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ESTIMATES FROM BELOW FOR CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS

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Let φ be the characteristic function of a probability law F that is analytic in $\mathbb{D}_R=\{z\colon |z|<R\}$, $0<R\le+\infty$, $M(r,\varphi)=\max\{|\varphi(z)|\colon |z|=r<R\}$ and $W_F(x)=1-F(x)+F(-x)$, $x\ge0$. A connection between the growth of $M(r,\varphi)$ and the decrease it of $W_F(x)$ is investigated in terms of estimates from below. For entire characteristic functions it is proved, for example, that if $\ln x_k\ge\lambda\ln(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)})$ for some increasing sequence (x_k) such that $x_{k+1}=O(x_k)$, $k\to\infty$, then $\ln\frac{\ln M(r,\varphi)}{r}\ge(1+o(1))\lambda\ln r$ as $r\to+\infty$.

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Пусть φ — аналитическая в $\mathbb{D}_R=\{z\colon |z|<R\}$, $0<R\le+\infty$, характеристическая функция вероятностного закона F, $M(r,\varphi)=\max\{|\varphi(z)|\colon |z|=r<R\}$ и $W_F(x)=1-F(x)+F(-x)$, $x\ge0$. В терминах оценок снизу изучена связь между ростом $M(r,\varphi)$ и убыванием $W_F(x)$. Например, для целых характеристических функций доказано, что если $\ln x_k\ge\lambda\ln(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)})$ для некоторой возрастающей последовательности (x_k) такой, что $x_{k+1}=O(x_k)$, $k\to\infty$, то $\ln\frac{\ln M(r,\varphi)}{r}\ge(1+o(1))\lambda\ln r$ при $r\to+\infty$.

1. Introduction. A non-decreasing function F continuous on the left on $(-\infty,\infty)$ is said ([1, p. 10]) to be a *probability law*, if $\lim\limits_{x\to+\infty}F(-x)=0$ and $\lim\limits_{x\to+\infty}F(x)=1$, and the function $\varphi(z)=\int_{-\infty}^{\infty}e^{izx}dF(x)$ defined for real z is called ([1, p. 12]) a *characteristic function of this law*. If φ has an analytic continuation on the disk $\mathbb{D}_R=\{z\colon |z|<R\}$, $0<R\le+\infty$, then we call φ an analytic in $\mathbb{D}_R$ characteristic function of the law F. Further we always assume that $\mathbb{D}_R$ is the maximal disk of the analyticity of φ. It is known that φ is an analytic in $\mathbb{D}_R$ characteristic function of the probability law F if and only if for every $r\in[0,R)$

$$W_F(x)=:1-F(x)+F(-x)=O\left(e^{-rx}\right),\ x\to+\infty.$$

Hence it follows that

$$\varliminf_{x\to+\infty}\frac{1}{x}\ln\frac{1}{W_F(x)}=R.$$

If we put $M(r,\varphi)=\max\{|\varphi(z)|\colon |z|=r\}$ for $r<R$ then $W_F(x)e^{rx}\le 2M(r,\varphi)$ for all $x\ge0$ and $r\in[0,R)$. For $R=+\infty$ this inequality is proved in [1, p.54] and for $R<+\infty$ the proof is analogous. Therefore, if we define (see also [2]) $\mu(r,\varphi)=\sup\{W_F(x)e^{rx}\colon x\ge0\}$

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then $\mu(r,\varphi) \le 2M(r,\varphi)$. Thus, the estimates from below for $\ln M(r,\varphi)$ follow from such estimates for $\ln \mu(r,\varphi)$. For entire characteristic functions N. I. Jakovleva ([3]) proved that, if $\overline{\lim\limits_{x\to+\infty}} \frac{\ln\ln(1/W_F(x))}{\ln x} = 1 + \frac{1}{\lambda}$ then $\varliminf\limits_{r\to+\infty} \frac{\ln\ln M(r,\varphi)}{\ln r} \ge 1+\lambda$. Hence it follows that if

$$\ln x \ge \lambda \ln\Big(\frac{1}{x}\ln\frac{1}{W_F(x)}\Big),\ x \ge x_0, \tag{1}$$

then

$$\ln\frac{\ln M(r,\varphi)}{r} \ge (1+o(1))\lambda\ln r,\ r\to+\infty. \tag{2}$$

The question arises, whether asymptotical inequality (2) is valid if condition (1) holds not necessarily for all $x \ge x_0$ but only for some increasing unbounded sequence (x_k). In view of the inequality $\ln M(r,\varphi) \ge \ln\mu(r,\varphi) - \ln 2$, the following assertion gives a positive answer to this question.

Proposition 1. *If there exists an increasing to $+\infty$ sequence (x_k) such that $\ln x_k \ge \lambda\ln\left(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)}\right)$ for all $k \ge 1$ and $x_{k+1} = O(x_k)$ as $k\to\infty$ then $\ln\frac{\ln\mu(r,\varphi)}{r} \ge (1+o(1))\lambda\ln r$ as $r\to+\infty$.*

We obtain Proposition 1 from main results proved below for characteristic functions that are entire or analytic in the disk characteristic functions.

2. Auxiliary results. If φ is an entire characteristic function and $\varphi \not\equiv \text{const}$ then ([1, p. 45]) $\lim\limits_{r\to+\infty} r^{-1}\ln M(r,\varphi) = \sigma \in (0,+\infty]$. If $\sigma < +\infty$ then the estimate $\ln M(r,\varphi)$ from below is trivial. It can be shown that $\sigma < +\infty$ provided $W_F(x) = 0$ for all $x \ge x_0$. Therefore we assume in what follows that $W_F(x) \ne 0$ for all $x \ge 0$ and, thus, $W_F(x) \searrow 0\ \ (x\to+\infty)$. Then $\frac{\ln\mu(r,\varphi)}{r} \to +\infty$ as $r\to+\infty$.

In the case, when $0 < R < +\infty$, the function $\mu(r,\varphi)$ may be bounded and it is easy to show that $\ln\mu(r,\varphi) \uparrow +\infty$ as $r \uparrow R$ if and only if

$$\overline{\lim_{x\to+\infty}} W_F(x)e^{Rx} = +\infty. \tag{3}$$

Further we assume that (3) holds and for the investigation of the growth of $\ln\mu(r,\varphi)$ we use the results from [4]. By $\Omega(0,R)$, $0 < R \le +\infty$, we denote the class of positive unbounded functions Φ on $[r_0,R)$ for some $r_0 \in [0\ ,R)$ such that the derivative Φ' is positively continuously differentiable and increasing to $+\infty$ on $[r_0,R)$. For $\Phi \in \Omega(0,R)$ let $\Psi(r) = r - \frac{\Phi(r)}{\Phi'(r)}$ be a function associated with Φ in the sense of Newton and ϕ be the inverse function to Φ'. For the numbers $\Phi'(r_0) < a < b < +\infty$ we put

$$G_1(a,b,\Phi) = \frac{ab}{b-a}\int_a^b \frac{\Phi(\phi(t))}{t^2}dt,\ \ G_2(a,b,\Phi) = \Phi\Big(\frac{1}{b-a}\int_a^b \phi(t)dt\Big).$$

Lemma 1. *Let $\Phi \in \Omega(0,R)$, $0 < R \le +\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F satisfying condition (3) and*

$$\ln W_F(x_k) \ge -x_k\Psi(\phi(x_k)) \tag{4}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then $(\forall r \in [\phi(x_k),\phi(x_{k+1})])$ and $(\forall k \ge k_0)$ the following estimates are valid

$$\ln\mu(r,\varphi) \ge \Phi(r) - \left(G_2(x_k,x_{k+1},\Phi) - G_1(x_k,x_{k+1},\Phi)\right), \tag{5}$$

$$\ln \mu(r,\varphi) \geq \Phi(r)\frac{G_1(x_k,x_{k+1},\Phi)}{G_2(x_k,x_{k+1},\Phi)} \tag{6}$$

and

$$\Phi^{-1}(\ln \mu(r,\varphi)) \geq r - (\Phi^{-1}(G_2(x_k,x_{k+1},\Phi)) - \Phi^{-1}(G_1(x_k,x_{k+1},\Phi))), \tag{7}$$

$$\Phi^{-1}(\ln \mu(r,\varphi)) \geq r\frac{\Phi^{-1}(G_2(x_k,x_{k+1},\Phi))}{\Phi^{-1}(G_1(x_k,x_{k+1},\Phi))}, \tag{8}$$

where Φ^{-1} is the inverse function to Φ.

