УДК 517.555

A. I. Bandura

A MODIFIED CRITERION OF BOUNDEDNESS L-INDEX IN DIRECTION

A. I. Bandura. *A modified criterion of boundedness of L-index in direction*, Mat. Stud. **39** (2013), 99–102.

A new criterion of boundedness of L-index in direction is obtained.

А. И. Бандура. *Модифицированный критерий ограниченности L-индекса по направлению* // Мат. Студії. – 2013. – Т.39, №1. – С.99–102.

Получен новый критерий ограниченности L-индекса по направлению.

1. Introduction. Let $L(z)$ be a positive continuous function on $\mathbb{C}^n$, and let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$. An entire function of $F(z)$, $z \in \mathbb{C}^n$, is called (see [1]–[3]) a function of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ the following inequality is valid:

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right|:\ 0 \le k \le m_0\right\}, \tag{1}$$

where

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z),\ \frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^{n}\frac{\partial F(z)}{\partial z_j}b_j = \langle \mathbf{grad}\ F,\ \overline{\mathbf{b}}\rangle, \frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\Big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\Big),\ k \ge 2.$$

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n$ *of the function $F(z)$ and is denoted by $N_{\mathbf{b}}(F,L) = m_0$.*

In the case of $n = 1$ and $L(z) = l(z),\ z \in \mathbb{C}$, we obtain the definition of a function of bounded l-index, and in the case $L(z) \equiv 1$ we get the definition of a function of bounded index.

For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{0\}$ and a function $L\colon\ \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta) = \inf\left\{\frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})}:\ |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})}\right\},$$

$\lambda_1^{\mathbf{b}}(z, \eta) = \inf\{\lambda_1^{\mathbf{b}}(z, t_0, \eta):\ t_0 \in \mathbb{C}\}$, $\lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta):\ z \in \mathbb{C}^n\}$, and also

$$\lambda_2^{\mathbf{b}}(z, t_0, \eta) = \sup\left\{\frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})}:\ |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})}\right\},$$

2010 *Mathematics Subject Classification:* 30D20, 32A15.
Keywords: entire function of several variables, bounded L-index in direction.

$\lambda_2^{\mathbf{b}}(z,\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta)\colon\ t_0 \in \mathbb{C}\}$, $\lambda_2^{\mathbf{b}}(\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,\eta)\colon\ z \in \mathbb{C}^n\}$.

By $Q_{\mathbf{b}}^n$ we denote the class of all functions L for which the following condition holds for all $\eta \geq 0$

$$0 < \lambda_1^{\mathbf{b}}(\eta) \leq \lambda_2^{\mathbf{b}}(\eta) < +\infty.$$

In a recent paper [1] the authors prove the following criterion of boundedness of the L-index in direction, which will be used several times throughout this paper.

Theorem 1 ([1])**.** *Let $L \in Q_{\mathbf{b}}^n$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for each $\eta > 0$ there exist numbers $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for every $t_0 \in \mathbb{C}$ and every $z \in \mathbb{C}^n$ there exists $k_0 = k_0(t_0,z) \in \mathbb{Z}_+$, with $0 \leq k_0 \leq n_0$ and the inequality*

$$\max\left\{\left|\frac{\partial^{k_0}F(z+t\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|\colon\ |t-t_0| \leq \frac{\eta}{L(z+t_0\mathbf{b})}\right\} \leq P_1\left|\frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|$$

holds.

The notation $L \asymp L^*$ means that for some θ_1, θ_2, $0 < \theta_1 \leq \theta_2 < +\infty$ and all $z \in \mathbb{C}^n$ the inequality $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$ holds.

We will need the following theorem from [1].

Theorem 2 ([1])**.** *Let $L, L^* \in Q_{\mathbf{b}}^n$, $L \asymp L^*$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is of bounded L^*-index in the direction $\mathbf{b}$ if and only if F is of bounded L-index in the direction $\mathbf{b}$.*

In [1] we prove the following criteria of boundedness of the L-index in direction in the form of an estimate of the maximum modulus on a larger circle by the maximum modulus on a smaller one.

