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ON ESTIMATES OF A FRACTIONAL COUNTERPART OF THE LOGARITHMIC DERIVATIVE OF A MEROMORPHIC FUNCTION

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We consider the problem of finding lower bounds for growth of solutions of a fractional differential equation in the complex plane. We estimate a fractional integral of the logarithmic derivative of a meromorphic function.

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Рассматривается проблема оценки снизу роста решений дифференциального уравнения дробного порядка в комплексной плоскости. Оценивается дробный интеграл логарифмической производной мероморфной функции.

Let f be a meromorphic function in $\mathbb{C}$, $f(0) \neq 0, \infty$. We use the standard notation of the Nevanlinna theory ([7]).

There are two standard ways to estimate the growth of solutions of ordinary differential equations in a complex domain. The first one is based on the Wiman-Valiron theory (see monographs [20], [19], [14]), which is very effective for the complex plane when solutions are of finite order of growth. However, there are some principal restrictions for application of the Wiman-Valiron method when the complex domain is not the whole plane, e.g. the unit disk, an angle, as well as when solutions are of infinite order. Concerning recent development of the Wiman-Valiron method we address the reader to the papers [15], [17], [18], [1], [6].

On the other hand, estimates of the logarithmic derivative $\frac{f'(z)}{f(z)}$ have been successfully applied in all mentioned cases for obtaining lower bounds for growth of solutions of ordinary differential equations (see e.g. [9, 2, 3, 4]). One of the typical, but not the best possible estimate is (cf. [8])

$$\int_r^{kr} dt \int_0^{2\pi} \left| \frac{f'(re^{i\varphi})}{f(re^{i\varphi})} \right| d\varphi = O(T(k^2 r, f)), \quad k > 1, \ r \to +\infty,$$

where $T(r, f)$ is the Nevanlinna characteristic of a meromorphic function f. Note that logarithmic derivative estimates have various applications in the theory of entire and meromorphic functions ([7]).

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In contrast to ordinary differential equations, the analytic theory of fractional differential equations with variable coefficients was initiated only recently ([11], [13]). Such equations are widely used for modeling of diffusion phenomena and anomalous relaxation ([5, 12]).

Let $h \in L(0, a)$, $a > 0$. The Riemann-Liouville fractional integral of order $\alpha > 0$ for h is defined as

$$D^{-\alpha}h(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} h(t)dt, \quad x \in (0, a),$$

$$D^0 h(x) \equiv h(x), D^\alpha h(x) = \frac{d^p}{dx^p}\{D^{-(p-\alpha)}h(x)\}, \quad \alpha \in (p-1, p], \ p \in \mathbb{N},$$

where $\Gamma(\alpha)$ is the Gamma function.

In the papers [11] and [13] the following equation is studied

$$\mathcal{D}^\alpha u(t) = a(t)u(t), \quad t > 0, \tag{1}$$

where $a(t) = A(t^\alpha)$, $A(z)$ is analytic in $|z| < r_0$, $\mathcal{D}^\alpha u(t) = D^\alpha u(t) - \frac{u(0)}{t^\alpha \Gamma(1-\alpha)}$ is the Caputo-Dzhrbashyan fractional derivative. In particular, in [13] the following theorem is proved.

Theorem A ([13]). *Suppose that A is a polynomial of degree $m \geq 0$. The solution $u(t)$ of equation* (1) *satisfying the initial condition $u(0) = u_0$ has the form $u(t) = v(t^\alpha)$, where v is an entire function whose order does not exceed $\frac{1+m}{\alpha}$.*

Essentially, lack of a fractional analogue of the logarithmic derivative estimate did not allow to obtain a lower bound for the growth of the solution of (1) in Theorem A.

Problem 1. *Find a reasonable estimate for $\frac{D^\alpha f}{f}(z)$ in the class of meromorphic (analytic) functions in $\mathbb{C}$ of finite order of growth.*

Remark 1. Writing $D^\alpha f(z)$ we mean that the operator is taken of variable $r = |z|$. Since $\frac{D^\alpha f}{f}(z)$ equals infinity at zeros (and poles) of f, it makes sense to consider the integral means with respect to the circles centered at the origin. Deriving uniform estimates for a meromorphic function f we have to omit even rays emanating from the origin and containing poles, because the Riemann-Liouville operator $D^\alpha f$ is represented by a divergent integral in this case.

