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JÓZEF SCHREIER "ON FINITE BASE IN TOPOLOGICAL GROUPS"

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One of the Ph.D. students in mathematics that belonged to Lviv Mathematical School of 1920–1939 years is Józef Schreier. In the article, the authors provide an overview of main results of the unpublished thesis "On finite base in topological groups" by Józef Schreier as well as the connection of the ideas of this work with modern investigations in the theory of topological groups and semigroups.

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Одним из докторантов математики, принадлежащих к львовской математической школе 1920–1939 годов является Юзеф Шрайер. В статье излагается обзор основных результатов неопубликованной диссертации "О конечном базисе в топологических группах" Юзефа Шрайера, а также связь идей этой работы с современными исследованиями в теории топологических групп и полугрупп.

1. Lviv mathematical school and doctorants in mathematics in 1920–1939. Considering the history of research in mathematics at the Lviv University, one often pays attention for the period 1919–1939. Then a creative team of mathematicians was formed in Lviv. In the history of science, it is known as the Lviv mathematical school. Stefan Banach and Hugo Steinhaus were undisputed leaders of this team. The extent and level of research of the Lviv mathematical school in the 30-ies were comparable with those of the Göttingen school. Here is how it was written in a letter to professors of mathematics and natural sciences department to the Ministry of Education and religious confessions in 1933; the letter was signed by the Dean S. Banach ([1]):

"One can safely say that no Polish university, and perhaps not many foreign universities organized teaching mathematics as thoroughly and efficiently as did the University of Lviv ..."

"In recent years the number of announced by the Lviv school proceedings counts in hundreds. Foreign scientists from Germany, France and America come to Lviv to impose contact with the local mathematical world ..."

"The number of scientific meetings and reports for the last period in the local division of the Polish Mathematical Society ... exceeds the number of those in Berlin and Göttingen for the same time period. Some branches of mathematics i.e. the theory of linear operations, topology, theory of summation, potential theory stand higher in Lviv than anywhere else".

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In 30-ies, there were four departments of mathematics in the mathematical and natural faculty of the University. Their leaders were Professor Eustachy Żyliński, Hugo Steinhaus, Stanisław Ruziewicz (until 1933), and Stefan Banach. In 1930, the department of logic was opened. The department was led by Leon Chwistek.

In this period, there were three departments at the Lviv Polytechnic University. They were guided by Prof. Wlodzimierz Stożek, Antoni Łomnicki, Kazimierz Kuratowski. The department of K. Kuratowski belonged to the so-called general faculty, which together with the department was closed in 1933. Stanisław Ulam was studying and defended his thesis in this faculty.

According to the then education system, until 1926, a student could finish university with either the "absolutorium" diploma, which gave the right to take the teacher exams, or the diploma of philosophy doctor. Since 1926, the Master degree was introduced and the master graduates performed the thesis. It was believed that the research work could be presented for Ph.D. only after obtaining the Master degree. However, there were exceptions. For example, Stanisław Mazur had enrolled at the university only for two semesters, but on the basis of submitted scientific thesis he was admitted to the doctoral exams and in 1932 received his Ph.D. The Faculty Council drew attention to the scientific level of the submitted theses and, perhaps, for this reason the number of doctoral theses was rather small.

During the period from 1920 to 1938 the following mathematicians received Ph.D. ([2]):
Stefan Banach (promoter Kazimier Twardowski, 1921)
Juliusz Schauder (promoter Eustachy Żyliński, 1924)
Stefan Kaczmarz (promoter Stanisław Ruziewicz, 1924)
Władysław Nikliborc (promoter Hugo Steinhaus, 1924)
Sala Weinlos (promoter Hugo Steinhaus, 1927)
Władysław Orlicz (promoter Eustachy Żyliński, 1928)
Zygmunt Birnbaum (promoter Hugo Steinhaus, 1929)
Myron Zarycki (promoter Stanisław Ruziewicz, 1930)
Herman Auerbach (promoter Hugo Steinhaus, 1930)
Stanisław Mazur (promoter Stefan Banach, 1932)
Józef Schreier (promoter Stefan Banach, 1934)
Mark Kac (promoter Hugo Steinhaus, 1937)
Meier Eidelheit (promoter Stefan Banach, 1938)
— Ph.D. in logic:
Władysław Hetper (promoter Leon Chwistek, 1937)
Józef Pepis (promoter Eustachy Żyliński, 1938).

