УДК 519.213.2+517.53

M. I. Parolya, M. M. Sheremeta

ESTIMATES FROM BELOW FOR CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS

M. I. Parolya, M. M. Sheremeta. *Estimates from below for characteristic functions of probability laws*, Mat. Stud. **39** (2013), 54–66.

Let φ be the characteristic function of a probability law F that is analytic in $\mathbb{D}_R = \{z\colon |z| < R\}$, $0 < R \le +\infty$, $M(r,\varphi) = \max\{|\varphi(z)|\colon |z| = r < R\}$ and $W_F(x) = 1-F(x)+F(-x)$, $x \ge 0$. A connection between the growth of $M(r,\varphi)$ and the decrease it of $W_F(x)$ is investigated in terms of estimates from below. For entire characteristic functions it is proved, for example, that if $\ln x_k \ge \lambda \ln(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)})$ for some increasing sequence (x_k) such that $x_{k+1} = O(x_k)$, $k \to \infty$, then $\ln\frac{\ln M(r,\varphi)}{r} \ge (1+o(1))\lambda \ln r$ as $r \to +\infty$.

М. И. Пароля, М. М. Шеремета. *Оценки снизу характеристических функций вероятностних законов* // Мат. Студії. – 2013. – Т.39, №1. – C.54–66.

Пусть φ — аналитическая в $\mathbb{D}_R = \{z\colon |z| < R\}$, $0 < R \le +\infty$, характеристическая функция вероятностного закона F, $M(r,\varphi) = \max\{|\varphi(z)|\colon |z| = r < R\}$ и $W_F(x) = 1 - F(x) + F(-x)$, $x \ge 0$. В терминах оценок снизу изучена связь между ростом $M(r,\varphi)$ и убыванием $W_F(x)$. Например, для целых характеристических функций доказано, что если $\ln x_k \ge \lambda \ln(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)})$ для некоторой возрастающей последовательности (x_k) такой, что $x_{k+1} = O(x_k)$, $k \to \infty$, то $\ln\frac{\ln M(r,\varphi)}{r} \ge (1+o(1))\lambda \ln r$ при $r \to +\infty$.

1. Introduction. A non-decreasing function F continuous on the left on $(-\infty,\infty)$ is said ([1, p. 10]) to be a *probability law*, if $\lim\limits_{x\to+\infty} F(-x) = 0$ and $\lim\limits_{x\to+\infty} F(x) = 1$, and the function $\varphi(z) = \int_{-\infty}^{\infty} e^{izx}dF(x)$ defined for real z is called ([1, p. 12]) a *characteristic function of this law*. If φ has an analytic continuation on the disk $\mathbb{D}_R = \{z\colon |z| < R\}$, $0 < R \le +\infty$, then we call φ an analytic in $\mathbb{D}_R$ characteristic function of the law F. Further we always assume that $\mathbb{D}_R$ is the maximal disk of the analyticity of φ. It is known that φ is an analytic in $\mathbb{D}_R$ characteristic function of the probability law F if and only if for every $r \in [0,R)$

$$W_F(x) =: 1 - F(x) + F(-x) = O\left(e^{-rx}\right),\ x \to +\infty.$$

Hence it follows that

$$\varliminf_{x\to+\infty} \frac{1}{x}\ln\frac{1}{W_F(x)} = R.$$

If we put $M(r,\varphi) = \max\{|\varphi(z)|\colon |z| = r\}$ for $r < R$ then $W_F(x)e^{rx} \le 2M(r,\varphi)$ for all $x \ge 0$ and $r \in [0,R)$. For $R = +\infty$ this inequality is proved in [1, p.54] and for $R < +\infty$ the proof is analogous. Therefore, if we define (see also [2]) $\mu(r,\varphi) = \sup\{W_F(x)e^{rx}\colon x \ge 0\}$

2010 *Mathematics Subject Classification:* 30D99, 60E10.
Keywords: characteristic function, probability law, lower estimate.

then $\mu(r,\varphi) \le 2M(r,\varphi)$. Thus, the estimates from below for $\ln M(r,\varphi)$ follow from such estimates for $\ln \mu(r,\varphi)$. For entire characteristic functions N. I. Jakovleva ([3]) proved that, if $\overline{\lim\limits_{x\to+\infty}} \frac{\ln\ln(1/W_F(x))}{\ln x} = 1+\frac{1}{\lambda}$ then $\lim\limits_{r\to+\infty} \frac{\ln\ln M(r,\varphi)}{\ln r} \ge 1+\lambda$. Hence it follows that if

$$\ln x \ge \lambda \ln\Big(\frac{1}{x}\ln\frac{1}{W_F(x)}\Big),\ x \ge x_0, \tag{1}$$

then

$$\ln\frac{\ln M(r,\varphi)}{r} \ge (1+o(1))\lambda \ln r,\ r\to+\infty. \tag{2}$$

The question arises, whether asymptotical inequality (2) is valid if condition (1) holds not necessarily for all $x \ge x_0$ but only for some increasing unbounded sequence (x_k). In view of the inequality $\ln M(r,\varphi) \ge \ln\mu(r,\varphi) - \ln 2$, the following assertion gives a positive answer to this question.

Proposition 1. *If there exists an increasing to $+\infty$ sequence (x_k) such that $\ln x_k \ge \lambda\ln\big(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)}\big)$ for all $k\ge 1$ and $x_{k+1} = O(x_k)$ as $k\to\infty$ then $\ln\frac{\ln\mu(r,\varphi)}{r} \ge (1+o(1))\lambda\ln r$ as $r\to+\infty$.*

We obtain Proposition 1 from main results proved below for characteristic functions that are entire or analytic in the disk characteristic functions.

2. Auxiliary results. If φ is an entire characteristic function and $\varphi \not\equiv \text{const}$ then ([1, p. 45]) $\lim\limits_{r\to+\infty} r^{-1}\ln M(r,\varphi) = \sigma \in (0,+\infty]$. If $\sigma < +\infty$ then the estimate $\ln M(r,\varphi)$ from below is trivial. It can be shown that $\sigma < +\infty$ provided $W_F(x) = 0$ for all $x \ge x_0$. Therefore we assume in what follows that $W_F(x) \ne 0$ for all $x \ge 0$ and, thus, $W_F(x) \searrow 0\ \ (x\to+\infty)$. Then $\frac{\ln\mu(r,\varphi)}{r} \to +\infty$ as $r\to+\infty$.

In the case, when $0 < R < +\infty$, the function $\mu(r,\varphi)$ may be bounded and it is easy to show that $\ln\mu(r,\varphi)\uparrow+\infty$ as $r\uparrow R$ if and only if

$$\overline{\lim_{x\to+\infty}}\, W_F(x)e^{Rx} = +\infty. \tag{3}$$

Further we assume that (3) holds and for the investigation of the growth of $\ln\mu(r,\varphi)$ we use the results from [4]. By $\Omega(0,R)$, $0 < R \le +\infty$, we denote the class of positive unbounded functions Φ on $[r_0,R)$ for some $r_0\in[0\ ,R)$ such that the derivative Φ' is positively continuously differentiable and increasing to $+\infty$ on $[r_0,R)$. For $\Phi\in\Omega(0,R)$ let $\Psi(r) = r - \frac{\Phi(r)}{\Phi'(r)}$ be a function associated with Φ in the sense of Newton and ϕ be the inverse function to Φ'. For the numbers $\Phi'(r_0) < a < b < +\infty$ we put

$$G_1(a,b,\Phi) = \frac{ab}{b-a}\int_a^b \frac{\Phi(\phi(t))}{t^2}dt,\ \ G_2(a,b,\Phi) = \Phi\Big(\frac{1}{b-a}\int_a^b \phi(t)dt\Big).$$