Here we give a contribution, which will probably help to solve Problem 1. According to formula (17.11) in [16, p. 241] we have

$$\frac{D^\alpha f}{f} = D^{\alpha-1}\left(\frac{f'}{f} + R_1\right),$$

where the error term R_1 has the form

$$R_1(x) = -\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{\alpha-2} D^\alpha f(t) \int_t^x \left(\frac{1}{f}\right)'(s)ds. \tag{2}$$

In view of these relationships, we search for an estimate of the value

$$\mathcal{I}_\alpha[f](r) = \int_0^{2\pi} D^{\alpha-1}\frac{|f'(re^{i\varphi})|}{|f(re^{i\varphi})|}d\varphi, \quad \alpha \in (0, 1).$$

Theorem 1. *Let f be a meromorphic function in $\mathbb{C}$, $f(0) \neq 0, \infty$, $\{c_q\}$ be the sequence of its zeros and poles, $\alpha \in (0,1)$, $\beta \in (1+\infty)$. Then for some $r_0 > 0$ and a constant $C(\alpha, \beta)$*

$$\mathcal{I}_\alpha[f](r) = \int\limits_0^{2\pi} D^{\alpha-1}\frac{|f'(re^{i\varphi})|}{|f(re^{i\varphi})|}d\varphi \leq C(\alpha,\beta)\Big(\frac{T(\beta r, f)}{r^\alpha} + \frac{1}{r^{\alpha-1}}\int\limits_0^{r/2}\frac{n(t,0,\infty,f)}{t^2}dt\Big), \quad r \geq r_0,$$

where $n(t,0,\infty,f)$ is the counting function of $\{c_q\}$.

Moreover, $C(\beta,\alpha) = O(\frac{1}{(\beta-1)(1-\alpha)})$ as $\beta \to 1+$, $\alpha \to 1-$.

Corollary 1. *For every $\varepsilon > 0$ we have $I_\alpha[f](r) = O(r^{(\rho-\alpha+\varepsilon)^+})$ as $r \to +\infty$.*

The following problem is left open.

Problem 2. *Find an estimate for $D^{\alpha-1}R(x,f)$, $\alpha \in (0,1)$, where $R(x,f)$ is given by (2).*

Proof of Theorem 1. Let $1 < k < \frac{3}{2}$. We have

$$D^{\alpha-1}\frac{|f'(re^{i\varphi})|}{|f(re^{i\varphi})|} = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^r \frac{|f'(te^{i\varphi})|}{|f(te^{i\varphi})|}(r-t)^{-\alpha}dt.$$

We use the following well-known estimate of the logarithmic derivative ([7])

$$\frac{|f'(z)|}{|f(z)|} \leq \frac{4sT(s,f)}{(s-|z|)^2} + 2\sum_{|c_q|<s}\frac{1}{|z-c_q|}, \quad |z| < s.$$

Therefore

$$\mathcal{I}_\alpha[f](r) = \int\limits_0^{2\pi}\int\limits_0^r \frac{1}{\Gamma(1-\alpha)}\frac{|f'(te^{i\varphi})|}{|f(te^{i\varphi})|}(r-t)^{-\alpha}dtd\varphi \leq$$

$$\leq \frac{2}{\Gamma(1-\alpha)}\Big(4\pi\int\limits_0^r \frac{sT(s,f)(r-t)^{-\alpha}}{(s-t)^2}dt + \sum_{|c_q|<s}\int\limits_0^{2\pi}\int\limits_0^r \frac{(r-t)^{-\alpha}}{|te^{i\varphi}-c_q|}dtd\varphi\Big) =:$$

$$=: \frac{2}{\Gamma(1-\alpha)}\Big(I + \sum_{|c_q|<s} J_q\Big). \tag{3}$$

We put $s = kr$. Then

$$I = 4\pi sT(s,f)\Bigg(\int\limits_0^{r-2(s-r)} + \int\limits_{r-2(s-r)}^r\Bigg)\frac{(r-t)^{-\alpha}}{(s-t)^2}dt \leq$$

$$\leq 4\pi sT(s,f)\Bigg(\frac{1}{2^\alpha(s-r)^\alpha}\int\limits_0^{r-2(s-r)}\frac{dt}{(s-t)^2} + \frac{1}{(s-r)^2}\int\limits_{r-2(s-r)}^r (r-t)^{-\alpha}dt\Bigg) =$$

$$= 4\pi sT(s,f)\Bigg(\frac{1}{2^\alpha(s-r)^\alpha}\frac{1}{(s-t)}\Big|_0^{r-2(s-r)} - \frac{1}{(1-\alpha)(s-r)^2}(r-t)^{1-\alpha}\Big|_{r-2(s-r)}^r\Bigg) \leq$$

$$\leq \frac{4\pi s}{2^\alpha}\Big(\frac{1}{3} + \frac{2}{1-\alpha}\Big)\frac{T(s,f)}{(s-r)^\alpha} \leq C(k,\alpha)\frac{T(kr,f)}{r^\alpha}. \tag{4}$$