Proof. Let P be an arbitrary function defined on $(0,+\infty)$ and different from $+\infty$ (it can take on the value $-\infty$ but $P \not\equiv -\infty$) and let $Q(r) = \sup\{P(x) + rx \colon x \geq 0\}$, $-\infty < r < R$, be the function conjugated to P in the sense of Young. By $\Omega(-\infty, R)$, as in [4], we denote the class of positive unbounded functions on $(-\infty, R)$ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty, R)$. The functions Ψ, ϕ and the quantities $G_1(a,b,\Phi)$, $G_2(a,b,\Phi)$ we define as above. Then from Theorem 1 in [4] it follows that if $P(x_k) \geq -x_k\Psi(\phi(x_k))$ for some increasing to $+\infty$ sequence (x_k) of positive numbers then for all $r \in [\phi(x_k), \phi(x_{k+1})]$ and all $k \geq 1$ the estimates (5)–(8) hold with $Q(r)$ instead of $\ln \mu(r,\varphi)$. It is clear that the functions $\ln \mu(r,\varphi)$ and $\ln W_F(x)$ are conjugated in the sense of Young, and in view of the definition of $\Omega(0,R)$ for each function $\Phi \in \Omega(0,R)$ there exists $\Phi_1 \in \Omega(-\infty, R)$ such that $\Phi_1(r) = \Phi(r)$ for $r \in [r_0, R)$. Since $\Psi_1(r) = \Psi(r)$ for $r \in [r_0, R)$ and $\phi_1(r) = \phi(r)$ for $x \geq x_0 = x_0(r_0)$, Proposition 1 follows from the quoted result in [4]. □

We note in passing that ([4]) $G_1(a,b,\Phi) < G_2(a,b,\Phi)$ and the following lemma is hold ([4]–[6]).

Lemma 2. *For $x > a$ let $G_*(x) = G_2(a,x,\Phi) - G_1(a,x,\Phi)$, $G_{**}(x) = \frac{G_2(a,x,\Phi)}{G_1(a,x,\Phi)}$ and for $x \in [a,b)$ let $G^*(x) = G_2(x,b,\Phi) - G_1(x,b,\Phi)$, $G^{**}(x) = \frac{G_2(x,b,\Phi)}{G_1(x,b,\Phi)}$. Then the functions G_* and G_{**} are increasing on $(a,+\infty)$ and the functions G^* and G^{**} are decreasing on (a,b).*

3. Estimates from below of $\ln \mu(r,\varphi)$ for the functions of finite order. We begin with a theorem, which is a generalization of Proposition 1.

Theorem 1. *Let φ be an entire characteristic function of a probability law F such that*

$$\ln W_F(x_k) \geq -\frac{\rho-1}{\rho}\Big(\frac{1}{T\rho}\Big)^{\frac{1}{\rho-1}} x_k^{\frac{\rho}{\rho-1}}, \quad \rho > 1,\ T > 0, \tag{9}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then

1) *if $x_{k+1} - x_k \leq h(x_k)$ for $k \geq 1$, where the function h is positive continuous and non-decreasing on $[0,+\infty)$ and $h(x) = o(x)$ as $x \to +\infty$, then*

$$\ln \mu(r,\varphi) \geq Tr^\rho - \frac{(1+o(1))}{8T\rho(\rho-1)} r^{2-\rho} h^2(T\rho r^{\rho-1}),\ r \to +\infty; \tag{10}$$

2) *if $x_{k+1} \leq x_k \omega(x_{k+1})$ for $k \geq 1$, where the function ω is continuous and non-decreasing on $[0,+\infty)$ and $\omega(x) > 1$ for all $x \geq 0$, then for all large enough r*

$$\ln \mu(r,\varphi) \geq \frac{T\rho^\rho r^\rho}{(\rho-1)^{\rho-1}} f\left(\omega(T\rho r^{\rho-1})\right), \quad f(\omega) = \frac{\omega(\omega-1)^{\rho-1}(\omega^{\frac{1}{\rho-1}}-1)}{(\omega^{\frac{\rho}{\rho-1}}-1)^\rho}.$$

Proof. It is easy to check that for the function $\Phi(r)=Tr^{\rho}$ with $\rho>1$ the following equalities are true $\phi(x)=(\frac{x}{\rho T})^{\frac{1}{\rho-1}}$, $x\Psi(\phi(x))=\frac{\rho-1}{\rho}(\frac{1}{T\rho})^{\frac{1}{\rho-1}}x^{\frac{\rho}{\rho-1}}$,

$$G_1(a,b,\Phi)=(\rho-1)\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}\frac{ab}{b-a}\Big(b^{\frac{1}{\rho-1}}-a^{\frac{1}{\rho-1}}\Big),$$

$$G_2(a,b,\Phi)=(\rho-1)^{\rho}\Big(\frac{1}{T\rho^{\rho^2}}\Big)^{\frac{1}{\rho-1}}\Big(\frac{b^{\frac{\rho}{\rho-1}}-a^{\frac{\rho}{\rho-1}}}{b-a}\Big)^{\rho}.$$

Therefore,

$$G_1(x_k,x_k+h(x_k),\Phi)=(\rho-1)\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big(1+\frac{h(x_k)}{x_k}\Big)\frac{x_k}{h(x_k)}\Big(\Big(1+\frac{h(x_k)}{x_k}\Big)^{\frac{1}{\rho-1}}-1\Big)=$$

$$=(\rho-1)\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big(1+\frac{h(x_k)}{x_k}\Big)\frac{x_k}{h(x_k)}\times$$

$$\times\Big\{\frac{1}{\rho-1}\frac{h(x_k)}{x_k}+\frac{2-\rho}{2(\rho-1)^2}\frac{h^2(x_k)}{x_k^2}+\frac{(2-\rho)(3-2\rho)}{6(\rho-1)^3}\frac{h^3(x_k)}{x_k^3}+O\Big(\frac{h^4(x_k)}{x_k^4}\Big)\Big\}=$$

$$=\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big(1+\frac{h(x_k)}{x_k}\Big)\Big\{1+\frac{2-\rho}{2(\rho-1)}\frac{h(x_k)}{x_k}+\frac{(2-\rho)(3-2\rho)}{6(\rho-1)^2}\frac{h^2(x_k)}{x_k^2}+$$

$$+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}=\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big\{1+\frac{\rho}{2(\rho-1)}\frac{h(x_k)}{x_k}+\frac{\rho(2-\rho)}{6(\rho-1)^2}\frac{h^2(x_k)}{x_k^2}+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\},$$