Lemma 1. *Let $\Phi\in\Omega(0,R)$, $0<R\le+\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F satisfying condition (3) and*

$$\ln W_F(x_k) \ge -x_k\Psi(\phi(x_k)) \tag{4}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then $(\forall r\in[\phi(x_k),\phi(x_{k+1})])$ and $(\forall k\ge k_0)$ the following estimates are valid

$$\ln\mu(r,\varphi) \ge \Phi(r) - \left(G_2(x_k,x_{k+1},\Phi) - G_1(x_k,x_{k+1},\Phi)\right), \tag{5}$$

$$\ln \mu(r,\varphi) \geq \Phi(r)\frac{G_1(x_k,x_{k+1},\Phi)}{G_2(x_k,x_{k+1},\Phi)} \tag{6}$$

and

$$\Phi^{-1}(\ln \mu(r,\varphi)) \geq r - (\Phi^{-1}(G_2(x_k,x_{k+1},\Phi)) - \Phi^{-1}(G_1(x_k,x_{k+1},\Phi))), \tag{7}$$

$$\Phi^{-1}(\ln \mu(r,\varphi)) \geq r\frac{\Phi^{-1}(G_2(x_k,x_{k+1},\Phi))}{\Phi^{-1}(G_1(x_k,x_{k+1},\Phi))}, \tag{8}$$

where Φ^{-1} *is the inverse function to* Φ.

Proof. Let P be an arbitrary function defined on $(0,+\infty)$ and different from $+\infty$ (it can take on the value $-\infty$ but $P \not\equiv -\infty$) and let $Q(r) = \sup\{P(x) + rx\colon x \geq 0\}$, $-\infty < r < R$, be the function conjugated to P in the sense of Young. By $\Omega(-\infty,R)$, as in [4], we denote the class of positive unbounded functions on $(-\infty,R)$ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty,R)$. The functions Ψ,ϕ and the quantities $G_1(a,b,\Phi)$, $G_2(a,b,\Phi)$ we define as above. Then from Theorem 1 in [4] it follows that if $P(x_k) \geq -x_k\Psi(\phi(x_k))$ for some increasing to $+\infty$ sequence (x_k) of positive numbers then for all $r \in [\phi(x_k),\phi(x_{k+1})]$ and all $k \geq 1$ the estimates (5)–(8) hold with $Q(r)$ instead of $\ln \mu(r,\varphi)$. It is clear that the functions $\ln \mu(r,\varphi)$ and $\ln W_F(x)$ are conjugated in the sense of Young, and in view of the definition of $\Omega(0,R)$ for each function $\Phi \in \Omega(0,R)$ there exists $\Phi_1 \in \Omega(-\infty,R)$ such that $\Phi_1(r) = \Phi(r)$ for $r \in [r_0,R)$. Since $\Psi_1(r) = \Psi(r)$ for $r \in [r_0,R)$ and $\phi_1(r) = \phi(r)$ for $x \geq x_0 = x_0(r_0)$, Proposition 1 follows from the quoted result in [4]. □

We note in passing that ([4]) $G_1(a,b,\Phi) < G_2(a,b,\Phi)$ and the following lemma is hold ([4]–[6]).

Lemma 2. *For* $x > a$ *let* $G_*(x) = G_2(a,x,\Phi) - G_1(a,x,\Phi)$, $G_{**}(x) = \frac{G_2(a,x,\Phi)}{G_1(a,x,\Phi)}$ *and for* $x \in [a,b)$ *let* $G^*(x) = G_2(x,b,\Phi) - G_1(x,b,\Phi)$, $G^{**}(x) = \frac{G_2(x,b,\Phi)}{G_1(x,b,\Phi)}$. *Then the functions* G_* *and* G_{**} *are increasing on* $(a,+\infty)$ *and the functions* G^* *and* G^{**} *are decreasing on* (a,b).

3. Estimates from below of $\ln \mu(r,\varphi)$ **for the functions of finite order.** We begin with a theorem, which is a generalization of Proposition 1.

Theorem 1. *Let* φ *be an entire characteristic function of a probability law* F *such that*

$$\ln W_F(x_k) \geq -\frac{\rho-1}{\rho}\Big(\frac{1}{T\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}, \quad \rho > 1,\ T > 0, \tag{9}$$

for some increasing to $+\infty$ *sequence* (x_k) *of positive numbers. Then*

1) *if* $x_{k+1} - x_k \leq h(x_k)$ *for* $k \geq 1$, *where the function* h *is positive continuous and non-decreasing on* $[0,+\infty)$ *and* $h(x) = o(x)$ *as* $x \to +\infty$, *then*

$$\ln \mu(r,\varphi) \geq Tr^{\rho} - \frac{(1+o(1))}{8T\rho(\rho-1)} r^{2-\rho} h^2(T\rho r^{\rho-1}),\ r \to +\infty; \tag{10}$$

2) *if* $x_{k+1} \leq x_k\omega(x_{k+1})$ *for* $k \geq 1$, *where the function* ω *is continuous and non-decreasing on* $[0,+\infty)$ *and* $\omega(x) > 1$ *for all* $x \geq 0$, *then for all large enough* r

$$\ln \mu(r,\varphi) \geq \frac{T\rho^{\rho} r^{\rho}}{(\rho-1)^{\rho-1}} f\left(\omega(T\rho r^{\rho-1})\right), \quad f(\omega) = \frac{\omega(\omega-1)^{\rho-1}(\omega^{\frac{1}{\rho-1}}-1)}{(\omega^{\frac{\rho}{\rho-1}}-1)^{\rho}}.$$

Proof. It is easy to check that for the function $\Phi(r) = Tr^\rho$ with $\rho > 1$ the following equalities are true $\phi(x) = (\frac{x}{\rho T})^{\frac{1}{\rho-1}}$, $x\Psi(\phi(x)) = \frac{\rho-1}{\rho}(\frac{1}{T\rho})^{\frac{1}{\rho-1}} x^{\frac{\rho}{\rho-1}}$,

$$G_1(a,b,\Phi) = (\rho-1)\Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} \frac{ab}{b-a}\Big(b^{\frac{1}{\rho-1}} - a^{\frac{1}{\rho-1}}\Big),$$

$$G_2(a,b,\Phi) = (\rho-1)^\rho\Big(\frac{1}{T\rho^{\rho^2}}\Big)^{\frac{1}{\rho-1}} \Big(\frac{b^{\frac{\rho}{\rho-1}} - a^{\frac{\rho}{\rho-1}}}{b-a}\Big)^\rho.$$

