. By Theorem 4, the semigroups G and $\mathbb{I}$ endowed with the canonical left and right unit operations are ditopological inverse semigroups and by Corollary 3, so is their reduced product $\hat{G} = \mathbb{I} \times_{\{0\}} G$ called *the cone over* G.

The 0-extensions and cones of topological groups will be essentially used in the forthcoming paper [2] devoted to constructing embeddings of Clifford ditopological inverse semigroups into Tychonoff products of topological semilattices and cones over topological groups.

6.4. Semidirect products of topological unosemigroups. In this subsection we shall consider the operation of a semidirect product of topological (left, right) unosemigroups. Let us mention that semidirect products of semigroups were studied in Chapter 2 of [4].

By a *continuous action* of a topological semigroup F on a topological semigroup S we understand a continuous function $\alpha\colon F \times S \to S$ having the following two properties:

- for each $f \in F$ the function $\alpha_f\colon S \to S$, $\alpha_f\colon x \mapsto \alpha(f,x)$, is a semigroup homomorphism of S;
- $\alpha_{fg} = \alpha_f \circ \alpha_g$ for each $f, g \in F$.

The action $\alpha\colon F\times S \to S$ induces a continuous associative binary operation $(s,f)\cdot(t,g) = (s\cdot\alpha_f(t), f\cdot g)$ on the product $S\times F$. The product $S\times F$ endowed with this binary operation is denoted by $S \times_\alpha F$ and called the *semidirect product* of the topological semigroups S and F.

We shall say that the action $\alpha\colon F \times S \to S$

- respects a (left, right) unit operation $u\colon F \to F$ on F if $\alpha(u(f), s) = s$ for all $(f,s) \in F \times S$;
- respects a (left, right) unit operation $u\colon S \to S$ on S if $\alpha(f, u(s)) = u(s)$ for all $(f,s) \in F \times S$.

If (F, λ_F) and (S, λ_S) are topological left unosemigroups and a continuous action $\alpha\colon F \times S \to S$ of F on S respects the left unit operation λ_F, then the unary operation $\lambda\colon S \times_\alpha F \to S \times_\alpha F$, $\lambda\colon (s,f) \mapsto (\lambda_S(s), \lambda_F(f))$, is a continuous left unit operation on the semidirect product $S \times_\alpha F$ as

$$(\lambda_S(s), \lambda_F(f)) \cdot (s,f) = (\lambda_S(s)\cdot\alpha(\lambda_F(f), s), \lambda_F(f)\cdot f) = (\lambda_S(s)\cdot s, f) = (s,f)$$

for all $(s,f) \in S \times F$.

Therefore, $\mathbf{S} \times_\alpha \mathbf{F} = (S \times_\alpha F, \lambda)$ is a topological left unosemigroup, called the *semidirect product* of the topological left unosemigroups $\mathbf{S} = (S, \lambda_S)$ and $\mathbf{F} = (F, \lambda_F)$.

Theorem 7. *Let $\mathbf{S} = (S, \lambda_S)$ and $\mathbf{F} = (F, \lambda_F)$ be topological left unosemigroups and $\alpha\colon F \times S \to F$ be a continuous action of F on S, which respects the left unit operation λ_F of the left unosemigroup F. The topological left unosemigroup $\mathbf{S} \times_\alpha \mathbf{F}$ is ditopological if and only if the topological left unosemigroups $\mathbf{S}$ and $\mathbf{F}$ are ditopological.*

Proof. To prove the "only if" part, assume that $\mathbf{S}\times_\alpha\mathbf{F}$ is a ditopological left unosemigroup. We need to prove that the topological left unosemigroups $\mathbf{S}$ and $\mathbf{F}$ are ditopological.

To prove the dicontinuity of the left unit operation λ_S, fix any point $s\in S$ and a neighborhood O_s of s in S. Fix any point $f\in F$ and consider the neighborhood $O_{(s,f)}=O_s\times F$ of (s,f) in the topological semigroup $S\times_\alpha F$. The dicontinuity of the left unit operation λ on $S\times_\alpha F$ yields the existence of neighborhoods $U_{(s,f)}\subset S\times_\alpha F$ and $W_{\lambda(s,f)}\subset\lambda(S\times_\alpha F)=\lambda_S(S)\times\lambda_F(F)$ of (s,f) and $\lambda(s,f)=(\lambda_S(s),\lambda_F(f))$ such that $(W_{\lambda(s,f)}\setminus U_{(s,f)})\cap\lambda^{-1}(W_{\lambda(s,f)})\subset O_{(s,f)}$. Clearly, we can assume that these neighborhoods are of the form $U_{(s,f)}=U_s\times U_f$ and $W_{\lambda(s,f)}=W_{\lambda_S(s)}\times W_{\lambda_F(f)}$ for some open sets $U_s\subset S$, $W_{\lambda_S(s)}\subset\lambda_S(S)$, $U_f\subset F$, and $W_{\lambda_F(f)}\subset\lambda_F(F)$.