$$G_2(x_k,x_k+h(x_k),\Phi)=(\rho-1)^{\rho}\Big(\frac{1}{T\rho^{\rho^2}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho^2}{\rho-1}}\frac{1}{h(x_k)^{\rho}}\Big\{\Big(1+\frac{h(x_k)}{x_k}\Big)^{\frac{\rho}{\rho-1}}-1\Big\}^{\rho}=$$

$$=(\rho-1)^{\rho}\Big(\frac{1}{T\rho^{\rho^2}}\Big)^{\frac{1}{\rho-1}}\frac{x_k^{\frac{\rho^2}{\rho-1}}}{h(x_k)^{\rho}}\Big\{\frac{\rho}{\rho-1}\frac{h(x_k)}{x_k}+\frac{\rho}{2(\rho-1)^2}\frac{h^2(x_k)}{x_k^2}+\frac{\rho(2-\rho)}{6(\rho-1)^2}\frac{h^3(x_k)}{x_k^3}+$$

$$+O\Big(\frac{h^4(x_k)}{x_k^4}\Big)\Big\}^{\rho}=\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big\{1+\frac{1}{2(\rho-1)}\frac{h(x_k)}{x_k}+\frac{2-\rho}{6(\rho-1)^2}\frac{h^2(x_k)}{x_k^2}+$$

$$+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}^{\rho}=\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big\{1+\frac{\rho}{2(\rho-1)}\frac{h(x_k)}{x_k}+\frac{\rho(5-\rho)}{24(\rho-1)^2}\frac{h^2(x_k)}{x_k^2}+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\},$$

as $k\to\infty$. Hence in view of the condition $x_{k+1}-x_k\le h(x_k)$ by Lemma 2 we have

$$G_2(x_k,x_{k+1},\Phi)-G_1(x_k,x_{k+1},\Phi)\le G_2(x_k,x_k+h(x_k),\Phi)-G_1(x_k,x_k+h(x_k),\Phi)=$$

$$=\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}}\Big(\frac{\rho}{8(\rho-1)}\frac{h^2(x_k)}{x_k^2}+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big)=\frac{\rho(1+o(1))}{8(\rho-1)}\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}x_k^{\frac{\rho}{\rho-1}-2}h^2(x_k),$$

as $k\to\infty$, and since (9) implies (4) (see inequality (5))

$$\ln\mu(r,\varphi)\ge Tr^{\rho}-\frac{(1+o(1))}{8(\rho-1)}\Big(\frac{1}{T\rho}\Big)^{\frac{1}{\rho-1}}h^2(x_k)x_k^{\frac{2-\rho}{\rho-1}},\ k\to\infty,$$

for all $r\in[(x_k/T\rho)^{\frac{1}{\rho-1}},(x_{k+1}/T\rho)^{\frac{1}{\rho-1}}]$, $k\ge k_0$. For such r we have $x_k\le T\rho r^{\rho-1}\le x_{k+1}$ and since the function h is non-decreasing and $x_{k+1}=(1+o(1))x_k$ as $k\to\infty$ then $\ln\mu(r,\varphi)\ge Tr^{\rho}-\frac{\rho(1+o(1))}{8(\rho-1)}\left(\frac{1}{T\rho}\right)^{\frac{1}{\rho-1}}h^2(T\rho r^{\rho-1})(T\rho r^{\rho-1})^{\frac{2-\rho}{\rho-1}}$ as $r\to+\infty$, whence (10) follows. The first part of Theorem 1 is proved.

For the proof of the second part we remark that

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})},x_{k+1},\Phi\Big)=(\rho-1)\Big(\frac{1}{T\rho^{\rho}}\Big)^{\frac{1}{\rho-1}}\frac{x_{k+1}^{\frac{\rho}{\rho-1}}}{\omega(x_{k+1})-1}\frac{\omega^{\frac{1}{\rho-1}}(x_{k+1})-1}{\omega^{\frac{1}{\rho-1}}(x_{k+1})},$$

$$G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big) = (\rho-1)^\rho\Big(\frac{1}{T\rho\rho^2}\Big)^{\frac{1}{\rho-1}} \frac{x_{k+1}^{\frac{\rho}{\rho-1}}}{\omega^{\frac{\rho}{\rho-1}}(x_{k+1})}\Big(\frac{\omega^{\frac{1}{\rho-1}}(x_{k+1})-1}{\omega(x_{k+1})-1}\Big)^\rho,$$

that is, by Lemma 2 in view of the condition $t_{k+1} \le t_k\omega(t_{k+1})$ we have

$$\frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \ge \frac{G_1(x_{k+1}/\omega(x_{k+1}), x_{k+1}, \Phi)}{G_2(x_{k+1}/\omega(x_{k+1}), x_{k+1}, \Phi)} = \frac{\rho^\rho}{(\rho-1)^{\rho-1}} f(\omega(t_{k+1})).$$

Therefore, by Lemma 1 $\ln\mu(r,\varphi) \ge Tr^\rho\frac{\rho^\rho}{(\rho-1)^{\rho-1}}f(\omega(x_{k+1}))$ for all r such as in the proof of the first part. Since $\omega(x_{k+1}) \le \omega(T\rho r^{\rho-1})$ we need to prove that the function $f(\omega)$ is decreasing on $[1,+\infty)$. It is easy to check that $f(\omega) = \xi^{\rho-1}(\omega) - \xi^\rho(\omega)$, where the function $\xi(\omega) = \frac{\omega-1}{\omega^{\frac{\rho}{\rho-1}}-1}$ decreases to 0 on $(1,+\infty)$ and $\xi(\omega) \uparrow \frac{\rho-1}{\rho}$ as $\omega \downarrow 1$, $\frac{\rho-1}{\rho} \ge \xi(\omega) > 0$ and $\xi'(\omega) < 0$ on $[1,+\infty)$. Hence it follows that $f'(\omega) = \rho\xi^{\rho-2}(\omega)\left(\frac{\rho-1}{\rho} - \xi(\omega)\right)\xi'(\omega) < 0$, i. e. f is a decreasing function. The second part of Theorem 1 is proved. □

Now we prove Proposition 1. By its assumption, $\ln W_F(x) \le -x_k^{\frac{\lambda+1}{\lambda}}$. Let $\rho > 1$ be an arbitrary number such that $\frac{\lambda+1}{\lambda} > \frac{\rho}{\rho-1}$. Then for every large enough k inequality (9) holds with $T = 1$. Since $x_{k+1} = O(x_k)$ as $k \to \infty$ we have $x_{k+1} \le Kx_k$, that is, the assumptions of Proposition 1 hold with $\omega(x) = K > 1$, and by Theorem 1 $\ln\mu(r,\varphi) \ge Ar^\rho$ for all large enough r, where A is a positive constant, whence it follows that $\ln\frac{\ln\mu(r,\varphi)}{r} \ge (\rho-1)\ln r + \ln A$. Letting here ρ to $\lambda+1$ we obtain the desired asymptotical inequality. Proposition 1 is proved.

From Theorem 1 the following proposition also follows.