Therefore,

$$G_1(x_k, x_k+h(x_k),\Phi) = (\rho-1)\Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big(1+\frac{h(x_k)}{x_k}\Big)\frac{x_k}{h(x_k)}\Big(\Big(1+\frac{h(x_k)}{x_k}\Big)^{\frac{1}{\rho-1}} - 1\Big) =$$
$$= (\rho-1)\Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big(1+\frac{h(x_k)}{x_k}\Big)\frac{x_k}{h(x_k)}\times$$
$$\times\Big\{\frac{1}{\rho-1}\frac{h(x_k)}{x_k} + \frac{2-\rho}{2(\rho-1)^2}\frac{h^2(x_k)}{x_k^2} + \frac{(2-\rho)(3-2\rho)}{6(\rho-1)^3}\frac{h^3(x_k)}{x_k^3} + O\Big(\frac{h^4(x_k)}{x_k^4}\Big)\Big\} =$$
$$= \Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big(1+\frac{h(x_k)}{x_k}\Big)\Big\{1 + \frac{2-\rho}{2(\rho-1)}\frac{h(x_k)}{x_k} + \frac{(2-\rho)(3-2\rho)}{6(\rho-1)^2}\frac{h^2(x_k)}{x_k^2} +$$
$$+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\} = \Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big\{1 + \frac{\rho}{2(\rho-1)}\frac{h(x_k)}{x_k} + \frac{\rho(2-\rho)}{6(\rho-1)^2}\frac{h^2(x_k)}{x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\},$$

$$G_2(x_k, x_k+h(x_k),\Phi) = (\rho-1)^\rho\Big(\frac{1}{T\rho^{\rho^2}}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho^2}{\rho-1}}\frac{1}{h(x_k)^\rho}\Big\{\Big(1+\frac{h(x_k)}{x_k}\Big)^{\frac{\rho}{\rho-1}} - 1\Big\}^\rho =$$
$$= (\rho-1)^\rho\Big(\frac{1}{T\rho^{\rho^2}}\Big)^{\frac{1}{\rho-1}} \frac{x_k^{\frac{\rho^2}{\rho-1}}}{h(x_k)^\rho}\Big\{\frac{\rho}{\rho-1}\frac{h(x_k)}{x_k} + \frac{\rho}{2(\rho-1)^2}\frac{h^2(x_k)}{x_k^2} + \frac{\rho(2-\rho)}{6(\rho-1)^2}\frac{h^3(x_k)}{x_k^3} +$$
$$+O\Big(\frac{h^4(x_k)}{x_k^4}\Big)\Big\}^\rho = \Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big\{1 + \frac{1}{2(\rho-1)}\frac{h(x_k)}{x_k} + \frac{2-\rho}{6(\rho-1)^2}\frac{h^2(x_k)}{x_k^2} +$$
$$+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}^\rho = \Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big\{1 + \frac{\rho}{2(\rho-1)}\frac{h(x_k)}{x_k} + \frac{\rho(5-\rho)}{24(\rho-1)^2}\frac{h^2(x_k)}{x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\},$$

as $k \to \infty$. Hence in view of the condition $x_{k+1} - x_k \le h(x_k)$ by Lemma 2 we have

$$G_2(x_k, x_{k+1},\Phi) - G_1(x_k, x_{k+1},\Phi) \le G_2(x_k, x_k+h(x_k),\Phi) - G_1(x_k, x_k+h(x_k),\Phi) =$$
$$= \Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}\Big(\frac{\rho}{8(\rho-1)}\frac{h^2(x_k)}{x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big) = \frac{\rho(1+o(1))}{8(\rho-1)}\Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}-2} h^2(x_k),$$

as $k \to \infty$, and since (9) implies (4) (see inequality (5))

$$\ln\mu(r,\varphi) \ge Tr^\rho - \frac{(1+o(1))}{8(\rho-1)}\Big(\frac{1}{T\rho}\Big)^{\frac{1}{\rho-1}} h^2(x_k) x_k^{\frac{2-\rho}{\rho-1}},\ k\to\infty,$$

for all $r \in [(x_k/T\rho)^{\frac{1}{\rho-1}}, (x_{k+1}/T\rho)^{\frac{1}{\rho-1}}]$, $k \ge k_0$. For such r we have $x_k \le T\rho r^{\rho-1} \le x_{k+1}$ and since the function h is non-decreasing and $x_{k+1} = (1+o(1))x_k$ as $k\to\infty$ then $\ln\mu(r,\varphi) \ge Tr^\rho - \frac{\rho(1+o(1))}{8(\rho-1)}\left(\frac{1}{T\rho}\right)^{\frac{1}{\rho-1}} h^2(T\rho r^{\rho-1})(T\rho r^{\rho-1})^{\frac{2-\rho}{\rho-1}}$ as $r\to+\infty$, whence (10) follows. The first part of Theorem 1 is proved.

For the proof of the second part we remark that

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1},\Phi\Big) = (\rho-1)\Big(\frac{1}{T\rho^\rho}\Big)^{\frac{1}{\rho-1}} \frac{x_{k+1}^{\frac{\rho}{\rho-1}}}{\omega(x_{k+1})-1}\frac{\omega^{\frac{1}{\rho-1}}(x_{k+1})-1}{\omega^{\frac{1}{\rho-1}}(x_{k+1})},$$

$$G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big) = (\rho-1)^{\rho}\Big(\frac{1}{T\rho\rho^2}\Big)^{\frac{1}{\rho-1}} \frac{x_{k+1}^{\frac{\rho}{\rho-1}}}{\omega^{\frac{\rho}{\rho-1}}(x_{k+1})}\Big(\frac{\omega^{\frac{1}{\rho-1}}(x_{k+1})-1}{\omega(x_{k+1})-1}\Big)^{\rho},$$

that is, by Lemma 2 in view of the condition $t_{k+1} \le t_k\omega(t_{k+1})$ we have

$$\frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \ge \frac{G_1(x_{k+1}/\omega(x_{k+1}), x_{k+1}, \Phi)}{G_2(x_{k+1}/\omega(x_{k+1}), x_{k+1}, \Phi)} = \frac{\rho^{\rho}}{(\rho-1)^{\rho-1}} f(\omega(t_{k+1})).$$

Therefore, by Lemma 1 $\ln\mu(r,\varphi) \ge Tr^{\rho}\frac{\rho^{\rho}}{(\rho-1)^{\rho-1}} f(\omega(x_{k+1}))$ for all r such as in the proof of the first part. Since $\omega(x_{k+1}) \le \omega(T\rho r^{\rho-1})$ we need to prove that the function $f(\omega)$ is decreasing on $[1,+\infty)$. It is easy to check that $f(\omega) = \xi^{\rho-1}(\omega) - \xi^{\rho}(\omega)$, where the function $\xi(\omega) = \frac{\omega-1}{\omega^{\frac{\rho}{\rho-1}}-1}$ decreases to 0 on $(1,+\infty)$ and $\xi(\omega) \uparrow \frac{\rho-1}{\rho}$ as $\omega \downarrow 1$, $\frac{\rho-1}{\rho} \ge \xi(\omega) > 0$ and $\xi'(\omega) < 0$ on $[1,+\infty)$. Hence it follows that $f'(\omega) = \rho\xi^{\rho-2}(\omega)\left(\frac{\rho-1}{\rho} - \xi(\omega)\right)\xi'(\omega) < 0$, i. e. f is a decreasing function. The second part of Theorem 1 is proved. □

Now we prove Proposition 1. By its assumption, $\ln W_F(x) \le -x_k^{\frac{\lambda+1}{\lambda}}$. Let $\rho > 1$ be an arbitrary number such that $\frac{\lambda+1}{\lambda} > \frac{\rho}{\rho-1}$. Then for every large enough k inequality (9) holds with $T = 1$. Since $x_{k+1} = O(x_k)$ as $k \to \infty$ we have $x_{k+1} \le Kx_k$, that is, the assumptions of Proposition 1 hold with $\omega(x) = K > 1$, and by Theorem 1 $\ln\mu(r,\varphi) \ge Ar^{\rho}$ for all large enough r, where A is a positive constant, whence it follows that $\ln\frac{\ln\mu(r,\varphi)}{r} \ge (\rho-1)\ln r + \ln A$. Letting here ρ to $\lambda+1$ we obtain the desired asymptotical inequality. Proposition 1 is proved.

From Theorem 1 the following proposition also follows.