Theorem 3 ([1])**.** *Let $L \in Q_{\mathbf{b}}^n$. An entire function $F(z)$ in $\mathbb{C}^n$ is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for any r_1 and r_2 such that $0 < r_1 < r_2 < +\infty$, there exists a number $P_1 = P_1(r_1,r_2) \geq 1$ such that for each $z^0 \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$ the next inequality is valid*

$$\max\left\{|F(z^0+t\mathbf{b})|\colon\ |t-t_0| = \frac{r_2}{L(z^0+t_0\mathbf{b})}\right\} \leq$$
$$\leq P_1 \max\left\{|F(z^0+t\mathbf{b})|\colon\ |t-t_0| = \frac{r_1}{L(z_0+t_0\mathbf{b})}\right\}. \qquad (2)$$

Theorem 4 ([1])**.** *Let $L \in Q_{\mathbf{b}}^n$. An entire function $F(z)$ in $\mathbb{C}^n$ is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist numbers r_1 and r_2, $0 < r_1 < 1 < r_2 < +\infty$, and $P_1 \geq 1$ such that for all $z^0 \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$ the inequality (2) holds.*

These theorems distinguish the universal quantifier and the existential quantifiers for r_1 and r_2 such that $0 < r_1 < r_2 < +\infty$.

Hence the following **question** arises naturally: *is it possible to change the quantifiers in the other criteria of boundedness of L-index in direction* (see [1]).

2. Main result. Using Fricke's idea from [4] a modification of Theorem 1 is obtained.

Theorem 5. *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist constants $\eta > 0$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for every $t_0 \in \mathbb{C}$ and every $z \in \mathbb{C}^n$ there exists $k_0 = k_0(t_0, z) \in \mathbb{Z}_+$, with $0 \leq k_0 \leq n_0$ and the inequality*

$$\max\left\{\left|\frac{\partial^{k_0} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right| : \ |t-t_0| \leq \frac{\eta}{L(z+t_0\mathbf{b})}\right\} \leq P_1 \left|\frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right|$$

holds.

Proof of Theorem 5. Necessity. If $F(z)$ is of bounded L-index in the direction $\mathbf{b}$, then it follows directly from Theorem 1 that the necessity condition of Theorem 5 is satisfied.
Sufficiency. Suppose that there exists $\eta > 0$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for each $t_0 \in \mathbb{C}$ and each $z \in \mathbb{C}^n$ there exists $k_0 = k_0(t_0, z) \in \mathbb{Z}_+$, with $0 \leq k_0 \leq n_0$ and

$$\max\left\{\left|\frac{\partial^{k_0} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right| : \ |t-t_0| \leq \frac{\eta}{L(z+t_0\mathbf{b})}\right\} \leq P_1 \left|\frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right|. \tag{3}$$

If $\eta > 1$ then we choose $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. But if $\eta \in (0; 1]$ then we choose $j_0 \in \mathbb{N}$ such that $\frac{j_0! k_0!}{(j_0+k_0)!} P_1 < 1$. Such a j_0 exists because

$$\frac{j_0! k_0!}{(j_0+k_0)!} P_1 = \frac{k_0!}{(j_0+1)(j_0+2)\cdot \ldots \cdot (j_0+k_0)} P_1 \to 0, \ j_0 \to \infty.$$

Applying Cauchy's formula to the function $F(z+t\mathbf{b})$ as a function of one complex variable t with $j \geq j_0$ we have that for each $t_0 \in \mathbb{C}$ and each $z \in \mathbb{C}^n$ there exists an integer $k_0 = k_0(t_0, z)$ with $0 \leq k_0 \leq n_0$

$$\frac{\partial^{k_0+j} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0+j}} = \frac{j!}{2\pi i} \int\limits_{|t-t_0| = \frac{\eta}{L(z+t_0\mathbf{b})}} \frac{1}{(t-t_0)^{j+1}} \frac{\partial^{k_0} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0}} dt.$$