Proposition 2. *If there exists an increasing to $+\infty$ sequence (x_k) such that (9) holds and $x_{k+1}/x_k \to 1$ $(k \to \infty)$ then $\ln M(r,\varphi) \ge (1+o(1))Tr^\rho$ as $r \to +\infty$.*

Indeed, since $x_{k+1} \le \omega x_k$ for an arbitrary $\omega > 1$ and all $k \ge k_0(\omega)$, by proposition 2) of Theorem 1 we have $\ln\mu(r,\varphi) \ge Tr^\rho\frac{\rho^\rho}{(\rho-1)^{\rho-1}}f(\omega)$. Since $\lim\limits_{\omega\downarrow1} f(\omega) = \frac{(\rho-1)^{\rho-1}}{\rho^\rho}$ we obtain hence the desired asymptotical inequality.

For analytic in $\mathbb{D}_R$ function the following theorem is an analog of Theorem 1.

Theorem 2. *Let φ be an analytic in $\mathbb{D}_R, 0 < R < +\infty$, characteristic function of probability law F such that*

$$\ln W_F(x_k) \ge -Rx_k + \frac{\rho+1}{\rho}(T\rho)^{\frac{\rho}{\rho+1}}x_k^{\frac{\rho}{\rho+1}},\ \rho > 0,\ T > 0, \tag{11}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then

1) if (x_k) satisfies the assumption of proposition 1) of Theorem 1 then

$$\ln\mu(r,\varphi) \ge \frac{T}{(R-r)^\rho} - \frac{(1+o(1))}{8T\rho(\rho+1)}(R-r)^{\rho+2}h^2\Big(\frac{T\rho}{(R-r)^{\rho+1}}\Big),\ r \uparrow R; \tag{12}$$

2) if (x_k) satisfies the assumption of proposition 2) of Theorem 1, then

$$\ln\mu(r,\varphi) \ge \frac{T(\rho+1)^{\rho+1}}{\rho^\rho(R-r)^\rho}f\Big(\omega\Big(\frac{T\rho}{(R-r)^{\rho+1}}\Big)\Big),\quad f(\omega) = \frac{(\omega^{\frac{1}{\rho+1}}-1)(\omega^{\frac{\rho}{\rho+1}}-1)^\rho}{\omega^{\frac{\rho}{\rho+1}}(\omega-1)^{\rho+1}}. \tag{13}$$

Proof. It is easy to check that for the function $\Phi(r) = T(R-r)^{-\rho}$ we have $\phi(x) = R-\left(\frac{T\rho}{x}\right)^{\frac{1}{\rho+1}}$, $x\Psi(\phi(x)) = Rx - \frac{\rho+1}{\rho}(T\rho)^{\frac{1}{\rho+1}}x^{\frac{\rho}{\rho+1}}$, $G_1(a,b,\Phi) = (\rho+1)\left(\frac{T}{\rho^\rho}\right)^{\frac{1}{\rho+1}}\frac{ab}{b-a}(a^{-\frac{1}{\rho+1}} - b^{-\frac{1}{\rho+1}})$ and $G_2(a,b,\Phi) = \frac{(T\rho^\rho)^{\frac{1}{\rho+1}}}{(\rho+1)^\rho}\left(\frac{b-a}{b^{\frac{\rho}{\rho+1}}-a^{\frac{\rho}{\rho+1}}}\right)^\rho$. Therefore, as in the proof of Theorem 1, it is possible to show that $G_1\Big(t_k, t_k+h(t_k),\Phi\Big)= \left(\frac{T}{\rho^\rho}\right)^{\frac{1}{\rho+1}}x_k^{\frac{\rho}{\rho+1}}\Big\{1-\frac{\rho}{2(\rho+1)}\frac{h(x_k)}{x_k}-\frac{\rho(2+\rho)}{6(\rho+1)^2}\frac{h^2(x_k)}{x_k^2}+ +O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}$, $G_2(t_k,t_k+h(t_k),\Phi) = \left(\frac{T}{\rho^{\rho^2}}\right)^{\frac{1}{\rho+1}}x_k^{\frac{\rho}{\rho+1}}\Big\{1-\frac{\rho}{2(\rho+1)}\frac{h(x_k)}{x_k}-\frac{\rho(5+\rho)}{24(\rho+1)^2}\frac{h^2(x_k)}{x_k^2}+O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\}$ and, thus, $G_2(x_k,x_k+h(x_k),\Phi) - G_1(x_k,x_k+h(x_k),\Phi) = \frac{1+o(1)}{8(\rho+1)}(T\rho)^{\frac{1}{\rho+1}}h^2(x_k)x_k^{-\frac{\rho+2}{\rho+1}}$, as $k\to\infty$, whence in view of the condition $x_{k+1}\le x_k+h(x_k)$ and lemmas 1 and 2 we obtain

$$\ln\mu(r,\varphi) \ge \frac{T}{(R-r)^\rho} - \frac{(T\rho)^{\frac{1}{\rho+1}}(1+o(1))}{8(\rho+1)}h^2(x_k)x_k^{-\frac{\rho+2}{\rho+1}}, \quad k\to\infty, \tag{14}$$

for all $r\in[R-(T\rho/x_k)^{\frac{1}{\rho+1}}, R-(T\rho/x_{k+1})^{\frac{1}{\rho+1}}]$ and all large enough k. For such r we have $x_k \le \frac{T\rho}{(R-r)^{\rho+1}} \le x_{k+1}$ and since the function h is non-decreasing and $x_{k+1} = (1+o(1))x_k$ as $k\to\infty$ (14) implies (12). The first part of Theorem 2 is proved.

We prove the second part. Since

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1},\Phi\Big) = (\rho+1)\Big(\frac{T}{\rho^\rho}\Big)^{\frac{1}{\rho+1}}x_{k+1}^{\frac{\rho}{\rho+1}}\frac{\omega^{\frac{1}{\rho+1}}(x_{k+1})-1}{\omega(x_{k+1})-1},$$
$$G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1},\Phi\Big) = \frac{(T\rho^{\rho^2})^{\frac{1}{\rho+1}}}{(\rho+1)^\rho}x_{k+1}^{\frac{\rho}{\rho+1}}\frac{\omega^{\frac{\rho}{\rho+1}}(x_{k+1})(\omega(x_{k+1})-1)^\rho}{(\omega^{\frac{\rho}{\rho+1}}(x_{k+1})-1)^\rho},$$

we have

$$\frac{G_1(x_{k+1}/\omega(x_{k+1}), x_{k+1},\Phi)}{G_2(x_{k+1}/\omega(x_{k+1}), x_{k+1},\Phi)} = \frac{(\rho+1)^{\rho+1}}{\rho^\rho}f(\omega(x_{k+1})),$$

and by Lemmas 1 and 2 $\ln\mu(r,\varphi) \ge \frac{T}{(R-r)^\rho}\frac{(\rho+1)^{\rho+1}}{\rho^\rho}f(\omega(x_{k+1}))$ for all r such as in Proposition 1) of this theorem. Since $\omega(x_{k+1}) \le \omega\big(\frac{T\rho}{(R-r)^{\rho+1}}\big)$ we need to prove, as above, that the function f is decreasing on $[1,+\infty)$. So, $f(\omega) = \frac{1}{\omega^{\frac{\rho}{\rho+1}}}\Big(\frac{\omega^{\frac{\rho}{\rho+1}}-1}{\omega-1}\Big)^\rho\frac{\omega^{\frac{1}{\rho+1}}-1}{\omega-1}$ and every factor is a decreasing function. □

From Theorem 2 the following two propositions follow.