We claim that the neighborhoods U_s and $W_{\lambda_S(s)}$ witness that λ_S is dicontinuous at s. Let $t\in(W_{\lambda_S(s)}\setminus U_s)\cap\lambda_S^{-1}(W_{\lambda_S(s)})$. This implies $\lambda_S(t)\in W_{\lambda_S(s)}$ and $wt\in U_s$ for some $w\in W_{\lambda_S(s)}$. Then for the elements $(t,f)\in S\times_\alpha F$ and $(w,\lambda_F(f))\in W_{\lambda(s,f)}$ we get

$$(w,\lambda_F(f))\cdot(t,f)=(w\cdot\alpha_{\lambda_F(f)}(t),\lambda_F(f)\cdot f)=(wt,f)\in U_s\times U_f=U_{(s,f)},$$
$$\lambda(s,f)=(\lambda_S(s),\lambda_F(f))\in W_{\lambda_S(s)}\times W_{\lambda_F(f)}=W_{\lambda(s,f)}.$$

Here we used the fact that the action α respects the left unit operation λ_F. The choice of the neighborhoods $U_{(s,f)}$ and $W_{\lambda(s,f)}$ guarantees that $(t,f)\in O_{(s,f)}$ and hence $t\in O_s$.

To check the dicontinuity of the left unit operation $\lambda_F\colon F\to F$, take any point $f\in F$ and a neighborhood $O_f\subset F$ of f in F. Fix any element $s\in S$ and consider the neighborhood $O_{(s,f)}=S\times O_f$ of (s,f) in $S\times_\alpha F$. The dicontinuity of the left unit operation λ on $\mathbf{S}\times_\alpha\mathbf{F}$ yields neighborhoods $U_{(s,f)}\subset S\times_\alpha F$ and $W_{\lambda(s,f)}\subset\lambda(S\times_\alpha F)$ of (s,f) and $\lambda(s,f)=(\lambda_S(s),\lambda_F(f))$ such that $(W_{\lambda(s,f)}\setminus U_{(s,f)})\cap\lambda^{-1}(W_{\lambda(s,f)})\subset O_{(s,f)}$. We lose no generality assuming that $U_{(s,f)}=U_s\times U_f$ and $W_{\lambda(s,f)}=W_{\lambda_S(s)}\times W_{\lambda_F(f)}$ for some open sets $U_s\subset S$, $W_{\lambda_S(s)}\subset\lambda_S(S)$, $U_f\subset F$, and $W_{\lambda_F(f)}\subset\lambda_F(F)$.

We claim that the neighborhoods U_f and $W_{\lambda_F(f)}$ witness the dicontinuity of λ_F at the point f. Given any point $g\in(W_{\lambda_F(f)}\setminus U_f)\cap\lambda_F^{-1}(W_{\lambda_F(f)})$, observe that $\lambda_F(g)\in W_{\lambda_F(f)}$ and $wg\in U_f$ for some $w\in W_{\lambda_F(f)}\subset\lambda_F(F)$. Taking into account that the action α respects the left unit operation λ_F and $w\in\lambda_F(F)$, we conclude that $\alpha_w(s)=s$. Then for the elements (s,g) and $(\lambda_S(s),w)\in W_{\lambda_S(s)}\times W_{\lambda_F(f)}=W_{\lambda(s,f)}$ we get

$$(\lambda_S(s),w)\cdot(s,g)=(\lambda_S(s)\cdot\alpha_w(s),wg)=(\lambda_S(s)\cdot s,wg)=(s,g)\in U_{(s,f)}$$

and $\lambda(s,g)=(\lambda_S(s),\lambda_F(g))\in W_{\lambda_S(s)}\times W_{\lambda_F(f)}=W_{\lambda(s,f)}$, which means that $(s,g)\in(W_{\lambda(s,f)}\setminus U_{(s,f)})\cap\lambda^{-1}(W_{\lambda(s,f)})\subset O_{(s,f)}=S\times O_f$ and hence $g\in O_f$. This completes the proof of the "only if" part of the theorem.

To prove the "if" part, assume that the topological left unosemigroups $\mathbf{S}$ and $\mathbf{F}$ are ditopological. We need to check that the left unit operation $\lambda\colon(s,f)\mapsto(\lambda_S(s),\lambda_F(f))$ on $S\times_\alpha F$ is dicontinuous at every point $(s,f)\in S\times_\alpha F$. Fix any open neighborhood $O_{(s,f)}$ of (s,f) in $S\times_\alpha F$. We lose no generality assuming that it is of basic form: $O_{(s,f)}=O_s\times O_f$ where O_s and O_f are open neighborhoods of s and f in S and F, respectively.