According to the statute of the University, the promoter was always a regular professor, who participated together with the rector and dean, in the ceremony of "promotion", represented the new doctor and signed the diploma. He could be a supervisor, but not always, as in the case of S. Banach.

S. Weinlos and J. Schreier are not mentioned as doctorant of the Lviv University in the biographical directory [3] and monograph [4]. Therefore, the research of results and scientific life of these mathematicians is of interest for the history of mathematics at the University.

2. Józef Schreier and his doctoral thesis. Józef Schreier was born on the 18th of February, 1909 in Drohobych in the family of local Rabbi Bernard Schreier. In 1919-27 he studied in the public school in Drohobych, where his father taught religion. In 1927, he took the examinations of mathematics and natural sciences for matriculation. In the same year, Schreier enrolled in mathematics and natural sciences department of the Lviv University. He received his MA diploma in March 1931. His master thesis "O pewnem zagadnieniu z organizacji turniejowej" was written under the supervision of H. Steinhaus. In the thesis, a problem was solved which was formulated by H. Steinhaus on seminars for many years. The results of the master thesis are published in the paper [18].

In the first half of 1930/31 school year, J. Schreier taught at a private high school in Drohobych. In the following year he had an unpaid teaching practice in Drohobych State himnasium. The pedagogical practice was required to the teacher exam.

Fig. 1: Józef Schreier during the finishing of gimnasium

Since 1932, he was in Lviv, conducted research, and prepared his doctoral thesis. In the 23rd of April, 1934, J. Schreier submitted to the evaluation the doctoral thesis "O skończonej bazie w grupach topologicznych" ([5]). At that time he already had 9 publications, five of them in collaboration with S. Ulam. S. Ulam in his memoirs highly appreciated J. Schreyer as a mathematician. In his review of the doctoral thesis, S. Banach and E. Żyliński noted that the thesis consists in a deepening the method that presented before in a joint paper with Mr. Ulam and is sufficient for doctoral thesis. The members of the doctoral exam commission in mathematics and logic were professors E. Żyliński, S. Banach, L. Chwistek, and Dean H. Steinhaus. All committee members rated the answer "with distinction". The solemnity of "promotion" of Ph.D. J. Schreier was held on June 30, 1934. The promoter S. Banach spoke about J. Schreier and his scientific achievements.

Further, J. Schreier continued to teach mathematics in the Drohobych high school. During the Nazi occupation, he hid with a group of Jews in Drohobych. At the time of their discovery by the police, he took poison and died in April 1943 ([3]).

In the sequel, we use modern terminology in the presentation of the main results of J. Schreier's thesis, as in nearly eighty years some old terms and concepts used by the author received a variety of options for interpretation.

First, let us touch on motivation of the themes of this thesis. Since we have no documentary information on which sources prompted by an interest in issues referred to in the thesis, we will be guided by hypothetical assumptions justifying their respective perceptions of the historical era of the late 30s of last century and the present retrospective view of the state of mathematical knowledge of this period of mathematics. Of course, any assumptions and motivations are subjective and we steadily accept all critical reader's reaction to this.

Thus, let us discuss the subject of this thesis. One of the fundamental ideas in mathematics of the nineteenth and twentieth centuries is the approximation in a broad sense and the construction of invariants that allow to classify the mathematical objects. Explain this point in the following examples: Weierstrass' theorem on approximation of continuous functions on the interval by polynomials and D. Hilbert's basis theorem in the invariant theory.