Proposition 2. *If there exists an increasing to $+\infty$ sequence (x_k) such that (9) holds and $x_{k+1}/x_k \to 1$ $(k \to \infty)$ then $\ln M(r,\varphi) \ge (1+o(1))Tr^{\rho}$ as $r \to +\infty$.*

Indeed, since $x_{k+1} \le \omega x_k$ for an arbitrary $\omega > 1$ and all $k \ge k_0(\omega)$, by proposition 2) of Theorem 1 we have $\ln\mu(r,\varphi) \ge Tr^{\rho}\frac{\rho^{\rho}}{(\rho-1)^{\rho-1}} f(\omega)$. Since $\lim\limits_{\omega\downarrow 1} f(\omega) = \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}}$ we obtain hence the desired asymptotical inequality.

For analytic in $\mathbb{D}_R$ function the following theorem is an analog of Theorem 1.

Theorem 2. *Let φ be an analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of probability law F such that*

$$\ln W_F(x_k) \ge -Rx_k + \frac{\rho+1}{\rho}(T\rho)^{\frac{\rho}{\rho+1}} x_k^{\frac{\rho}{\rho+1}}, \ \rho > 0, \ T > 0, \tag{11}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then

1) if (x_k) satisfies the assumption of proposition 1) of Theorem 1 then

$$\ln\mu(r,\varphi) \ge \frac{T}{(R-r)^{\rho}} - \frac{(1+o(1))}{8T\rho(\rho+1)}(R-r)^{\rho+2}h^2\Big(\frac{T\rho}{(R-r)^{\rho+1}}\Big), \ r \uparrow R; \tag{12}$$

2) if (x_k) satisfies the assumption of proposition 2) of Theorem 1, then

$$\ln\mu(r,\varphi) \ge \frac{T(\rho+1)^{\rho+1}}{\rho^{\rho}(R-r)^{\rho}} f\Big(\omega\Big(\frac{T\rho}{(R-r)^{\rho+1}}\Big)\Big), \ \ f(\omega) = \frac{(\omega^{\frac{1}{\rho+1}}-1)(\omega^{\frac{\rho}{\rho+1}}-1)^{\rho}}{\omega^{\frac{\rho}{\rho+1}}(\omega-1)^{\rho+1}}. \tag{13}$$

Proof. It is easy to check that for the function $\Phi(r) = T(R-r)^{-\rho}$ we have $\phi(x) = R - \left(\frac{T\rho}{x}\right)^{\frac{1}{\rho+1}}$, $x\Psi(\phi(x)) = Rx - \frac{\rho+1}{\rho}(T\rho)^{\frac{1}{\rho+1}} x^{\frac{\rho}{\rho+1}}$, $G_1(a,b,\Phi) = (\rho+1)\left(\frac{T}{\rho^\rho}\right)^{\frac{1}{\rho+1}} \frac{ab}{b-a}(a^{-\frac{1}{\rho+1}} - b^{-\frac{1}{\rho+1}})$ and $G_2(a,b,\Phi) = \frac{(T\rho^\rho)^{\frac{1}{\rho+1}}}{(\rho+1)^\rho}\left(\frac{b-a}{b^{\frac{\rho}{\rho+1}} - a^{\frac{\rho}{\rho+1}}}\right)^\rho$. Therefore, as in the proof of Theorem 1, it is possible to show that $G_1\Big(t_k, t_k + h(t_k), \Phi\Big) = \left(\frac{T}{\rho^\rho}\right)^{\frac{1}{\rho+1}} x_k^{\frac{\rho}{\rho+1}}\Big\{1 - \frac{\rho}{2(\rho+1)}\frac{h(x_k)}{x_k} - \frac{\rho(2+\rho)}{6(\rho+1)^2}\frac{h^2(x_k)}{x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}$, $G_2(t_k, t_k + h(t_k), \Phi) = \left(\frac{T}{\rho^{\rho^2}}\right)^{\frac{1}{\rho+1}} x_k^{\frac{\rho}{\rho+1}}\Big\{1 - \frac{\rho}{2(\rho+1)}\frac{h(x_k)}{x_k} - \frac{\rho(5+\rho)}{24(\rho+1)^2}\frac{h^2(x_k)}{x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}$ and, thus, $G_2(x_k, x_k + h(x_k), \Phi) - G_1(x_k, x_k + h(x_k), \Phi) = \frac{1+o(1)}{8(\rho+1)}(T\rho)^{\frac{1}{\rho+1}} h^2(x_k) x_k^{-\frac{\rho+2}{\rho+1}}$, as $k \to \infty$, whence in view of the condition $x_{k+1} \le x_k + h(x_k)$ and lemmas 1 and 2 we obtain

$$\ln \mu(r,\varphi) \ge \frac{T}{(R-r)^\rho} - \frac{(T\rho)^{\frac{1}{\rho+1}}(1+o(1))}{8(\rho+1)} h^2(x_k) x_k^{-\frac{\rho+2}{\rho+1}}, \quad k \to \infty, \tag{14}$$

for all $r \in [R - (T\rho/x_k)^{\frac{1}{\rho+1}}, R - (T\rho/x_{k+1})^{\frac{1}{\rho+1}}]$ and all large enough k. For such r we have $x_k \le \frac{T\rho}{(R-r)^{\rho+1}} \le x_{k+1}$ and since the function h is non-decreasing and $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ (14) implies (12). The first part of Theorem 2 is proved.

We prove the second part. Since

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big) = (\rho+1)\Big(\frac{T}{\rho^\rho}\Big)^{\frac{1}{\rho+1}} x_{k+1}^{\frac{\rho}{\rho+1}} \frac{\omega^{\frac{1}{\rho+1}}(x_{k+1}) - 1}{\omega(x_{k+1}) - 1},$$

$$G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big) = \frac{(T\rho^{\rho^2})^{\frac{1}{\rho+1}}}{(\rho+1)^\rho} x_{k+1}^{\frac{\rho}{\rho+1}} \frac{\omega^{\frac{\rho}{\rho+1}}(x_{k+1})(\omega(x_{k+1}) - 1)^\rho}{(\omega^{\frac{\rho}{\rho+1}}(x_{k+1}) - 1)^\rho},$$

we have

$$\frac{G_1(x_{k+1}/\omega(x_{k+1}), x_{k+1}, \Phi)}{G_2(x_{k+1}/\omega(x_{k+1}), x_{k+1}, \Phi)} = \frac{(\rho+1)^{\rho+1}}{\rho^\rho} f(\omega(x_{k+1})),$$

and by Lemmas 1 and 2 $\ln \mu(r,\varphi) \ge \frac{T}{(R-r)^\rho}\frac{(\rho+1)^{\rho+1}}{\rho^\rho} f(\omega(x_{k+1}))$ for all r such as in Proposition 1) of this theorem. Since $\omega(x_{k+1}) \le \omega\Big(\frac{T\rho}{(R-r)^{\rho+1}}\Big)$ we need to prove, as above, that the function f is decreasing on $[1, +\infty)$. So, $f(\omega) = \frac{1}{\omega^{\frac{\rho}{\rho+1}}}\Big(\frac{\omega^{\frac{\rho}{\rho+1}} - 1}{\omega - 1}\Big)^\rho \frac{\omega^{\frac{1}{\rho+1}} - 1}{\omega - 1}$ and every factor is a decreasing function. □

From Theorem 2 the following two propositions follow.