Proposition 3. *If a probability law F satisfies the condition*

$$\ln\ln(W_F(x_k)e^{Rx_k}) \ge \frac{\lambda}{\lambda+1}\ln x_k, \quad \lambda>0, \tag{15}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers and $x_{k+1} = O(x_k)$, $k\to\infty$, then for its characteristic function φ we have the following asymptotic inequality

$$\ln\ln M(r,\varphi) \ge (1+o(1))\lambda\ln\frac{1}{R-r}, \quad r\uparrow R. \tag{16}$$

Indeed, (15) implies $\ln W_F(x_k) \ge -Rx_k + x_k^{\frac{\lambda}{\lambda+1}} \ge -Rx_k + \frac{\rho+1}{\rho}\rho^{\frac{1}{\rho+1}}x_k^{\frac{\rho}{\rho+1}}$ for every $\rho<\lambda$ and all large enough k, that is, (11) holds and since $x_{k+1}\le Kx_k$ for all k by item 2) of Theorem 2 we have $\ln\mu(r,\varphi) \ge \frac{A}{(R-r)^\rho}$, where A is a positive constant, whence $\ln\ln\mu(r,\varphi) \ge \rho\ln\frac{1}{R-r}+O(1)$, $r\uparrow R$. In view of the arbitrariness of ρ we obtain (16).

Proposition 4. *If for a probability law F condition (11) holds and $x_{k+1} = (1 + o(1))x_k$ as $k \to \infty$ then $\ln M(r, \varphi) \geq \frac{(1+o(1))T}{(R-r)^\rho}$ as $r \uparrow R$.*

Proposition 4 easy follows from item 2) of Theorem 2, because $\lim\limits_{\omega\downarrow 1} f(\omega) = \frac{\rho^\rho}{(\rho+1)^{\rho+1}}$. We remark that if in item 1) of Theorems 1–2 $x_{k+1} - x_k = h \equiv \text{const}$ and in item 2) of these theorems $x_{k+1}/x_k = \omega \equiv \text{const}$ then we need not use Lemma 2, that is, we need not estimate of $G_2 - G_1$ and G_1/G_2. Therefore, in view of the optimality of estimates (5) and (6), which we used in the proof of theorems 1–2, in the cases where $x_{k+1} - x_k = h$ and $x_{k+1}/x_k = \omega$ estimates (10), (12) and corresponding (1), (13) are unimprovable.

4. Generalized results. Since we not always can find G_1 and G_2 in an explicit way, the following theorem is useful.

Theorem 3. *Let $0 < R \leq +\infty$, $\Phi \in \Omega(0, R)$ be such that $\Phi(r)\Phi'(r)^{-1-\eta}$ non-increase on $[r_0, R)$ for some $r_0 \in (0, R)$ and $\eta \in [0, +\infty)$, let φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F, which satisfies condition (3) and let inequality (4) hold for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then*

1) *if $x_{k+1} - x_k \leq h(x_k)$, $k \geq 1$, where a positive and continuous on $(0, +\infty)$ function h is such that $h(x) = o(x)$ as $x \to \infty$, the function $x + h(x)$ increases and the function $x^\eta h(x)$ non-decreases on $(0, +\infty)$, then*

$$\ln \mu(r, \varphi) \geq \Phi(r) - (1 + o(1))\frac{1+\eta}{2}\frac{\Phi(r)}{\Phi'(r)}h(\Phi'(r)), \quad r \uparrow R; \tag{17}$$

2) *if $x_{k+1} \leq x_k\omega(x_{k+1})$, $k \geq 1$, where a continuous and non-decreasing on $(0, +\infty)$ function ω is such that $\omega(x) > 1$ for $x > 0$, then*

$$\ln \mu(r, \varphi) \geq \frac{\omega^\eta(\Phi'(r)) - 1}{\eta\omega^\eta(\Phi'(r))(\omega(\Phi'(r)) - 1)}\Phi(r) \tag{18}$$

for all $r < R$ close enough to R.

Proof. At first we assume that $\eta > 0$ and prove item 1). From the non-increase of $\frac{\Phi(r)}{\Phi'(r)^{1+\eta}}$ we have

$$\begin{gathered}
G_1(x_k, x_k + h(x_k), \Phi) = \frac{x_k(x_k + h(x_k))}{h(x_k)}\int_{x_k}^{x_k+h(x_k)} \frac{\Phi(\phi(x))}{x^{1+\eta}}x^{\eta-1}dx \geq \\
\geq \frac{x_k(x_k + h(x_k))}{h(x_k)}\frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^{1+\eta}}\frac{(x_k + h(x_k))^\eta - x_k^\eta)}{\eta} = \\
= \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^\eta}\frac{x_k^{1+\eta}}{\eta h(x_k)}\Big\{\Big(1 + \frac{h(x_k)}{x_k}\Big)^\eta - 1\Big\} = \\
= \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^\eta}\frac{x_k^{1+\eta}}{\eta h(x_k)}\Big\{\frac{\eta h(x_k)}{x_k} + \frac{\eta(\eta-1)h^2(x_k)}{2x_k^2} + O\Big(\frac{h^3(x_k)}{x_k^3}\Big)\Big\} = \\
= \frac{\Phi(\phi(x_k + h(x_k)))}{(x_k + h(x_k))^\eta}x_k^\eta\Big\{1 + \frac{(\eta-1)h(x_k)}{2x_k} + O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big\}, \quad k \to \infty, \\
G_2(x_k, x_k + h(x_k), \Phi) = \Phi\Big(\frac{1}{h(x_k)}\int_{x_k}^{x_k+h(x_k)} \phi(t)dt\Big) \leq \Phi(\phi(x_k + h(x_k))).
\end{gathered}$$

Therefore,

$$G_2(x_k, x_k+h(x_k),\Phi)-G_1(x_k,x_k+h(x_k),\Phi)\le$$
$$\le \Phi(\phi(x_k+h(x_k)))\Big\{1-\Big(\frac{x_k}{x_k+h(x_k)}\Big)^\eta\Big(1+\frac{(\eta-1)h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big)\Big\}=$$
$$=\Phi(\phi(x_k+h(x_k)))\left\{1-\frac{1+\frac{(\eta-1)h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)}{1+\eta\frac{h(x_k)}{x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)}\right\}=\Phi(\phi(x_k+h(x_k)))\times$$
$$\times\Big\{1-\Big(1+\frac{(\eta-1)h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big)\Big(1-\frac{\eta h(x_k)}{x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big)\Big\}=$$
$$=\Phi(\phi(x_k+h(x_k)))\Big\{\frac{1+\eta}{2}\frac{h(x_k)}{x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big\}=$$
$$=\frac{\Phi(\phi(x_k+h(x_k)))}{(x_k+h(x_k))^{1+\eta}}\frac{1+\eta}{2}h(x_k)x_k^\eta(1+o(1)),\quad k\to\infty.$$

Hence in view of the condition $x_{k+1}\le x_k+h(x_k)$ using Lemma 2 (growth of G_*) and inequality (5) we obtain

$$\ln\mu(r,\varphi)\ge\Phi(r)-\frac{\Phi(\phi(x_k+h(x_k)))}{(x_k+h(x_k))^{1+\eta}}\frac{1+\eta}{2}x_k^\eta h(x_k)(1+o(1)),\quad k\to\infty,\tag{19}$$

for all $r\in[\phi(x_k),\phi(x_{k+1})]$. Since $\Phi(\varphi(t))t^{-\eta-1}$ non-increases, $x^\eta h(x)$ non-decreases and the inequalities $\phi(x_k)\le r\le\phi(x_{k+1})$ imply the inequalities $x_k\le\Phi'(r)\le x_{k+1}$, we obtain

$$\ln\mu(r,\varphi)\ge\Phi(r)-\frac{\Phi(\phi(x_k))}{x_k^{1+\eta}}\frac{1+\eta}{2}x_k^\eta h(x_k)(1+o(1))\ge$$
$$\ge\Phi(r)-\frac{\Phi(r)}{\Phi'(r)^{1+\eta}}\frac{1+\eta}{2}\Phi'(r)^\eta h(\Phi'(r))(1+o(1))$$

i.e., inequality (17) holds.