In the finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$ the approximation problem is identified with the problem of classification: any vector can be represented by a linear combination of basis vectors that are defined by means of the maximal linearly independent system. The number of vectors of this system is called the dimension of space and is its invariant which allows to classify the finite-dimensional spaces up to isomorphism.

The situation becomes more complicated when considering objects with algebraic structure poorer than the structure of vector space. In such objects as modules, groups, rings, semigroups the concept of linear independence, dimension, the system of generating elements receive different interpretations, and in solving problems of classification they are not universal. A good choice of concepts is determining in the overall situation. A striking example is the above Hilbert's basis theorem, where the concepts of dimension interpreted by the Noetherness, i.e. finiteness of the growing chain of ideals. Prior to the 20th years of XX century the notion of topological group was perceived only as a group of transformations (symmetries) of geometric objects. Only in the late 20th–early 30-ies in the papers of Leja [6], Van Danzing ([7]), O. Schreier ([8]) the topological group are defined axiomatically and the general theory of topological groups is developing. At that time, the theory of infinite dimensional vector spaces also obtains the rapid development along with the theory of topological groups (D. Hilbert [9], J. von Neumann [10], S. Banach [11]). S. Banach in his theory of normed linear spaces, later called with his name, paid special attention to the problem of existence of countable base of a separable Banach space. Of course, he was interested in this issue in more general algebraic-topological objects: topological groups and semigroups. This assumption is based on two observations: the appearance in the journal Fund. Math. in 1934–35 a number of articles authored by Banach ([12]), Ulam-Schreier ([13]). Sierpiński ([15]), Jarnik-Knichal ([16]) devoted to this subject as well as notes in the Scottish book: 20, 29, 30, 31, 90, 95, 96, 98 and others([14]). One can assume with great certainty that the concept of bases in general topological-algebraic objects constituted the subject of discussion in the Scottish café, whose active member was J. Schreier.

In the article [13] of Ulam and Schreier, there are formulated two general problems concerning the concept of base in topological semigroups and groups of special type. Solutions to these problems are discussed in J. Schreier's thesis.

Let S be a topological semigroups (group), X be a subset in S. By $\langle X\rangle$ we denote the smallest subsemigroup (group) which contains X. If the closure of $\langle X\rangle$ contains S, we say that X is a topological basis in S. A topological group (semigroups) S has a finite basis, if S or, in the case of the group, the connected component of the unit in S has a finite basis of X. If X is a topological space, then by $C(X)$ we denote the semigroup of continuous functions on X with respect to the operation of composition. If X is a locally compact Hausdorff space, then $C(X)$ is a topological semigroup in the compact-open topology.

In [13], Ulam and Schreier proved that $C([0,1]^m)$, $m \geqslant 1$, has a base consisting of 5 elements. In the second chapter of J. Schreier's thesis it is proved that the automorphism group of a compact connected oriented surface has a finite base.

The idea of proving the existence of finite base in various spaces of maps, automorphisms, functions is extremely impressive and versatile. Let us illustrate it with a result from J. Schreier's thesis (p. 26).

By $\operatorname{Sym}\mathbb{N}$ we denote the group of bijections of natural numbers.

Theorem. *There exist elements φ and $\chi \in \operatorname{Sym}\mathbb{N}$ such that the group $\langle\varphi,\chi\rangle$ acts n-tran-*

sitively on $\operatorname{Sym}\mathbb{N}$ *for arbitrary* n.

Corollary 1. *The group* $\operatorname{Sym}\mathbb{N}$ *in the topology of pointwise convergence has a base consisting of two elements.*

Proof of Corollary 1. If $\alpha \in \operatorname{Sym}\mathbb{N}$, then the neighborhood base of a point α in the topology of pointwise convergence on $\operatorname{Sym}\mathbb{N}$ consists of all the possible sets of the form

$$U(i_1,\dots,i_n;l_1,\dots,l_n) = \{\alpha \in \operatorname{Sym}\mathbb{N} \mid \alpha(i_k) = l_k\},$$

where $i_1,\dots,i_n;l_1,\dots,l_n$ are arbitrary finite collections of natural numbers. □

Proof of Theorem. Let $l_1,\dots,l_r$ be an arbitrary finite collection of r natural numbers. Prove that there exist two bijections φ and χ of the set of natural numbers such that, for some bijection $\alpha \in \langle\varphi,\chi\rangle$ the equality $\alpha(i) = l_i$ holds for $i \in \{1,\dots,r\}$.