Proposition 3. *If a probability law F satisfies the condition*

$$\ln\ln(W_F(x_k)e^{Rx_k}) \ge \frac{\lambda}{\lambda+1}\ln x_k, \quad \lambda > 0, \tag{15}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers and $x_{k+1} = O(x_k)$, $k \to \infty$, then for its characteristic function φ we have the following asymptotic inequality

$$\ln\ln M(r,\varphi) \ge (1+o(1))\lambda \ln\frac{1}{R-r}, \quad r \uparrow R. \tag{16}$$

Indeed, (15) implies $\ln W_F(x_k) \ge -Rx_k + x_k^{\frac{\lambda}{\lambda+1}} \ge -Rx_k + \frac{\rho+1}{\rho}\rho^{\frac{1}{\rho+1}} x_k^{\frac{\rho}{\rho+1}}$ for every $\rho < \lambda$ and all large enough k, that is, (11) holds and since $x_{k+1} \le Kx_k$ for all k by item 2) of Theorem 2 we have $\ln \mu(r,\varphi) \ge \frac{A}{(R-r)^\rho}$, where A is a positive constant, whence $\ln\ln \mu(r,\varphi) \ge \rho \ln\frac{1}{R-r} + O(1)$, $r \uparrow R$. In view of the arbitrariness of ρ we obtain (16).

Proposition 4. *If for a probability law F condition (11) holds and $x_{k+1} = (1 + o(1))x_k$ as $k \to \infty$ then $\ln M(r, \varphi) \geq \frac{(1+o(1))T}{(R-r)^\rho}$ as $r \uparrow R$.*

Proposition 4 easy follows from item 2) of Theorem 2, because $\lim\limits_{\omega\downarrow 1} f(\omega) = \frac{\rho^\rho}{(\rho+1)^{\rho+1}}$. We remark that if in item 1) of Theorems 1–2 $x_{k+1} - x_k = h \equiv \text{const}$ and in item 2) of these theorems $x_{k+1}/x_k = \omega \equiv \text{const}$ then we need not use Lemma 2, that is, we need not estimate of $G_2 - G_1$ and G_1/G_2. Therefore, in view of the optimality of estimates (5) and (6), which we used in the proof of theorems 1–2, in the cases where $x_{k+1} - x_k = h$ and $x_{k+1}/x_k = \omega$ estimates (10), (12) and corresponding (1), (13) are unimprovable.

4. Generalized results. Since we not always can find G_1 and G_2 in an explicit way, the following theorem is useful.

Theorem 3. *Let $0 < R \leq +\infty$, $\Phi \in \Omega(0, R)$ be such that $\Phi(r)\Phi'(r)^{-1-\eta}$ non-increase on $[r_0, R)$ for some $r_0 \in (0, R)$ and $\eta \in [0, +\infty)$, let φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F, which satisfies condition (3) and let inequality (4) hold for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then*

1) *if $x_{k+1} - x_k \leq h(x_k)$, $k \geq 1$, where a positive and continuous on $(0, +\infty)$ function h is such that $h(x) = o(x)$ as $x \to \infty$, the function $x + h(x)$ increases and the function $x^\eta h(x)$ non-decreases on $(0, +\infty)$, then*

$$\ln \mu(r, \varphi) \geq \Phi(r) - (1 + o(1))\frac{1+\eta}{2}\frac{\Phi(r)}{\Phi'(r)}h(\Phi'(r)), \quad r \uparrow R; \tag{17}$$

2) *if $x_{k+1} \leq x_k\omega(x_{k+1})$, $k \geq 1$, where a continuous and non-decreasing on $(0, +\infty)$ function ω is such that $\omega(x) > 1$ for $x > 0$, then*

$$\ln \mu(r, \varphi) \geq \frac{\omega^\eta(\Phi'(r)) - 1}{\eta\omega^\eta(\Phi'(r))(\omega(\Phi'(r)) - 1)}\Phi(r) \tag{18}$$

for all $r < R$ close enough to R.

Proof. At first we assume that $\eta > 0$ and prove item 1). From the non-increase of $\frac{\Phi(r)}{\Phi'(r)^{1+\eta}}$ we have

$$\begin{aligned} G_1(x_k, x_k + h(x_k), \Phi) &= \frac{x_k(x_k + h(x_k))}{h(x_k)} \int_{x_k}^{x_k+h(x_k)} \frac{\Phi(\phi(x))}{x^{1+\eta}} x^{\eta-1}dx \geq \\ &\geq \frac{x_k(x_k + h(x_k))}{h(x_k)} \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^{1+\eta}} \frac{(x_k + h(x_k))^\eta - x_k^\eta)}{\eta} = \\ &= \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^\eta} \frac{x_k^{1+\eta}}{\eta h(x_k)} \Big\{\Big(1 + \frac{h(x_k)}{x_k}\Big)^\eta - 1\Big\} = \\ &= \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^\eta} \frac{x_k^{1+\eta}}{\eta h(x_k)} \Big\{\frac{\eta h(x_k)}{x_k} + \frac{\eta(\eta-1)h^2(x_k)}{2x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\} = \\ &= \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^\eta} x_k^\eta \Big\{1 + \frac{(\eta-1)h(x_k)}{2x_k} + O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big\}, \quad k \to \infty, \end{aligned}$$

$$G_2(x_k, x_k + h(x_k), \Phi) = \Phi\Big(\frac{1}{h(x_k)} \int_{x_k}^{x_k+h(x_k)} \phi(t)dt\Big) \leq \Phi(\phi(x_k + h(x_k))).$$

Therefore,

$$\begin{gathered}G_2(x_k, x_k+h(x_k), \Phi) - G_1(x_k, x_k+h(x_k), \Phi) \le \\ \le \Phi(\phi(x_k+h(x_k)))\Big\{1-\Big(\frac{x_k}{x_k+h(x_k)}\Big)^{\eta}\Big(1+\frac{(\eta-1)h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big)\Big\}= \\ = \Phi(\phi(x_k+h(x_k)))\left\{1-\frac{1+\frac{(\eta-1)h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)}{1+\eta\frac{h(x_k)}{x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)}\right\}= \Phi(\phi(x_k+h(x_k)))\times \\ \times\Big\{1-\Big(1+\frac{(\eta-1)h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big)\Big(1-\frac{\eta h(x_k)}{x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big)\Big\}= \\ = \Phi(\phi(x_k+h(x_k)))\Big\{\frac{1+\eta}{2}\frac{h(x_k)}{x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big\}= \\ = \frac{\Phi(\phi(x_k+h(x_k)))}{(x_k+h(x_k))^{1+\eta}}\frac{1+\eta}{2}h(x_k)x_k^{\eta}(1+o(1)), \quad k\to\infty.\end{gathered}$$

Hence in view of the condition $x_{k+1} \le x_k + h(x_k)$ using Lemma 2 (growth of G_*) and inequality (5) we obtain

$$\ln\mu(r,\varphi) \ge \Phi(r) - \frac{\Phi(\phi(x_k+h(x_k)))}{(x_k+h(x_k))^{1+\eta}}\frac{1+\eta}{2}x_k^{\eta}h(x_k)(1+o(1)), \quad k\to\infty, \tag{19}$$

for all $r \in [\phi(x_k), \phi(x_{k+1})]$. Since $\Phi(\varphi(t))t^{-\eta-1}$ non-increases, $x^{\eta}h(x)$ non-decreases and the inequalities $\phi(x_k) \le r \le \phi(x_{k+1})$ imply the inequalities $x_k \le \Phi'(r) \le x_{k+1}$, we obtain

$$\begin{gathered}\ln\mu(r,\varphi) \ge \Phi(r) - \frac{\Phi(\phi(x_k))}{x_k^{1+\eta}}\frac{1+\eta}{2}x_k^{\eta}h(x_k)(1+o(1)) \ge \\ \ge \Phi(r) - \frac{\Phi(r)}{\Phi'(r)^{1+\eta}}\frac{1+\eta}{2}\Phi'(r)^{\eta}h(\Phi'(r))(1+o(1))\end{gathered}$$

i.e., inequality (17) holds.