If $\eta=0$ then by analogy we have

$$G_1(x_k,x_k+h(x_k),\Phi)\ge\frac{x_k(x_k+h(x_k))}{h(x_k)}\frac{\Phi(\phi(x_k+h(x_k)))}{x_k+h(x_k)}\ln\Big(1+\frac{h(x_k)}{x_k}\Big)=$$
$$=\Phi(\phi(x_k+h(x_k)))\Big(1-\frac{h(x_k)}{2x_k}+O\Big(\frac{h^2(x_k)}{x_k^2}\Big)\Big),\quad k\to\infty,$$
$$G_2(x_k,x_k+h(x_k),\Phi)-G_1(x_k,x_k+h(x_k),\Phi)\le\frac{\Phi(\phi(x_k+h(x_k)))}{x_k+h(x_k)}\frac{h(x_k)}{2}(1+o(1)),\ k\to\infty,$$

whence we obtain (19) with $\eta=0$. Hence, as above estimate (17) follows. The first part of Theorem 3 is proved.

We prove second part. For $\eta>0$ we have

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})},x_{k+1},\Phi\Big)=\frac{x_{k+1}}{\omega(x_{k+1})-1}\int_{\frac{x_{k+1}}{\omega(x_{k+1})}}^{x_{k+1}}\frac{\Phi(\phi(x))}{x^{1+\eta}}x^{\eta-1}dx\ge$$
$$\ge\frac{x_{k+1}}{\omega(x_{k+1})-1}\frac{\Phi(\phi(x_{k+1}))}{x_{k+1}^{1+\eta}}\frac{1}{\eta}\Big(x_{k+1}^\eta-\frac{x_{k+1}^\eta}{\omega^\eta(x_{k+1})}\Big)=\frac{\Phi(\phi(x_{k+1}))}{\eta(\omega(x_{k+1})-1)}\Big(1-\frac{1}{\omega^\eta(x_{k+1})}\Big),$$
$$G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})},x_{k+1},\Phi\Big)\le\Phi(\phi(x_{k+1})).$$

Therefore, in view of the condition $x_{k+1} \le x_k\omega(x_{k+1})$ using Lemma 2 and inequality (6) for all $r \in [\phi(x_k), \phi(x_{k+1})]$ and all $k \ge k_0$ we have

$$\ln \mu(r, \phi) \ge \Phi(r)\frac{\omega^\eta(x_{k+1}) - 1}{\eta\omega^\eta(x_{k+1})(\omega(x_{k+1}) - 1)} \ge \Phi(r)\frac{\omega^\eta(\Phi'(r)) - 1}{\eta\omega^\eta(\Phi'(r))(\omega(\Phi'(r)) - 1)},$$

because the function $f(x) = \frac{x^\eta - 1}{x^\eta(x-1)}$ decreases on $[1, +\infty)$ and $x_{k+1} \ge \Phi'(r)$. The inequality (18) is proved.

If $\eta = 0$ then, by analogy, we have

$$G_1\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big) \ge \frac{x_{k+1}}{\omega(x_{k+1}) - 1}\frac{\Phi(\phi(x_{k+1}))}{x_{k+1}} \ln\omega(x_{k+1}) = \Phi(\phi(x_{k+1}))\frac{\ln\omega(x_{k+1})}{\omega(x_{k+1}) - 1}$$

and in view of the estimates $G_2\Big(\frac{x_{k+1}}{\omega(x_{k+1})}, x_{k+1}, \Phi\Big) \le \Phi(\phi(x_{k+1}))$, as above, we obtain

$$\ln \mu(r, \varphi) \ge \Phi(r)\frac{\ln\omega(x_{k+1})}{\omega(x_{k+1}) - 1} \ge \Phi(r)\frac{\ln\omega(\Phi'(r))}{\omega(\Phi'(r)) - 1} \tag{20}$$

for all $r < R$ close enough to R. Since $\frac{\omega^\eta - 1}{\eta\omega^\eta(\omega - 1)} \to \frac{\ln\omega}{\omega - 1}$ as $\eta \to 0$ estimate (20) coincides with estimate (18) with $\eta = 0$. □

The condition of the non-increase of $\Phi(r)(\Phi'(r))^{-1-\eta}$ can be removed if we use estimates (7) and (8) from Lemma 1. We get the following theorem.

Theorem 4. *Let $\Phi \in \Omega(0, R), 0 < R \le +\infty$, and φ be an analytic characteristic function of a probability law, which satisfies conditions (3) and (4) for some increasing to $+\infty$ sequence (x_k) of positive numbers. Then*

1) *if $\phi(x_{k+1}) - \phi(x_k) \le h(x_{k+1})$, where h is a positive continuous and non-increasing function on $(0, +\infty)$ such that $R > \phi(x) - h(x) \to R$ as $x \to +\infty$, then for all $r < R$ close enough to R*

$$\ln \mu(r, \varphi) \ge \Phi(r - h(\Phi'(r))); \tag{21}$$

2) *if $\phi(x_{k+1}) \le \phi(x_k)\omega(x_{k+1})$, where ω is a positive continuous and non-increasing function on $(0, +\infty)$ such that $R > \frac{\phi(x)}{\omega(x)} \to R$ as $x \to +\infty$, then for all $r < R$ close enough to R*

$$\ln \mu(r, \varphi) \ge \Phi\Big(\frac{r}{\omega(\Phi'(r))}\Big). \tag{22}$$

Proof. Since the function $\Phi(\phi(t))$ increases we have

$$\Phi^{-1}(G_1(x_k, x_{k+1}, \Phi)) \ge \Phi^{-1}\Big(\frac{x_k x_{k+1}}{x_{k+1} - x_k}\Phi(\phi(x_k))\int_{x_k}^{x_{k+1}}\frac{dx}{x^2}\Big) = \phi(x_k),$$

$$\Phi^{-1}(G_2(x_k, x_{k+1}, \Phi)) = \frac{1}{x_{k+1} - x_k}\int_{x_k}^{x_{k+1}}\phi(t)dt \le \phi(x_{k+1}).$$

Therefore, from (7) and (8) for all $r \in [\phi(x_k), \phi(x_{k+1})]$ we obtain respectively

$$\Phi^{-1}(\ln\mu(r, \varphi)) \ge r - (\phi(x_{k+1}) - \phi(x_k)) \ge r - h(x_{k+1}) \ge r - h(\Phi'(r)),$$

$$\Phi^{-1}(\ln\mu(r, \varphi)) \ge r\frac{\varphi(x_k)}{\varphi(x_{k+1})} \ge \frac{r}{\omega(x_{k+1})} \ge \frac{r}{\omega(\Phi'(r))},$$

whence the inequalities (21) and (22) follows. □

5. Corollaries. Let L be a class of continuous increasing functions α such that $\alpha(x) \geq 0$ for $x \geq x_0$, $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and on $[x_0, +\infty)$ the function α increases to $+\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$; further $\alpha \in L_{\text{si}}$ if $\alpha(cx) = (1 + o(1))\alpha(x))$ as $x \to +\infty$ for any $c \in (0, +\infty)$. It is easy to see that $L_{\text{si}} \subset L^0$.