By the dicontinuity of the left unit operation λ_F at f, there are neighborhoods $U_f\subset F$ and $W_{\lambda_F(f)}\subset\lambda_F(F)$ of f and $\lambda_F(f)$ such that $(W_{\lambda_F(f)}\setminus U_f)\cap\lambda_F^{-1}(W_{\lambda_F(f)})\subset O_f$. By the dicontinuity of the left unit operation λ_S at s, there are neighborhoods $U_s\subset S$ and $W_{\lambda_S(s)}\subset\lambda_S(S)$ of s and $\lambda_S(s)$ such that $(W_{\lambda_S(s)}\setminus U_s)\cap\lambda_S^{-1}(W_{\lambda_S(s)})\subset O_s$.

We claim that the neighborhoods $U_{(s,f)}=U_s\times U_f$ and $W_{(s,f)}=W_{\lambda_S(s)}\times W_{\lambda_F(f)}$ of (s,f) and $\lambda(s,f)$ witness that the left unit operation λ is dicontinuous at (s,f). Given any pair $(t,g)\in (W_{\lambda(s,f)}\leftthreetimes U_{(s,f)})\cap\lambda^{-1}(W_{\lambda(s,f)})$, we need to show that $(t,g)\in O_{(s,f)}$. It follows that $(w,h)\cdot(t,g)\in U_{(s,g)}$ for some pair $(w,h)\in W_{\lambda(s,f)}$. Taking into account that the action α respects the left unit operation λ_F and $h\in W_{\lambda_F(f)}\subset\lambda_F(F)$, we conclude that $(w,h)\cdot(t,g)=(w\cdot\alpha_h(t),hg)=(wt,hg)$ and hence $(wt,hg)\in U_{(s,f)}=U_s\times U_f$ and

$$(t,g)\in\big((W_{\lambda_S(s)}\leftthreetimes U_s)\cap\lambda_S^{-1}(W_{\lambda_S(s)})\big)\times\big((W_{\lambda_F(f)}\leftthreetimes U_f)\cap\lambda_F^{-1}(W_{\lambda_F(f)})\big)\subset O_s\times O_f=O_{(s,f)}.$$

□

Now, given two topological right unosemigroups $\mathbf{S}=(S,\rho_S)$ and $\mathbf{F}=(F,\rho_F)$ and a continuous action $\alpha\colon F\times S\to S$ of F on S, we shall define a right unit operation on the semidirect product $S\times_\alpha F$. This can be done under the additional assumption that the action α is ρ_S-*invertible* in the sense that for every $f\in F$ the restriction $\bar\alpha_f=\alpha_f|\rho_S(S)$ is a bijective map of $\rho_S(S)$ and the map $\alpha^-\colon F\times\rho_S(S)\to\rho_S(S)$, $\alpha^-\colon(f,s)\mapsto\bar\alpha_f^{-1}(s)$, is continuous.

In this case the map $\rho\colon S\times_\alpha F\to S\times_\alpha F$ defined by $\rho(s,f)=(\bar\alpha_f^{-1}(\rho_S(s)),\rho_F(f))=(\alpha^-(f,\rho_S(s)),\rho_F(f))$ is continuous.

Since

$$\begin{aligned}(s,f)\cdot\rho(s,f)&=(s,f)\cdot(\bar\alpha_f^{-1}(\rho_S(s)),\rho_F(f))=(s\cdot\alpha_f\circ\bar\alpha_f^{-1}(\rho_S(s)),f\cdot\rho_F(f))=\\&=(s\cdot\bar\alpha_f\circ\bar\alpha_f^{-1}(\rho_S(s)),f)=(s\cdot\rho_S(s),f)=(s,f),\end{aligned}$$

the map ρ is a continuous right unit operation on $S\times_\alpha F$. Therefore, $\mathbf{S}\times_\alpha\mathbf{F}=(S\times_\alpha F,\rho)$ is a topological right unosemigroup, called the *semidirect product* of the topological right unosemigroups $\mathbf{S}=(S,\rho_S)$ and $\mathbf{F}=(F,\rho_F)$.

Let us observe that if the action α respects the right unit operation ρ_S, then for every $f\in F$ the restriction $\bar\alpha_f=\alpha_f|\rho_S(S)$ is an identity map of $\rho_S(S)$ and hence the action α is ρ_S-invertible. Moreover, in this case $\rho(s,f)=(\rho_S(s),\rho_F(f))$ for all $(s,f)\in S\times F$.

The following propositions will help us to detect ρ_S-invertible actions.