If $\eta = 0$ then by analogy we have

$$\begin{gathered}G_1(x_k, x_k+h(x_k), \Phi) \ge \frac{x_k(x_k+h(x_k))}{h(x_k)}\frac{\Phi(\phi(x_k+h(x_k)))}{x_k+h(x_k)}\ln\Big(1+\frac{h(x_k)}{x_k}\Big)= \\ = \Phi(\phi(x_k+h(x_k)))\Big(1-\frac{h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big), \quad k\to\infty,\end{gathered}$$

$$G_2(x_k, x_k+h(x_k), \Phi) - G_1(x_k, x_k+h(x_k), \Phi) \le \frac{\Phi(\phi(x_k+h(x_k)))}{x_k+h(x_k)}\frac{h(x_k)}{2}(1+o(1)), \ k\to\infty,$$

whence we obtain (19) with $\eta = 0$. Hence, as above estimate (17) follows. The first part of Theorem 3 is proved.

We prove second part. For $\eta > 0$ we have

$$\begin{gathered}G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big)= \frac{x_{k+1}}{\omega(x_{k+1})-1}\int_{\frac{x_{k+1}}{\omega(x_{k+1})}}^{x_{k+1}}\frac{\Phi(\phi(x))}{x^{1+\eta}}x^{\eta-1}dx \ge \\ \ge \frac{x_{k+1}}{\omega(x_{k+1})-1}\frac{\Phi(\phi(x_{k+1}))}{x_{k+1}^{1+\eta}}\frac{1}{\eta}\Big(x_{k+1}^{\eta}-\frac{x_{k+1}^{\eta}}{\omega^{\eta}(x_{k+1})}\Big)= \frac{\Phi(\phi(x_{k+1}))}{\eta(\omega(x_{k+1})-1)}\Big(1-\frac{1}{\omega^{\eta}(x_{k+1})}\Big), \\ G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big)\le \Phi(\phi(x_{k+1})).\end{gathered}$$

Therefore, in view of the condition $x_{k+1}\le x_k\omega(x_{k+1})$ using Lemma 2 and inequality (6) for all $r\in[\phi(x_k),\phi(x_{k+1})]$ and all $k\ge k_0$ we have

$$\ln\mu(r,\phi)\ge\Phi(r)\frac{\omega^\eta(x_{k+1})-1}{\eta\omega^\eta(x_{k+1})(\omega(x_{k+1})-1)}\ge\Phi(r)\frac{\omega^\eta(\Phi'(r))-1}{\eta\omega^\eta(\Phi'(r))(\omega(\Phi'(r))-1)},$$

because the function $f(x)=\frac{x^\eta-1}{x^\eta(x-1)}$ decreases on $[1,+\infty)$ and $x_{k+1}\ge\Phi'(r)$. The inequality (18) is proved.

If $\eta=0$ then, by analogy, we have

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})},x_{k+1},\Phi\Big)\ge\frac{x_{k+1}}{\omega(x_{k+1})-1}\frac{\Phi(\phi(x_{k+1}))}{x_{k+1}}\ln\omega(x_{k+1})=\Phi(\phi(x_{k+1}))\frac{\ln\omega(x_{k+1})}{\omega(x_{k+1})-1}$$

and in view of the estimates $G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})},x_{k+1},\Phi\Big)\le\Phi(\phi(x_{k+1}))$, as above, we obtain

$$\ln\mu(r,\varphi)\ge\Phi(r)\frac{\ln\omega(x_{k+1})}{\omega(x_{k+1})-1}\ge\Phi(r)\frac{\ln\omega(\Phi'(r))}{\omega(\Phi'(r))-1}\tag{20}$$

for all $r<R$ close enough to R. Since $\frac{\omega^\eta-1}{\eta\omega^\eta(\omega-1)}\to\frac{\ln\omega}{\omega-1}$ as $\eta\to0$ estimate (20) coincides with estimate (18) with $\eta=0$. □

The condition of the non-increase of $\Phi(r)(\Phi'(r))^{-1-\eta}$ can be removed if we use estimates (7) and (8) from Lemma 1. We get the following theorem.

Theorem 4. *Let $\Phi\in\Omega(0,R), 0<R\le+\infty$, and φ be an analytic characteristic function of a probability law, which satisfies conditions (3) and (4) for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then*

1) *if $\phi(x_{k+1})-\phi(x_k)\le h(x_{k+1})$, where h is a positive continuous and non-increasing function on $(0,+\infty)$ such that $R>\phi(x)-h(x)\to R$ as $x\to+\infty$, then for all $r<R$ close enough to R*
$$\ln\mu(r,\varphi)\ge\Phi(r-h(\Phi'(r)));\tag{21}$$
2) *if $\phi(x_{k+1})\le\phi(x_k)\omega(x_{k+1})$, where ω is a positive continuous and non-increasing function on $(0,+\infty)$ such that $R>\frac{\phi(x)}{\omega(x)}\to R$ as $x\to+\infty$, then for all $r<R$ close enough to R*
$$\ln\mu(r,\varphi)\ge\Phi\Big(\frac{r}{\omega(\Phi'(r))}\Big).\tag{22}$$

Proof. Since the function $\Phi(\phi(t))$ increases we have

$$\Phi^{-1}(G_1(x_k,x_{k+1},\Phi))\ge\Phi^{-1}\Big(\frac{x_kx_{k+1}}{x_{k+1}-x_k}\Phi(\phi(x_k))\int_{x_k}^{x_{k+1}}\frac{dx}{x^2}\Big)=\phi(x_k),$$
$$\Phi^{-1}(G_2(x_k,x_{k+1},\Phi))=\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\phi(t)dt\le\phi(x_{k+1}).$$

Therefore, from (7) and (8) for all $r\in[\phi(x_k),\phi(x_{k+1})]$ we obtain respectively

$$\Phi^{-1}(\ln\mu(r,\varphi))\ge r-(\phi(x_{k+1})-\phi(x_k))\ge r-h(x_{k+1})\ge r-h(\Phi'(r)),$$
$$\Phi^{-1}(\ln\mu(r,\varphi))\ge r\frac{\varphi(x_k)}{\varphi(x_{k+1})}\ge\frac{r}{\omega(x_{k+1})}\ge\frac{r}{\omega(\Phi'(r))},$$

whence the inequalities (21) and (22) follows. □

5. Corollaries. Let L be a class of continuous increasing functions α such that $\alpha(x) \ge 0$ for $x \ge x_0$, $\alpha(x) = \alpha(x_0)$ for $x \le x_0$ and on $[x_0, +\infty)$ the function α increases to $+\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$; further $\alpha \in L_{\text{si}}$ if $\alpha(cx) = (1+o(1))\alpha(x))$ as $x \to +\infty$ for any $c \in (0, +\infty)$. It is easy to see that $L_{\text{si}} \subset L^0$.