Therefore, in view of (3), we have

$$\begin{aligned}\frac{1}{j!}\left|\frac{\partial^{k_0+j} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0+j}}\right| \leq \frac{L^j(z+t_0\mathbf{b})}{\eta^j} \max\left\{\left|\frac{\partial^{k_0} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right| : \ |t-t_0| = \frac{\eta}{L(z+t_0\mathbf{b})}\right\} \leq \\ \leq P_1 \frac{L^j(z+t_0\mathbf{b})}{\eta^j} \left|\frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right|,\end{aligned} \tag{4}$$

that is by the choice of j_0 for $\eta > 1$

$$\begin{aligned}\frac{1}{(k_0+j)! L^{k_0+j}(z+t_0\mathbf{b})} \left|\frac{\partial^{k_0+j} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0+j}}\right| \leq \frac{j! k_0!}{(j+k_0)!} \frac{P_1}{\eta^j} \frac{1}{k_0! L^{k_0}(z+t_0\mathbf{b})} \left|\frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right| \leq \\ \leq \eta^{j_0-j} \frac{1}{k_0! L^{k_0}(z+t_0\mathbf{b})} \left|\frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right| \leq \frac{1}{k_0! L^{k_0}(z+t_0\mathbf{b})} \left|\frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right|\end{aligned} \tag{5}$$

for all $j \geq j_0$.

Since by the assumptions of the theorem the numbers $n_0 = n_0(\eta)$, $j_0 = j_0(\eta)$ are independent of z and t_0, and $z \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$ are arbitrary, the inequality (5) is equivalent to the assertion that $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F, L) \le n_0 + j_0$.

If $\eta \in (0,1)$ then in view of (4) we have for all $j \ge j_0$

$$\frac{1}{(k_0+j)!L^{k_0+j}(z+t_0\mathbf{b})}\left|\frac{\partial^{k_0+j}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0+j}}\right| \le \frac{j!k_0!P_1}{(j+k_0)!}\frac{1}{\eta^j k_0!L^{k_0}(z+t_0\mathbf{b})}\left|\frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right| \le$$
$$\le \frac{1}{\eta^j k_0!L^{k_0}(z+t_0\mathbf{b})}\left|\frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|$$

or, by the choice of j_0,

$$\frac{1}{(k_0+j)!}\frac{\eta^{k_0+j}}{L^{k_0+j}(z+t_0\mathbf{b})}\left|\frac{\partial^{k_0+j}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0+j}}\right| \le \frac{1}{k_0!}\frac{\eta^{k_0}}{L^{k_0}(z+t_0\mathbf{b})}\left|\frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|.$$

Thus the function F is of bounded $\tilde{L}$-index in the direction $\mathbf{b}$, where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then, by Theorem 2, F is of bounded L-index in the direction $\mathbf{b}$. □

REFERENCES

1. Bandura A.I., Skaskiv O.B. *Entire function of bounded L-index in direction*// Mat. Stud. – 2007. – V.27, №1. – P. 30–52. (in Ukrainian)
2. Bandura A.I., Skaskiv O.B. *Entire function of bounded and unbounded L-index in direction*// Mat. Stud. – 2007. – V.27, №1. – P. 211–215. (in Ukrainian)
3. Bandura A.I., Skaskiv O.B. *Sufficient sets for boundedness L-index in direction for entire function*// Mat. Stud. – 2008. – V.30, №2. – P. 177–182.
4. Fricke G.H. *Entire functions of locally slow growth*// Journal d'Analyse Mathematique. – 1975. – V.28, №1. – P. 101–122.

Department of higher mathematics
Ivano-Frankivs'k National Technical University of Oil and Gas
andriykopanytsia@gmail.com

Received 25.11.2012