Proposition 6. *A continuous action $\alpha\colon F\times S\to S$ of a topological right unosemigroup (F,ρ_F) on a topological right unosemigroup (S,ρ_S) is ρ_S-invertible if*

1) *α respects the right unit operation ρ_F;*
2) *$\alpha_f(\rho_S(S))=\rho_S(S)$ for all $f\in F$;*
3) *there is a continuous unary operation $(\,)^{-1}\colon F\to F$ such that $\rho_F(f)=f^{-1}f$ for all $f\in F$.*

Proof. Taking into account that the action α preserves the right unit operation $\rho_F\colon F\to F$, $\rho_F\colon f\mapsto f^{-1}f$, we conclude that for every $f\in F$ and $s\in S$ we get $s=\alpha(\rho_F(f),s)=\alpha(f^{-1}f,s)=\alpha_{f^{-1}f}(s)=\alpha_{f^{-1}}\circ\alpha_f(s)$, which implies that the homomorphism $\alpha_f\colon S\to S$ is injective and hence has the inverse $\alpha_f^{-1}\colon\alpha_f(S)\to S$. It follows from $\alpha_f(\rho_S(S))=\rho_S(S)$ that the restriction $\bar\alpha_f=\alpha_f|\rho_S(S)$ is a bijective map of $\rho_S(S)$.

It remains to check that the map $\alpha^-\colon F\times\rho_S(S)\to\rho_S(S)$, $\alpha^-\colon(f,s)\mapsto\bar\alpha_f^{-1}(s)$, is continuous. For this purpose, observe that the function α^- coincides with the continuous function $\beta\colon F\times\rho_S(S)\to\rho_S(S)$, $\beta(f,s)\mapsto\alpha(f^{-1},s)$. Indeed, given any $f\in F$ and $s\in\rho_S(S)$, we can find a unique point $x\in\rho_S(S)$ with $s=\bar\alpha_f(x)=\alpha_f(x)$ and conclude that

$$\begin{aligned}\beta(f,s)&=\alpha(f^{-1},s)=\alpha_{f^{-1}}(s)=\alpha_{f^{-1}}\circ\alpha_f(x)=\alpha_{f^{-1}f}(x)=\\&=\alpha_{\rho_F(f)}(x)=x=\bar\alpha_f^{-1}(s)=\alpha^-(f,s).\end{aligned}$$

□

Now we study the semidirect products of ditopological right unosemigroups.

Theorem 8. *Let $\mathbf{S} = (S, \rho_S)$ and $\mathbf{F} = (F, \rho_F)$ be topological right unosemigroups and $\alpha\colon F \times S \to S$ be a ρ_S-invertible continuous action of F on S. The semidirect product $\mathbf{S} \times_\alpha \mathbf{F}$ is a ditopological right unosemigroup if and only if the topological right unosemigroups $\mathbf{S}$ and $\mathbf{F}$ are ditopological.*

Proof. To prove the "only if" part, assume that the topological right unosemigroup $\mathbf{S} \times_\alpha \mathbf{F} = (S \times_\alpha F, \rho)$ is ditopological. We need to show that the right unit operations ρ_S and ρ_F are dicontinuous.

To prove the dicontinuity of the right unit operation ρ_S, fix any point $s \in S$ and a neighborhood $O_s \subset S$ of s. Fix any point $f \in \rho_F(F) \subset F$ and consider the homomorphism $\alpha_f\colon S \to S$, whose restriction $\bar{\alpha}_f$ is a bijective map of the set $\rho_S(S)$. We claim that $\bar{\alpha}_f$ is the identity map of $\rho_S(S)$. It follows from $f \in \rho_F(F)$ that $f = \rho_F(g)$ for some $g \in F$. The equality $g = g \cdot \rho_F(g) = gf$ implies that $\bar{\alpha}_g = \bar{\alpha}_g \circ \bar{\alpha}_f$, which is possible only in the case of the identity map $\bar{\alpha}_f$.

Now consider the point $(s, f) \in S \times_\alpha F$ and its neighborhood $O_{(s,f)} = O_s \times F$. It follows that $\rho(s, f) = (\bar{\alpha}_f^{-1}(\rho_S(s)), \rho_S(f)) = (\rho_S(s), \rho_F(f))$. The dicontinuity of the right unit operation ρ on $S \times_\alpha F$ yields the existence of neighborhoods $U_{(s,f)} \subset S \times_\alpha F$ and $W_{\rho(s,f)} \subset \rho(S \times_\alpha F) = \rho_S(S) \times \rho_F(F)$ of (s, f) and $\rho(s, f) = (\rho_S(s), \rho_F(f))$ such that $(U_{(s,f)} \mathbin{\swarrow} W_{\rho(s,f)}) \cap \rho^{-1}(W_{\rho(s,f)}) \subset O_{(s,f)}$. We lose no generality assuming that $U_{(s,f)} = U_s \times U_f$ and $W_{\rho(s,f)} = W_{\rho_S(s)} \times W_{\rho_F(f)}$ for some open sets $U_s \subset S$, $W_{\rho_S(s)} \subset \rho_S(S)$, $U_f \subset F$, and $W_{\rho_F(f)} \subset \rho_F(F)$.