Corollary 1. *Let either $\alpha \in L_{\text{si}}$ and $\beta \in L^0$ or $\alpha \in L^0$ and $\beta \in L_{\text{si}}$ and φ be an entire characteristic function of a probability law F such that*

$$\beta\Big(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)}\Big) \le \alpha(x_k) \tag{23}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers, which satisfies the condition $\beta^{-1}(c\alpha(x_{k+1}))/\beta^{-1}(c\alpha(x_k)) \to 1$ as $k \to \infty$ for any $c \in (1, +\infty)$. Then

$$\alpha\Big(\frac{\ln \mu(r,\varphi)}{r}\Big) \ge (1+o(1))\beta(r), \quad r \to \infty. \tag{24}$$

Proof. Let at first $\alpha \in L_{\text{si}}$, $\beta \in L^0$ and $\varepsilon \in (0,1)$ be an arbitrary number. Since $\beta \in L^0$, we have ([7]) $\beta(\frac{x}{1-\varepsilon}) \le (1+\delta_1(\varepsilon))\beta(x)$, where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ and, thus, $\beta^{-1}(x) \le (1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))x)$, and the condition $\alpha \in L_{\text{si}}$ implies $\alpha(\varepsilon x) = (1+o(1))\alpha(x)$ as $x \to +\infty$, that is, for any $\delta_2 > 0$ and all large enough x the inequality $\alpha(\varepsilon x) \ge \frac{1}{1+\delta_2}\alpha(x)$ is true. Therefore, $x\beta^{-1}(\alpha(x)) \le x(1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))\alpha(x)) \le (1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(\varepsilon x))$ for all large enough x. On the other hand

$$\int_{x_0}^{x} \beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt \ge \int_{\varepsilon x}^{x} \beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt \ge$$
$$\ge \beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(\varepsilon x))(1-\varepsilon)x.$$

Hence it follows from (23) that

$$\ln W_F(x_k) \ge -x_k\beta^{-1}(\alpha(x_k)) \ge -(1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(\varepsilon x_k)) \ge$$
$$\ge -\int_{x_0}^{x_k} \beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt \tag{25}$$

for each $\varepsilon \in (0,1)$, $\delta_2 > 0$ and all $k \ge k_0 = k_0(\varepsilon, \delta_2)$.

We put $\Phi(r) = \int_{r_0}^{r} \alpha^{-1}\big(\frac{\beta(t)}{1+\delta}\big)dt$, where $1+\delta < (1+\delta_1(\varepsilon))(1+\delta_2)$. Then $\Phi'(r) = \alpha^{-1}\Big(\frac{\beta(r)}{1+\delta}\Big)$, $\phi(x) = \beta^{-1}((1+\delta)\alpha(x))$ and

$$x\Psi(\phi(x)) = \int_{x_0}^{x} \phi(t)dt + \text{const} \le \int_{x_0}^{x} \beta^{-1}((1+\delta)\alpha(t))dt + \text{const} \le$$
$$\le \int_{x_0}^{x} \beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt.$$

Therefore, inequality (25) implies (4) for all large enough k.

Further, since $\frac{\beta^{-1}((1+\delta)\alpha(x_{k+1}))}{\beta^{-1}((1+\delta)\alpha(x_k))} \to 1$ $(k \to \infty)$, there exists a decreasing to 1 continuous function ω such that $\frac{\phi(x_{k+1})}{\phi(x_k)} \le \omega(x_{k+1})$ for all k. Therefore, by item 2) of Theorem 4 inequality (22) is true, that is, in view of the condition $\beta \in L^0$ we have

$$\ln \mu(r,\varphi) \ge \Phi\Big(\frac{r}{\omega(\Phi(r))}\Big) = \Phi((1+o(1))r) = \int_{r_0}^{r(1+o(1))} \alpha^{-1}\Big(\frac{\beta(x)}{1+\delta}\Big)dx \ge$$
$$\ge \int_{(1-\varepsilon)r}^{r} \alpha^{-1}\Big(\frac{\beta(x)}{(1+\delta)^2}\Big)dx \ge \alpha^{-1}\Big(\frac{\beta((1-\varepsilon)r)}{(1+\delta)^2}\Big)\varepsilon r$$

for all large enough r. Since $\alpha\in L_{\rm si}$, $\beta\in L^0$ and the numbers ε,δ_2 and δ are arbitrary, from the latter inequality we easily obtain (24).

If $\alpha\in L^0$ and $\beta\in L_{\rm si}$ then $\alpha((1-\varepsilon)x)\ge\frac{1}{1+\delta_1(\varepsilon)}\alpha(x)$, where $\delta_1(\varepsilon)\to0$ as $\varepsilon\to0$ and $\beta(\varepsilon x)\ge\frac{1}{1+\delta_2}\beta(x)$ for all large enough x. Therefore, as above

$$\int_{x_0}^{x}\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt\ge\varepsilon x\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha((1-\varepsilon)x))\ge$$
$$\ge\varepsilon x\beta^{-1}((1+\delta_2)\alpha(x))\ge x\beta^{-1}\Big(\frac{1}{1+\delta_2}\beta(\beta^{-1}((1+\delta_2)\alpha(x))\Big)=x\beta^{-1}(\alpha(x)).$$

Hence it follows from (23) that $\ln W_F(x_k)\ge-\int_{x_0}^{x_k}\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt$ for any $\varepsilon\in(0,1)$, $\delta_2>0$ and all $k\ge k_0=k_0(\varepsilon,\delta_2)$. Therefore, choosing $\Phi(r)$, as above, and repeating the arguments, we again arrive at inequality (24). □

For analytic functions in $\mathbb{D}_R$, $0<R<+\infty$, the following corollary is an analog of Corollary 1.

Corollary 2. *Let $\alpha\in L_{\rm si}$, $\beta\in L_{\rm si}$, $\frac{d\ln\beta^{-1}(\alpha(x))}{d\ln x}\le q<1$ for all large enough x and $\alpha\Big(\frac{x}{\beta^{-1}(\alpha(x))}\Big)=(1+o(1))\alpha(x)$ as $x\to+\infty$, and φ be an analytic in $\mathbb{D}_R$, $0<R<+\infty$, characteristic function of a probability law F, for which*

$$\beta\Big(\frac{x_k}{\ln(W_F(x_k)e^{Rx_k})}\Big)\le\alpha(x_k)\tag{26}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\beta^{-1}(\alpha(x_{k+1}))=O(\beta^{-1}(\alpha(x_k)))$ as $k\to\infty$. Then

$$\alpha(\ln\mu(r,\varphi))\ge(1+o(1))\beta\Big(\frac{1}{R-r}\Big),\quad r\uparrow R.\tag{27}$$

Proof. From (26) it follows that $\ln W_F(x_k)\ge-Rx_k+\frac{x_k}{\beta^{-1}(\alpha(x_k))}$. Since $\frac{d\ln\beta^{-1}(\alpha(x))}{d\ln x}\le q<1$, we have $\frac{x}{\beta^{-1}(\alpha(x))}\uparrow+\infty$ $(r_0\le x\to+\infty)$, and using L'Hospital's rule it is easy to show that

$$\frac{x}{\beta^{-1}(\alpha(x))}\ge(1+o(1))(1-q)\int_{x_0}^{x}\frac{dt}{\beta^{-1}(\alpha(t))},\quad x\to\infty.$$

Therefore,

$$\ln W_F(x_k)\ge-Rx_k+(1-q_1)\int_{x_0}^{x_k}\frac{dt}{\beta^{-1}(\alpha(t))}\tag{28}$$

for any $q_1\in(q,1)$ and all large enough k. We put

$$\Phi(r)=\int_{r_0}^{r}\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{R-x}\Big)\Big)dx,\tag{29}$$

where $q_2\in(q_1,1)$. Then $\Phi'(r)=\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{R-r}\Big)\Big)$, $\phi(x)=R-\frac{1-q_2}{\beta^{-1}(\alpha(x))}$ and

$$x\Psi(\phi(x))=Rx-(1-q_2)\int_{x_0}^{x}\frac{dt}{\beta^{-1}(\alpha(t))}+\text{const},$$

that is, in view of (28) and $q_1 < q_2$ we obtain (4). Since $\beta^{-1}(\alpha(x_{k+1})) \le K\beta^{-1}(\alpha(x_k))$, $K > 1$, for all $k \ge 1$, we have $\frac{1}{\beta^{-1}(\alpha(x_k))} - \frac{1}{\beta^{-1}(\alpha(x_{k+1}))} \le \frac{K-1}{\beta^{-1}(\alpha(x_{k+1}))}$. Therefore, if we put $h(x) = \frac{(K-1)(1-q_2)}{\beta^{-1}(\alpha(x))}$ then $\phi(x) - h(x) = R - \frac{K(1-q_2)}{\beta^{-1}(\alpha(x))} \to R$ as $x \to +\infty$, $h(\Phi'(r)) = (K-1)(R-r)$ and $\phi(x_{k+1}) - \phi(x_k) \le h(x_{k+1})$ for $k \ge 1$.