Let $\{S_i \mid i \in \mathbb{N}\}$ be a decomposition of the set $\mathbb{N}$ into a countable number of infinite disjoint subsets. The map φ is "infinitely expansive in iterations":

$$\varphi(S_1) = S_1 \cup S_2,\ \varphi(S_2) = S_3,\ \varphi(S_3) = S_4,\ \dots$$
$$\varphi^i(S_1) = S_1 \cup S_2 \cup \dots \cup S_{i+1} \ \text{ for }\ i = 1,\dots$$
$$\varphi^i(S_2) = S_{i+2}.$$

We now construct a "decoding" map χ. Let $\{\Gamma^{(s)}\}$ be a family of all finite subsets $A_1^{(s)},\dots,A_r^{(s)},B_1^{(s)},\dots,B_r^{(s)}$ of the set S_2, and $A_\mu^{(s)} \neq A_\nu^{(s)}$, $B_\mu^{(s)} \neq B_\nu^{(s)}$ for $\mu \neq \nu$. Denote

$$C_\nu^{(i)} = \varphi^i(A_\nu^{(i)}), \quad D_\nu^{(i)} = \varphi^i(B_\nu^{(i)}).$$

The sets $C_\nu^{(i)}$ and $D_\nu^{(i)}$ belong to the set S_{i+2}. Denote by $\chi(n)$ an arbitrary bijection from $\mathbb{N}$ into $\mathbb{N}$ such that

$$\chi(S_1) = S_2,\ \chi(S_2) = S_1,\ \chi(S_i) = S_i \ \text{ and for }\ i > 2,$$
$$\chi(C_\nu^{(i)}) = D_\nu^{(i)}.$$

Let now $l_1,\dots,l_r$ be an arbitrary finite collection of distinct natural numbers. Choose ρ large enough so that the numbers $1,2,\dots,r,l_1,\dots,l_r$ belonged to $S_1 \cup S_2 \cup \dots \cup S_\rho$. The numbers $\chi\varphi^{-\rho}(1),\dots,\chi\varphi^{-\rho}(r),\chi\varphi^{-\rho}(l_1),\dots,\chi\varphi^{-\rho}(l_r)$ comprise a collection $\Gamma^{(s)}$: $\chi\varphi^{-\rho}(\nu) = A_\nu^{(s)}$, $\chi\varphi^{-\rho}(l_\nu) = B_\nu^{(s)}$. Therefore, the bijection

$$\alpha(n) = \varphi^\rho\chi^{-1}\varphi^{-s}\chi\varphi^s\chi\varphi^{-\rho}(n)$$

satisfies the condition $\alpha(\nu) = l_\nu$, $\nu \in \{1,\dots,r\}$. □

In this example, a beautiful idea of the game with infinite sets emerges transparently. Since the image of the map φ^ρ can absorb an arbitrary finite subset from $\mathbb{N}$ for suitable ρ, the map $\varphi^{-\rho}$ will map it into S_1, where the constructed map χ realizes the substitution

$$\begin{pmatrix} 1 & 2 & \dots & r \\ l_1 & l_2 & \dots & l_r \end{pmatrix}.$$

The way back $\varphi^\rho\chi^{-1}$ is obvious.

The following theorem is just another example of the striking fact. The monster Sym $\mathbb{N}$ is topologically generated by two elements.