We claim that the neighborhoods U_s and $W_{\rho_S(s)}$ have the required property: $(U_s \mathbin{\swarrow} W_{\rho_S(s)}) \cap \rho_S^{-1}(W_{\rho_S(s)}) \subset O_s$. Given any point $t \in (U_s \mathbin{\swarrow} W_{\rho_S(s)}) \cap \rho_S^{-1}(W_{\rho_S(s)})$, find a point $w \in W_{\rho_S(s)} \subset \rho_S(S)$ such that $tw \in U_s$. Consider the point $(t, f) \in S \times_\alpha F$ and $(w, \rho_F(f)) \in W_{\rho_S(s)} \times W_{\rho_F(f)} = W_{\rho(s,f)}$ and observe that $(t, f) \cdot (w, \rho_F(f)) = (t \cdot \bar{\alpha}_f(w), f \cdot \rho_S(f)) = (tw, f) \in U_s \times U_f$. Since

$$\rho(t, f) = (\bar{\alpha}_f^{-1}(\rho_S(t)), \rho_F(f)) = (\rho_S(t), \rho_F(f)) \in W_{\rho_S(s)} \times W_{\rho_F(f)} = W_{\rho(s,f)},$$

we get the desired inclusion $(t, f) \in (U_{(s,f)} \mathbin{\swarrow} W_{\rho(s,f)}) \cap \rho^{-1}(W_{\rho(s,f)}) \subset O_{(s,f)} = O_s \times F$, which implies $t \in O_s$.

Next, we show that the right unit operation ρ_F on F is dicontinuous at every point $f \in F$. Fix any neighborhood O_f of f in F. Fix any point $s \in S$ and consider the pair (s, f) and its neighborhood $O_{(s,f)} = S \times O_f$ in $S \times_\alpha F$. The dicontinuity of the right unit operation ρ on $S \times_\alpha F$ yields the existence of neighborhoods $U_{(s,f)} \subset S \times_\alpha F$ and $W_{\rho(s,f)} \subset \rho(S \times_\alpha F) = \rho_S(S) \times \rho_F(F)$ of the elements (s, f) and $\rho(s, f)$ such that $(U_{(s,f)} \mathbin{\swarrow} W_{\rho(s,f)}) \cap \rho^{-1}(W_{\rho(s,f)}) \subset O_{(s,f)}$. Consider the point $r = \bar{\alpha}_f^{-1}(\rho_S(s))$ and observe that $\rho(s, f) = (r, \rho_F(f))$. Without loss of generality we can assume that $W_{\rho(s,f)} = W_r \times W_{\rho_F(f)}$ and $U_{(s,f)} = U_s \times U_f$ for some open sets $W_r \subset \rho_S(S)$, $W_{\rho_F(f)} \subset \rho_F(F)$, $U_s \subset S$ and $U_f \subset F$.

Consider the continuous functions $\beta\colon F \to \rho_S(S)$, $\beta\colon g \mapsto \bar{\alpha}_g^{-1}(\rho_S(s)) = \alpha^-(g, \rho_S(s))$, and $\gamma\colon F \to S$, $\gamma\colon g \mapsto s \cdot \alpha(g, r)$, and observe that $\beta(f) = r$ and

$$\gamma(f) = s \cdot \alpha_f(r) = s \cdot \bar{\alpha}_f \circ \bar{\alpha}_f^{-1}(\rho_S(s)) = s \cdot \rho_S(s) = s.$$

Using the continuity of the functions β and γ, find a neighborhood $U'_f \subset U_f$ of f such that $\beta(U'_f) \subset W_r$ and $\gamma(U'_f) \subset U_s$.

We claim that the neighborhoods U'_f and $W_{\rho_F(f)}$ have the required property: $(U'_f \leftthreetimes W_{\rho_F(f)}) \cap \rho_F^{-1}(W_{\rho_F(f)}) \subset O_f$. Given any point $g \in (U'_f \leftthreetimes W_{\rho_F(f)}) \cap \rho_F^{-1}(W_{\rho_F(f)})$, find a point $h \in W_{\rho_F(f)} \subset \rho_F(F)$ with $gh \in U'_f$. Consider the points $(s,g) \in S \times_\alpha F$ and $(r,h) \in W_r \times W_{\rho_F(f)} = W_{\rho(s,f)}$. Since $h \in \rho_F(F)$, the map $\bar\alpha_h$ is an identity homeomorphism of $\rho_S(S)$. Hence, $\alpha_h(r) = r$ and

$$s \cdot \alpha_g(r) = s \cdot \alpha_g(\alpha_h(r)) = s \cdot \alpha_{gh}(r) = \gamma(gh) \in \gamma(U'_f) \subset U_s.$$