Corollary 1. *Let either $\alpha \in L_{\text{si}}$ and $\beta \in L^0$ or $\alpha \in L^0$ and $\beta \in L_{\text{si}}$ and φ be an entire characteristic function of a probability law F such that*

$$\beta\Big(\frac{1}{x_k} \ln \frac{1}{W_F(x_k)}\Big) \leq \alpha(x_k) \tag{23}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers, which satisfies the condition $\beta^{-1}(c\alpha(x_{k+1}))/\beta^{-1}(c\alpha(x_k)) \to 1$ as $k \to \infty$ for any $c \in (1, +\infty)$. Then

$$\alpha\Big(\frac{\ln \mu(r, \varphi)}{r}\Big) \geq (1 + o(1))\beta(r), \quad r \to \infty. \tag{24}$$

Proof. Let at first $\alpha \in L_{\text{si}}$, $\beta \in L^0$ and $\varepsilon \in (0, 1)$ be an arbitrary number. Since $\beta \in L^0$, we have ([7]) $\beta(\frac{x}{1-\varepsilon}) \leq (1 + \delta_1(\varepsilon))\beta(x)$, where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ and, thus, $\beta^{-1}(x) \leq (1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))x)$, and the condition $\alpha \in L_{\text{si}}$ implies $\alpha(\varepsilon x) = (1+o(1))\alpha(x)$ as $x \to +\infty$, that is, for any $\delta_2 > 0$ and all large enough x the inequality $\alpha(\varepsilon x) \geq \frac{1}{1+\delta_2}\alpha(x)$ is true. Therefore, $x\beta^{-1}(\alpha(x)) \leq x(1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))\alpha(x)) \leq (1-\varepsilon)\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(\varepsilon x))$ for all large enough x. On the other hand

$$\int_{x_0}^{x} \beta^{-1}((1 + \delta_1(\varepsilon))(1 + \delta_2)\alpha(t))dt \geq \int_{\varepsilon x}^{x} \beta^{-1}((1 + \delta_1(\varepsilon))(1 + \delta_2)\alpha(t))dt \geq$$
$$\geq \beta^{-1}((1 + \delta_1(\varepsilon))(1 + \delta_2)\alpha(\varepsilon x))(1 - \varepsilon)x.$$

Hence it follows from (23) that

$$\ln W_F(x_k) \geq -x_k\beta^{-1}(\alpha(x_k)) \geq -(1 - \varepsilon)\beta^{-1}((1 + \delta_1(\varepsilon))(1 + \delta_2)\alpha(\varepsilon x_k)) \geq$$
$$\geq -\int_{x_0}^{x_k} \beta^{-1}((1 + \delta_1(\varepsilon))(1 + \delta_2)\alpha(t))dt \tag{25}$$

for each $\varepsilon \in (0, 1)$, $\delta_2 > 0$ and all $k \geq k_0 = k_0(\varepsilon, \delta_2)$.

We put $\Phi(r) = \int_{r_0}^{r} \alpha^{-1}\big(\frac{\beta(t)}{1+\delta}\big)dt$, where $1+\delta < (1+\delta_1(\varepsilon))(1+\delta_2)$. Then $\Phi'(r) = \alpha^{-1}\Big(\frac{\beta(r)}{1+\delta}\Big)$, $\phi(x) = \beta^{-1}((1 + \delta)\alpha(x))$ and

$$x\Psi(\phi(x)) = \int_{x_0}^{x} \phi(t)dt + \text{const} \leq \int_{x_0}^{x} \beta^{-1}((1 + \delta)\alpha(t))dt + \text{const} \leq$$
$$\leq \int_{x_0}^{x} \beta^{-1}((1 + \delta_1(\varepsilon))(1 + \delta_2)\alpha(t))dt.$$

Therefore, inequality (25) implies (4) for all large enough k.

Further, since $\frac{\beta^{-1}((1+\delta)\alpha(x_{k+1}))}{\beta^{-1}((1+\delta)\alpha(x_k))} \to 1$ $(k \to \infty)$, there exists a decreasing to 1 continuous function ω such that $\frac{\phi(x_{k+1})}{\phi(x_k)} \leq \omega(x_{k+1})$ for all k. Therefore, by item 2) of Theorem 4 inequality (22) is true, that is, in view of the condition $\beta \in L^0$ we have

$$\ln \mu(r, \varphi) \geq \Phi\Big(\frac{r}{\omega(\Phi(r))}\Big) = \Phi((1 + o(1))r) = \int_{r_0}^{r(1+o(1))} \alpha^{-1}\Big(\frac{\beta(x)}{1 + \delta}\Big)dx \geq$$
$$\geq \int_{(1-\varepsilon)r}^{r} \alpha^{-1}\Big(\frac{\beta(x)}{(1 + \delta)^2}\Big)dx \geq \alpha^{-1}\Big(\frac{\beta((1 - \varepsilon)r)}{(1 + \delta)^2}\Big)\varepsilon r$$

for all large enough r. Since $\alpha\in L_{\rm si}$, $\beta\in L^0$ and the numbers ε,δ_2 and δ are arbitrary, from the latter inequality we easily obtain (24).

If $\alpha\in L^0$ and $\beta\in L_{\rm si}$ then $\alpha((1-\varepsilon)x)\ge\frac{1}{1+\delta_1(\varepsilon)}\alpha(x)$, where $\delta_1(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $\beta(\varepsilon x)\ge\frac{1}{1+\delta_2}\beta(x)$ for all large enough x. Therefore, as above

$$\int_{x_0}^{x}\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt\ge\varepsilon x\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha((1-\varepsilon)x))\ge$$
$$\ge\varepsilon x\beta^{-1}((1+\delta_2)\alpha(x))\ge x\beta^{-1}\Big(\frac{1}{1+\delta_2}\beta(\beta^{-1}((1+\delta_2)\alpha(x))\Big)=x\beta^{-1}(\alpha(x)).$$

Hence it follows from (23) that $\ln W_F(x_k)\ge-\int_{x_0}^{x_k}\beta^{-1}((1+\delta_1(\varepsilon))(1+\delta_2)\alpha(t))dt$ for any $\varepsilon\in(0,1)$, $\delta_2>0$ and all $k\ge k_0=k_0(\varepsilon,\delta_2)$. Therefore, choosing $\Phi(r)$, as above, and repeating the arguments, we again arrive at inequality (24). □

For analytic functions in $\mathbb{D}_R$, $0<R<+\infty$, the following corollary is an analog of Corollary 1.