In developing this idea of "expanding" and "decoding" mappings, in the third section of the thesis, J. Schreier presents a kaleidoscope of theorems on the existence of finite bases for semigroups and groups of selfmaps of the Cantor set, the set of measurable mappings on the line, the homeomorphism group of the interval, the spaces l^p and L^p for $p \neq 2$.

Further development of the idea of finite bases in topological-algebraic structures is rather dramatic. As is already mentioned, in the article [13], S. Ulam and J. Schreier constructed for $m \geq 1$ in $C([0,1]^m)$ the base of five elements; in the case $m = 1$ W. Sierpiński ([15]) reduced the number of base elements to four, and V. Jarnik and V. Knichal ([16]) to two. At the same time W. Sierpiński ([19]) proves that the semigroups of all selfmaps of an infinite set X has the following property: for any countably set of maps $f_1, f_2, \dots \in X^X$ there are two maps g_1, g_2 such that $f_i \in \langle g_1, g_2 \rangle$. In the following volume of [12], S. Banach gave a simple proof of Sierpiński's Theorem. There was a natural question, whether it is possible to replace the word "map" by "bijections" in Sierpiński's Theorem. This problem is called the problem of Stan Wagon:

Let $f_1, f_2, \dots \in \mathrm{Sym}(X)$ be a finite family of bijections in the symmetric group of an infinite set X. Are there $g_1, g_2 \in \mathrm{Sym}(X)$ such that $f_1, f_2, \dots \in \langle g_1, g_2 \rangle$?

Stan Wagon's problem received a positive solution only in 1995. Having developed the idea of Banach-Sierpiński-Ulam-Schreier, Fred Galvin ([20]) received an affirmative solution to this problem. From Galvin's Theorem it follows that any countable group embeds in a group generated by two generators. Previously, this fact was proved by Higmann-B. Neumann-G. Neumann ([21]) by means of another construction (HNN-extensions).

Let us elaborate on topological aspects of this issue. The topological groups with a base consisting of one generator are called monothetic groups. The monothetic locally compact groups can be described by the Pontryagin alternative: they are either discrete or compact. On the other hand, the monothetic compact groups are the groups whose group of characters is a discrete subgroup of the circle. In Schreier's thesis it is proved that in a connected complete metrizable monothetic group the set of generators is the set of second category (page 10). Outside the class of locally compact groups the structure of monothetic groups is complicate. This lead J. Mycielski to the following problem ([22]): what can be said on the complete metrizable monothetic groups? The wideness of the class of monothetic groups is described by the theorem of S. Morris and V. Pestov ([23]): a topological group G is isomorphic to a topological subgroup of a monothetic group if and only if G is abelian, $\aleph_0$-bounded ([24]) and of weight not exceeding the continuum. In the topological case of the base of two elemants, there is also a description ([25]): a topological group G embeds in a group generated by two elements, if it is $\aleph_0$-bounded and of weight not exceeding continuum.

When viewing records in "The Scottish book" ([14], p. 178) we pay attention to problem 96 of S. Ulam: does the group $\mathrm{Sym}(\mathbb{N})$ admit a a locally compact group topology? There is a note of S. Ulam and J. Schreier, which is dated November 1935, that $\mathrm{Sym}(\mathbb{N})$ does not admit a compact group topology. The negative answer to problem 96 issue is given by E. D. Gaughan ([26]) in 1967. The question of the existence of locally compact non-discrete topologization of a normal subgroup of symmetric group $\mathrm{Sym}_\omega X$ was left open. The negative answer is obtained in 2011 in [27].

We did not touched all aspects of the ideas of Ju. Schreier's thesis. We suggest the article [28] concerning different classes of semigroups of maps, and [17] concerning the autohomeomorphism groups.