Also $\bar\alpha_{gh} = \bar\alpha_g \circ \bar\alpha_h = \bar\alpha_g$ implies that

$$\begin{aligned}\rho(s,g) = (\bar\alpha_g^{-1}(\rho_S(s)), \rho_F(g)) = (\bar\alpha_{gh}^{-1}(\rho_S(s)), \rho_F(g)) =\\ = (\beta(gh), \rho_F(g)) \in \beta(U'_f) \times W_{\rho_F(f)} \subset W_r \times W_{\rho_F(f)} = W_{\rho(s,f)}.\end{aligned}$$

Since $(s,g) \cdot (r,h) = (s \cdot \alpha_g(r), gh) \in U_s \times U'_f \subset U_{(s,f)}$, we get

$$(s,g) \in (U_{(s,f)} \leftthreetimes W_{\rho(s,f)}) \cap \rho^{-1}(W_{\rho(s,f)}) \subset O_{(s,f)} = S \times O_f$$

and hence $g \in O_f$, which completes the proof of the "only if" part of the theorem.

To prove the "if" part, assume that the topological right unosemigroups (S, ρ_S) and (F, ρ_F) are ditopological. We need to check that the right unit operation

$$\rho\colon S \times_\alpha F \to S \times_\alpha F,\ \rho\colon (s,f) \mapsto (\bar\alpha_f^{-1}(\rho_S(s)), \rho_F(f)),$$

is dicontinuous at each point $(s,f) \in S \times_\alpha F$.

Fix any neighborhood $O_{(s,f)}$ of the point (s,f) in the topological semigroup $S \times_\alpha F$. We lose no generality assuming that this neighborhood is of the basic form $O_{(s,f)} = O_s \times O_f$, where O_s and O_f are open neighborhoods of s and f in S and F, respectively. The dicontinuity of the right unit operation ρ_S yields the existence of neighborhoods $U_s \subset S$ and $W_{\rho_S(s)} \subset \rho_S(S)$ of s and $\rho_S(s)$ such that $(U_s \leftthreetimes W_{\rho_S(s)}) \cap \rho_S^{-1}(W_{\rho_S(s)}) \subset O_s$.

Now consider the point $r = \bar\alpha_f^{-1}(\rho_S(s)) \in \rho_S(S)$ and observe that

$$\alpha(f,r) = \bar\alpha_f(r) = \bar\alpha_f \circ \bar\alpha_f^{-1}(\rho_S(s)) = \rho_S(s) \in W_{\rho_S(s)}.$$

The continuity of the action $\alpha|(F \times \rho_S(S))$ yields the existence of a neighborhood $O'_f \subset O_f$ of f and a neighborhood $W_r \subset \rho_S(S)$ of r such that $\alpha(O'_f \times W_r) \subset W_{\rho_S(s)}$.

Since the right unit operation ρ_F is dicontinuous at f, there are neighborhoods $U_f \subset F$ and $W_{\rho_F(f)} \subset \rho_F(F)$ of f and $\rho_F(f)$ such that $(U_f \leftthreetimes W_{\rho_F(f)}) \cap \rho_F^{-1}(W_{\rho_F(f)}) \subset O'_f$.

The ρ_S-invertibility of the action α guarantees that

$$\rho(S \times_\alpha F) = \{(\bar\alpha_f^{-1}(\rho_S(s)), \rho_F(f))\colon (s,f) \in S \times_\alpha F\} = \rho_S(S) \times \rho_F(F),$$

which implies that the product $W_{\rho(s,f)} = W_r \times W_{\rho_F(f)}$ is a neighborhood of the point $\rho(s,f) = (\bar\alpha_f^{-1}(\rho_S(s)), \rho_F(f)) = (r, \rho_F(f))$ in $\rho(S \times_\alpha F) = \rho_S(S) \times \rho_F(F)$. We claim that the neighborhoods $U_{(s,f)} = U_s \times U_f$ and $W_{\rho(s,f)} = W_r \times W_{\rho_F(f)}$ have the required property: $(U_{(s,f)} \leftthreetimes W_{\rho(s,f)}) \cap \rho^{-1}(W_{\rho(s,f)}) \subset O_{(s,f)}$. Fix any point $(t,g) \in (U_{(s,f)} \leftthreetimes W_{\rho(s,f)}) \cap \rho^{-1}(W_{\rho(s,f)})$. It follows that $(t,g)(w,h) \in U_s \times U_f$ for some $(w,h) \in W_{\rho(s,f)}$.