Corollary 2. *Let $\alpha\in L_{\rm si}$, $\beta\in L_{\rm si}$, $\frac{d\ln\beta^{-1}(\alpha(x))}{d\ln x}\le q<1$ for all large enough x and $\alpha\Big(\frac{x}{\beta^{-1}(\alpha(x))}\Big)=(1+o(1))\alpha(x)$ as $x\to+\infty$, and φ be an analytic in $\mathbb{D}_R$, $0<R<+\infty$, characteristic function of a probability law F, for which*

$$\beta\Big(\frac{x_k}{\ln(W_F(x_k)e^{Rx_k})}\Big)\le\alpha(x_k)\tag{26}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\beta^{-1}(\alpha(x_{k+1}))=O(\beta^{-1}(\alpha(x_k)))$ as $k\to\infty$. Then

$$\alpha(\ln\mu(r,\varphi))\ge(1+o(1))\beta\Big(\frac{1}{R-r}\Big),\quad r\uparrow R.\tag{27}$$

Proof. From (26) it follows that $\ln W_F(x_k)\ge-Rx_k+\frac{x_k}{\beta^{-1}(\alpha(x_k))}$. Since $\frac{d\ln\beta^{-1}(\alpha(x))}{d\ln x}\le q<1$, we have $\frac{x}{\beta^{-1}(\alpha(x))}\uparrow+\infty$ $(r_0\le x\to+\infty)$, and using L'Hospital's rule it is easy to show that

$$\frac{x}{\beta^{-1}(\alpha(x))}\ge(1+o(1))(1-q)\int_{x_0}^{x}\frac{dt}{\beta^{-1}(\alpha(t))},\quad x\to\infty.$$

Therefore,

$$\ln W_F(x_k)\ge-Rx_k+(1-q_1)\int_{x_0}^{x_k}\frac{dt}{\beta^{-1}(\alpha(t))}\tag{28}$$

for any $q_1\in(q,1)$ and all large enough k. We put

$$\Phi(r)=\int_{r_0}^{r}\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{R-x}\Big)\Big)dx,\tag{29}$$

where $q_2\in(q_1,1)$. Then $\Phi'(r)=\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{R-r}\Big)\Big)$, $\phi(x)=R-\frac{1-q_2}{\beta^{-1}(\alpha(x))}$ and

$$x\Psi(\phi(x))=Rx-(1-q_2)\int_{x_0}^{x}\frac{dt}{\beta^{-1}(\alpha(t))}+\text{const},$$

that is, in view of (28) and $q_1 < q_2$ we obtain (4). Since $\beta^{-1}(\alpha(x_{k+1})) \le K\beta^{-1}(\alpha(x_k)),\ K > 1$, for all $k \ge 1$, we have $\frac{1}{\beta^{-1}(\alpha(x_k))} - \frac{1}{\beta^{-1}(\alpha(x_{k+1}))} \le \frac{K-1}{\beta^{-1}(\alpha(x_{k+1}))}$. Therefore, if we put $h(x) = \frac{(K-1)(1-q_2)}{\beta^{-1}(\alpha(x))}$ then $\phi(x) - h(x) = R - \frac{K(1-q_2)}{\beta^{-1}(\alpha(x))} \to R$ as $x \to +\infty$, $h(\Phi'(r)) = (K-1)(R-r)$ and $\phi(x_{k+1}) - \phi(x_k) \le h(x_{k+1})$ for $k \ge 1$.

Finally, for function (29) and $r > \max\{r_0, R/2\}$ we have

$$\Phi(r) \ge \int_{2r-R}^{r} \alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{R-x}\Big)\Big)dx \ge (R-r)\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{2(R-r)}\Big)\Big).$$

Therefore, by item 1) of Theorem 4

$$\ln\mu(r,\varphi) \ge (R - r + h(\Phi'(r)))\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{2(R-r+h(\Phi'(r)))}\Big)\Big) = \\ = K(R-r)\alpha^{-1}\Big(\beta\Big(\frac{1-q_2}{2K(R-r)}\Big)\Big)$$

for all $r < R$ close enough to R. But from the condition $\alpha(\frac{x}{\beta^{-1}(\alpha(x))}) = (1+o(1))\alpha(x)$ as $x \to +\infty$ it follows that $\alpha(\frac{\alpha^{-1}(\beta(t))}{t}) = (1+o(1))\beta(t)$ as $t \to \infty$ and since $\alpha \in L_{\text{si}}$, $\beta \in L_{\text{si}}$ the last inequality implies (27). □

We remark that under the other conditions of Corollary 2 the condition $\beta^{-1}(\alpha(x_{k+1})) = O(\beta^{-1}\ (\alpha(x_k)))$ as $k \to \infty$ holds provided $x_{k+1} = O(x_k)$ as $k \to \infty$.

The conditions on α and β in Corollary 2 assume that the function α increases slower than the function β. In the case where α increases quicker than β, the following corollary is true.

Corollary 3. *Let $\alpha \in L_{\text{si}}$, $\beta \in L_{\text{si}}$, $\frac{d\ln\alpha^{-1}(\beta(x))}{d\ln x} \le q < 1$ for all large enough x, $\frac{d\alpha^{-1}(\beta(x))}{dx} = \frac{1}{f(x)} \downarrow 0$ and $\alpha^{-1}(\beta(f(x))) = O(\alpha^{-1}(\beta(x)))$ as $x \to +\infty$, and φ be an analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of a probability law F, for which*

$$\alpha(\ln(W_F(x_k)e^{Rx_k})) \ge \beta(x_k) \tag{30}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\overline{\lim\limits_{k\to\infty}} \frac{f(x_{k+1})}{f(x_k)} < 2$. Then asymptotical inequality (27) holds.

Proof. If we put $x\Psi(\phi(x)) = Rx - \alpha^{-1}(\beta(x))$ then (30) implies (4) and $\phi(x) = (x\Psi(\phi(x)))' = R - \frac{d\alpha^{-1}(\beta(x))}{dx} = R - \frac{1}{f(x)}$. Hence it follows that $\Phi'(r) = f^{-1}(\frac{1}{R-r})$,

$$\Phi(r) - \Phi(r_0) = \int_{r_0}^{r} f^{-1}\Big(\frac{1}{R-x}\Big)dx = \int_{r_1}^{f^{-1}(\frac{1}{R-r})} td\Big(-\frac{1}{f(t)}\Big) = \\ = -(R-r)f^{-1}\Big(\frac{1}{R-r}\Big) + \alpha^{-1}\Big(\beta\Big(f^{-1}\Big(\frac{1}{R-r}\Big)\Big)\Big) \ge (1-q)\alpha^{-1}\Big(\beta\Big(f^{-1}\Big(\frac{1}{R-r}\Big)\Big)\Big).$$

But from the condition $\alpha^{-1}(\beta(f(x))) = O(\alpha^{-1}(\beta(x)))$ as $x \to +\infty$ it follows that

$$\alpha^{-1}\Big(\beta\Big(\frac{1}{R-r}\Big)\Big) \le K\alpha^{-1}\Big(\beta\Big(f^{-1}\Big(\frac{1}{R-r}\Big)\Big)\Big),$$

where K is a positive constant. Therefore, $\Phi(r) \geq K_1\alpha^{-1}(\beta(\frac{1}{R-r}))$, where K_1 is a positive constant, for all $r < R$ close enough to R, and if $h(x) = a(R - \phi(x))$, $0 < a < 1$, then

$$\Phi(r - h(\Phi'(r))) \geq K_1\alpha^{-1}\Big(\beta\Big(\frac{1}{(1+a)(R-r)}\Big)\Big). \tag{31}$$

Under such a choice of the function h the condition $\phi(x_{k+1}) - \phi(x_k) \leq h(x_{k+1})$ is equivalent to the condition $f(x_{k+1}) \leq (1+a)f(x_k)$, and the latter condition follows from the condition $\overline{\lim\limits_{k\to\infty}} \frac{f(x_{k+1})}{f(x_k)} < 2$. Therefore, by item 1) of Theorem 4 inequality (21) is true and in view of (31) and the conditions $\alpha \in L_{\text{si}}$, $\beta \in L_{\text{si}}$ we obtain (27). □

We remark that from Corollaries 1–3 one can obtain analogues of Propositions 1–4, but we shall not discuss this here.

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