REFERENCES

1. State Archives of Lviv region, 26, 5, rec. 1655.
2. Prytula Ya. Doktoraty z matematyki i logiki na Uniwersytecie Jana Kazimierza we Lwowie w latach 1920–1938. – Dzieje matematyki polskiej pod red. W. Więsława, Wrocław 2012. 12 stron + 18 skanow dokumentow.
3. Domoradzki S., Pawlikowska-Brożek Z., Węglowska D., Słownik biograficzny matematyków polskich. – Tarnobrzeg, 2003, 286 p.
4. Duda R., Lwowska szkoła matematyczna. – Wrocław, 2007, 256 p.
5. State Archives of Lviv region, 26, 4, rec. 1227.
6. Leja F. *Sur la notion du gruppe abstrait topologique*// Fund. Math. – 1927. – V.9. – P. 37–44.
7. Danzing D. van. Studien over topologische Algebra. – Dissertation, Amsterdam, 1931.
8. Schreier Otto. *Abstrakte kontinuierliche Gruppen*// Abh. Math. Sem. Univ. Hamburg. – 1926. – V.4. – P. 15–32.
9. Hilbert D. Grundzüge einer allgemainen Theorie der linearen Integralgleichungen. – Leipzig, 1912.
10. Neumann J. *Allgemeine Eigenwerttheorie Hermitescher Functionaloperatoren*// Math. Ann. – 1929. – V.102. – P. 49–131.
11. Banach S. Sur les opérations linéaires. – Warszawe, 1932.
12. Banach S. *Sur un théoremé de W. Sierpinski*// Fund. Math. – 1935. – V.25. – P. 5–6.
13. Schreier J., Ulam S. *Über topologische Abbildungen der euklidischen Sphäreu*// Fund. Math. – 1934. – V.23. – P. 102–118.
14. Mauldin D. (ed.) The Scottisch Book. Mathematics from the Scottish Cafe. – Birkhäuser, Boston, Mass., 1981.
15. Sierpiński W. *Sur l'approximation des fonctions continues par les superpositions de quatre fonctions*// Fund. Math. – 1934. – V.23. – P. 119–120.
16. Jarnik V., Knichal V. *Sur l'approximation des fonctions continues par les superpositions de deux fonctions*// Fund. Math. – 1935. – V.24. – P. 206–208.
17. Pestov V.G. *Topological groups: where to from here?*// in: Proceedings, 14th Summer Conf. on General Topology and its Appl. (C.W. Post Campus of Long Island University, August 1999), Topology Proceedings 24. – 1999. – P. 421–502.
18. Schreier J. *O systemach elininacji w turniejach*// Mathesis Polska 7. – 1932. – z.9–10. – P. 154–160.
19. Sierpiński W. *Sul les suites infinies de fonctions definies dans les ensembles quelconques*// Fund. Math. – 1935. – V.24. – P. 209–212.
20. Galvin F. *Generating countable sets of permutations*// J. London Math. Soc. – 1995. – V.51. – P. 230–242.
21. Higman G., Neumann B.H., Neumann H. *Embedding theorems for groups*// J. London Math. Soc. – 1949. – V.24. – P. 247–254.
22. Mycielski J. Some problems on complete metric groups, Unpublished manuscript, 1975, 7 p.
23. Morris S.A., Pestov V. *Subgroups of monothetic groups*// arXiv: math./ 9907128v1 [math.GN], 1999.
24. Guran I.I. *On topological groups, close to being Lindelof*// Soviet Math. Dokl. – 1981. – V.23. – P. 173–175.
25. Morris S.A., Pestov V. *A topological generalization of the Higman-Neumann-Neumann theorem*// Journal Group Theory. – 1998. – V.2. – P. 181–187.
26. Gaughan E.D. *Topological Group Structures of infinite symmetric Groups*// Proc. N. A. of Sciences U.S.A. – 1967. – V.58. – P. 907–910.
27. Banakh T., Guran I. *Perfectly supportable semigroups are σ-discrete in each hausdorff schift-invariant topology*// arXiv: 1112.5727vq [math.GN], 2011.
28. Mitchell J.D., Péresse Y. *Generating countable sets of surjective functions*// Fund. Math. – 2011. – V.213. – P. 67–93.

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