Observe that $W_r \times W_{\rho_F(f)} = W_{\rho(s,f)} \ni \rho(t,g) = (\bar\alpha_g^{-1}(\rho_S(t)), \rho_F(g))$ implies $\rho_F(g) \in W_{\rho_F(f)}$ and $\bar\alpha_g^{-1}(\rho_S(t)) \in W_r$. Also $(w,h) \in W_{\rho(s,f)} = W_r \times W_{\rho_F(f)}$ yields $w \in W_r$ and $h \in W_{\rho_F(f)}$.

It follows from $\rho_F(g), h \in W_{\rho_F(f)}$ and $gh \in U_f$ that $g \in O'_f \subset O_f$. Then

$$\rho_S(t) = \bar{\alpha}_g \circ \bar{\alpha}_g^{-1}(\rho_S(t)) = \alpha(g, \bar{\alpha}_g^{-1}(\rho_S(t))) \in \alpha(O'_f \times W_r) \subset W_{\rho_S(s)}.$$

By the same reason, $\alpha_g(w) = \alpha(g, w) \in \alpha(O'_f \times W_r) \subset W_{\rho_S(s)}$. Observe also that

$$(t \cdot \alpha_g(w), gh) = (t, g) \cdot (w, h) \in U_{(s,t)} = U_s \times U_f$$

yields $gh \in U_f$ and $t \cdot \alpha_g(w) \in U_s$. The choice of the neighborhoods U_s and $W_{\rho_S(s)} \supset \{\rho_S(t), \alpha_g(w)\}$ guarantees that $t \in O_s$. So, $(t, g) \in O_s \times O_f = O_{(s,f)}$. □

The above discussion implies that for topological unosemigroups $\mathbf{F} = (F, \lambda_F, \rho_F)$, $\mathbf{S} = (S, \lambda_S, \rho_S)$ and a λ_F-respecting ρ_S-invertible continuous action $\alpha\colon F \times S \to S$ of F on S, the triple $\mathbf{S} \times_\alpha \mathbf{F} = (S \times_\alpha F, \lambda, \rho)$ is a topological unosemigroup. This topological unosemigroup will be called the *semidirect product* of the topological unosemigroups $\mathbf{F}$ and $\mathbf{S}$. By Theorems 7 and 8 the topological unosemigroup $\mathbf{S} \times_\alpha F$ is ditopological if and only if so are the topological unosemigroups $\mathbf{S}$ and $\mathbf{F}$.

6.5. Hartman-Mycielski Extension. Given a topological space X, for every $n \in \mathbb{N}$ by $HM_n(X)$ we denote the set of all functions $f\colon [0,1) \to X$ for which there exists a sequence $0 = a_0 < a_1 < \dots < a_n = 1$ such that f is constant on each interval $[a_i, a_{i+1}), 0 \le i < n$. The union $HM(X) = \bigcup_{n\in\mathbb{N}} HM_n(X)$ is called the *Hartman-Mycielski extension* of the space X, see [6], [3].

A neighborhood sub-base of the topology of $HM(X)$ at $f \in HM(X)$ consists of the sets $N(a, b, V, \varepsilon)$, where

1) $0 \le a < b \le 1$, f is constant on $[a, b)$, V is a neighborhood of $f(a)$ in X, and $\varepsilon > 0$;
2) $g \in N(a, b, V, \varepsilon)$ means that $|\{t \in [a, b)\colon g(t) \notin V\}| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure.

If X is a topological semigroup then $HM(X)$ also is a topological semigroup with respect to the pointwise operations of multiplication of functions. Moreover, for any continuous left (right) unit operation $u_X\colon X \to X$ on X, the unary operation $u_{HM(X)}\colon HM(X) \to HM(X)$, $u_{HM(X)}\colon f \mapsto u \circ f$, is a continuous left (right) unit operation on $HM(X)$, see [1, Proposition 2].

Theorem 9. *If $\mathbf{X} = (X, \lambda_X)$ is a ditopological left unosemigroup then so is its Hartman-Mycielski extension $HM(\mathbf{X}) = (HM(X), \lambda_{HM(X)})$.*

Proof. For simplicity of notation, we write λ instead of $\lambda_{HM(X)}$. Assume that X is a ditopological left unosemigroup. To show that the topological left unosemigroup $HM(X)$ is ditopological, fix any element $f \in HM(X)$ and its sub-basic neighborhood in $HM(X)$ $O_f = N(a, b, O_{f(a)}, \varepsilon)$ such that f is constant on the half-interval $[a, b) \subset [0, 1)$ and $O_{f(a)}$ is an open neighborhood of $f(a)$ in X. Since the topological left unosemigroup X is ditopological, there are neighborhoods $U_x \subset S$ and $W_{\lambda_X(x)} \subset \lambda_X(X)$ of the points $x = f(a)$ and $\lambda_X(x) = \lambda_X(f(a))$ such that $(W_{\lambda_X(x)} \searrow U_x) \cap \lambda_X^{-1}(W_{\lambda_X(x)}) \subset O_x$.