Finally, for function (29) and $r > \max\{r_0, R/2\}$ we have

$$\Phi(r) \ge \int_{2r-R}^{r} \alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{R-x}\Big)\Big)dx \ge (R-r)\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{2(R-r)}\Big)\Big).$$

Therefore, by item 1) of Theorem 4

$$\ln \mu(r,\varphi) \ge (R - r + h(\Phi'(r)))\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{2(R-r+h(\Phi'(r)))}\Big)\Big) =$$
$$= K(R-r)\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{2K(R-r)}\Big)\Big)$$

for all $r < R$ close enough to R. But from the condition $\alpha(\frac{x}{\beta^{-1}(\alpha(x))}) = (1+o(1))\alpha(x)$ as $x \to +\infty$ it follows that $\alpha(\frac{\alpha^{-1}(\beta(t))}{t}) = (1+o(1))\beta(t)$ as $t \to \infty$ and since $\alpha \in L_{\text{si}}$, $\beta \in L_{\text{si}}$ the last inequality implies (27). □

We remark that under the other conditions of Corollary 2 the condition $\beta^{-1}(\alpha(x_{k+1})) = O(\beta^{-1}(\alpha(x_k)))$ as $k \to \infty$ holds provided $x_{k+1} = O(x_k)$ as $k \to \infty$.

The conditions on α and β in Corollary 2 assume that the function α increases slower than the function β. In the case where α increases quicker than β, the following corollary is true.

Corollary 3. *Let $\alpha \in L_{\text{si}}$, $\beta \in L_{\text{si}}$, $\frac{d\ln\alpha^{-1}(\beta(x))}{d\ln x} \le q < 1$ for all large enough x, $\frac{d\alpha^{-1}(\beta(x))}{dx} = \frac{1}{f(x)} \downarrow 0$ and $\alpha^{-1}(\beta(f(x))) = O(\alpha^{-1}(\beta(x)))$ as $x \to +\infty$, and φ be an analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of a probability law F, for which*

$$\alpha(\ln(W_F(x_k)e^{Rx_k})) \ge \beta(x_k) \tag{30}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\overline{\lim\limits_{k\to\infty}} \frac{f(x_{k+1})}{f(x_k)} < 2$. Then asymptotical inequality (27) holds.

Proof. If we put $x\Psi(\phi(x)) = Rx - \alpha^{-1}(\beta(x))$ then (30) implies (4) and $\phi(x) = (x\Psi(\phi(x)))' = R - \frac{d\alpha^{-1}(\beta(x))}{dx} = R - \frac{1}{f(x)}$. Hence it follows that $\Phi'(r) = f^{-1}(\frac{1}{R-r})$,

$$\Phi(r) - \Phi(r_0) = \int_{r_0}^{r} f^{-1}\Big(\frac{1}{R-x}\Big)dx = \int_{r_1}^{f^{-1}(\frac{1}{R-r})} td\Big(-\frac{1}{f(t)}\Big) =$$
$$= -(R-r)f^{-1}\Big(\frac{1}{R-r}\Big) + \alpha^{-1}\Big(\beta\Big(f^{-1}\Big(\frac{1}{R-r}\Big)\Big)\Big) \ge (1-q)\alpha^{-1}\Big(\beta\Big(f^{-1}\Big(\frac{1}{R-r}\Big)\Big)\Big).$$

But from the condition $\alpha^{-1}(\beta(f(x))) = O(\alpha^{-1}(\beta(x)))$ as $x \to +\infty$ it follows that

$$\alpha^{-1}\Big(\beta\Big(\frac{1}{R-r}\Big)\Big) \le K\alpha^{-1}\Big(\beta\Big(f^{-1}\Big(\frac{1}{R-r}\Big)\Big)\Big),$$

where K is a positive constant. Therefore, $\Phi(r) \geq K_1\alpha^{-1}(\beta(\frac{1}{R-r}))$, where K_1 is a positive constant, for all $r < R$ close enough to R, and if $h(x) = a(R - \phi(x))$, $0 < a < 1$, then

$$\Phi(r - h(\Phi'(r))) \geq K_1\alpha^{-1}\Big(\beta\Big(\frac{1}{(1+a)(R-r)}\Big)\Big). \tag{31}$$

Under such a choice of the function h the condition $\phi(x_{k+1}) - \phi(x_k) \leq h(x_{k+1})$ is equivalent to the condition $f(x_{k+1}) \leq (1+a)f(x_k)$, and the latter condition follows from the condition $\overline{\lim\limits_{k\to\infty}} \frac{f(x_{k+1})}{f(x_k)} < 2$. Therefore, by item 1) of Theorem 4 inequality (21) is true and in view of (31) and the conditions $\alpha \in L_{\text{si}}$, $\beta \in L_{\text{si}}$ we obtain (27). □

We remark that from Corollaries 1–3 one can obtain analogues of Propositions 1–4, but we shall not discuss this here.

REFERENCES

1. Linnik Yu.V., Ostrovskii I.V. Decomposition of random variables and vectors. – Moscow: Nauka, 1972. – 479 p. (in Russian)
2. Skaskiv O.B., Sorokivs'kyj V.M. *On the maximum of module of characteristic functions of probabilistic law*// Kraj. Zadachi Dyfer. Rivnyan'. – 2001. – V.7. – P. 286–289. (in Ukrainian)
3. Jakovleva N.I. *On the growth of entire characteristic functions of probability laws*// Problems of mathematical physics and funct. analysis. – K.: Naukova dumka, 1976. – P. 43–54. (in Russian)
4. Sheremeta M.M., Sumyk O.M. *A connection between the growth of Young conjugated functons*// Mat. Stud. – 1999. – V.11, №1. – P. 41–47. (in Ukrainian)
5. Sheremeta M.M., Sumyk O.M. *Lower estimates for the maximal term of a Dirichlet series*// Izv. Vyssh. Uchebn. Zaved., Mat. – 2001. – №4. – 53–57. (in Russian)
6. Sumyk O.M. *Lower estimates for maximal term of Dirichlet series*// Visn. L'viv. Univ., ser. mekh.-math. – 1999. – V.53. – P. 40–44. (in Ukrainian)
7. Sheremeta M.M. *On two classes of positive functions and belonging to them of main characteristics of entire functions*// Mat. Stud. – 2003. – V.19, №11. – P. 74–82.

Ivan Franko Natonal University of L'viv
marta0691@rambler.ru
m_m_sheremeta@list.ru

Received 10.05.2012