Consider the open neighborhoods $U_f = N(a, b, U_x, \frac{\varepsilon}{3}) \subset HM(X)$ of f and

$$W_{\lambda(f)} = \lambda(HM(X)) \cap N\Big(a, b, W_{\lambda_X(x)}, \frac{\varepsilon}{3}\Big) \subset \lambda(HM(X))$$

of $\lambda(f)$. We claim that each function $g \in (W_{\lambda(f)} \leftthreetimes U_f) \cap \lambda^{-1}(W_{\lambda(f)}) \subset HM(X)$ belongs to the neighborhood O_f. The inclusion $\lambda(g) \in W_{\lambda(f)} = N(a, b, W_{\lambda_X(x)}, \frac{\varepsilon}{3})$ guarantees that the set $A = \{t \in [a, b)\colon \lambda_X(f(t)) \notin W_{\lambda_X(x)}\}$ has Lebesgue measure $< \frac{\varepsilon}{3}$. On the other hand, the inclusion $wg \in U_f = N(a, b, U_x, \frac{\varepsilon}{3})$ implies that the set $B = \{t \in [a, b)\colon w(t) \cdot g(t) \notin U_x\}$ has Lebesgue measure $< \frac{\varepsilon}{3}$. Finally, the inclusion $w \in W_{\lambda(f)}$ implies that the set $C = \{t \in [a, b)\colon w(t) \notin W_{\lambda_X(x)}\}$ has Lebesgue measure $< \frac{\varepsilon}{3}$. Then the union $A \cup B \cup C$ has Lebesgue measure $< \varepsilon$ and for each $t \in [a, b) \setminus (A \cup B \cup C)$ we get $\lambda_X(g(t)) \in W_{\lambda_X(x)}$, $w(t) \in W_{\lambda_X(x)}$ and $w(t)g(t) \in U_x$. Then the choice of the neighborhoods U_x and $W_{\lambda_X(x)}$ guarantees that $g(t) \in O_x = O_{f(a)}$ and hence $g \in N(a, b, O_{f(a)}, \varepsilon) = O_f$. □

By analogy we can prove the following result.

Theorem 10. *If $\mathbf{X} = (X, \rho_X)$ is a ditopological right unosemigroup then so is its Hartman-Mycielski extension $HM(\mathbf{X}) = (HM(X), \rho_{HM(X)})$.*

Theorems 9 and 10 imply

Theorem 11. *If $\mathbf{X} = (X, \lambda_X, \rho_X)$ is a ditopological unosemigroup then so is its Hartman-Mycielski extension $HM(\mathbf{X}) = (HM(X), \lambda_{HM(X)}, \rho_{HM(X)})$.*

Since the Hartman-Mycielski space $HM(X)$ is contractible ([6]), Theorems 9–11 imply

Corollary 4. *Each topological (left, right) unosemigroup is topologically isomorphic to the (left, right) subunosemigroup of a contractible topological (left, right) unosemigroup.*

7. Acknowledgements. The authors would like to thank Oleg Gutik for fruitful valuable remarks concerning this paper.

REFERENCES

1. T. Banakh, O. Hryniv, *On closed embeddings of free topological algebras*, Visnyk Lviv Univ., **61** (2003), 21–25; available at http://arxiv.org/abs/1202.4480.
2. T. Banakh, I. Pastukhova, *Embedding ditopological inverse semigroups into the products of cones over groups*, http://arxiv.org/abs/1304.2558.
3. R. Brown, S. Morris, *Embeddings in contractible or compact objects*, Colloq. Math., **38** (1977/78), №2, 213–222.
4. J.H. Carruth, J.A. Hildebrant, R.J. Koch, The Theory of Topological Semigroups, II. Marcell Dekker, Inc., New York and Basel, 1986.
5. J. Desharnais, P. Jipsen, G. Struth, Domain and antidomain semigroups, in: Relations and Kleene algebra in computer science, 73–87, Lecture Notes in Comput. Sci., Springer, Berlin, 2009.
6. S. Hartman, J. Mycielski, *On the embedding of topological groups into connected topological groups*, Colloq. Math., **5** (1958), 167–169.
7. O. Hryniv, *Universal objects in some classes of Clifford topological inverse semigroups*, Semigroup Forum, **75** (2007), №3, 683–689.

Ivan Franko National University of Lviv
t.o.banakh@gmail.com
irynkapastukhova@gmail.com

Received 14.03.2013