We have to estimate $J_q = \int_0^r \int_0^{2\pi} \frac{(r-t)^{-\alpha}}{|te^{i\varphi}-c_q|} d\varphi dt$. Without loss of generality we may assume that $c_q > 1$. By Theorem 1.7 ([10, p. 7–8]) we have for some $C > 0$

$$\int\limits_0^{2\pi} \frac{d\varphi}{|1 - ze^{-i\varphi}|} \le C \ln \frac{2}{1-|z|}, \quad |z| < 1.$$

Hence

$$\int\limits_0^{2\pi} \frac{d\varphi}{|te^{i\varphi} - c_q|} \le \begin{cases} \frac{C}{t} \ln \frac{2}{1-\frac{t}{c_q}}, & t > c_q; \\ \frac{C}{c_q} \ln \frac{2}{1-\frac{t}{c_q}}, & t < c_q. \end{cases} \tag{5}$$

Note that the choice of k yields the inequality $\frac{2}{3}c_q < \frac{2}{3}kr < r$. We split the integral J_q in the following way

$$J_q = \left(\int\limits_0^{2c_q/3} + \int\limits_{2c_q/3}^{r} \right) (r-t)^{-\alpha} dt \int\limits_0^{2\pi} \frac{d\varphi}{|te^{i\varphi} - c_q|} =: J_{q1} + J_{q2}. \tag{6}$$

Using (5) we get

$$J_{q1} \le C \int\limits_0^{2c_q/3} \frac{(r-t)^{-\alpha}}{c_q} \ln \frac{2}{1-\frac{t}{c_q}} dt \le \frac{C \ln 6}{c_q} \int\limits_0^{2c_q/3} (r-t)^{-\alpha} dt \le \frac{C \ln 6}{1-\alpha} \frac{r^{1-\alpha}}{c_q}. \tag{7}$$

In order to estimate J_{q2} we consider three cases: i) $c_q \le \frac{r}{2}$; ii) $\frac{r}{2} < c_q \le r$, and iii) $r < c_q \le kr$.

In the case i) applying (5) and using a property of the fractional integral ([16, (2.44)]) we obtain

$$\int\limits_{3c_q/2}^{r} \int\limits_0^{2\pi} \frac{(r-t)^{-\alpha}}{|te^{i\varphi} - c_q|} d\varphi dt \le C \int\limits_{3c_q/2}^{r} \frac{(r-t)^{-\alpha}}{t} \ln \frac{2}{1-\frac{c_q}{t}} dt \le C \ln 6 \Gamma(1-\alpha) D^{\alpha-1}\left(\frac{1}{r}\right) \le \frac{C(\alpha)}{r^\alpha}. \tag{8}$$

Similarly

$$\int\limits_{2c_q/3}^{\frac{3c_q}{2}} \int\limits_0^{2\pi} \frac{(r-t)^{-\alpha}}{|te^{i\varphi} - c_q|} d\varphi dt \le C \int\limits_{2c_q/3}^{c_q} \frac{(r-t)^{-\alpha}}{c_q} \ln \frac{2}{1-\frac{t}{c_q}} dt + C \int\limits_{c_q}^{3c_q/2} \frac{(r-t)^{-\alpha}}{t} \ln \frac{2}{1-\frac{c_q}{t}} dt \le$$

$$\le \frac{C}{(r-c_q)^\alpha} \int\limits_{2/3}^{1} \ln \frac{2}{1-\tau} d\tau + \frac{C}{(r-\frac{3c_q}{2})^\alpha} \int\limits_1^{3/2} \frac{1}{\tau} \ln \frac{2\tau}{\tau-1} d\tau \le \frac{C(\alpha)}{r^\alpha}. \tag{9}$$

It now follows from (6), (8), and (9) that

$$J_q \le \frac{C(\alpha) r^{1-\alpha}}{c_q}, \quad c_q < \frac{r}{2}. \tag{10}$$

We then consider the case iii), i.e. $r < c_q \le kr$. Making use of (5) and Hölder's inequality, where $\alpha p < 1 < p$, $\frac{1}{p} + \frac{1}{p'} = 1$, we deduce

$$J_{q2} \le C \int_{\frac{2}{3}c_q}^{r} \frac{(r-t)^{-\alpha}}{c_q} \ln \frac{2}{1-\frac{t}{c_q}} dt = \frac{C}{c_q^\alpha} \int_{\frac{2}{3}}^{\frac{r}{c_q}} \left(\frac{r}{c_q} - \tau\right)^{-\alpha} \ln \frac{2}{1-\tau} d\tau \le$$

$$\le \frac{C}{c_q^\alpha}\Big(\int_{\frac{2}{3}}^{\frac{r}{c_q}} (\frac{r}{c_q}-\tau)^{-\alpha p} d\tau\Big)^{\frac{1}{p}} \Big(\int_{\frac{2}{3}}^{\frac{r}{c_q}} \Big(\ln\frac{2}{1-\tau}\Big)^{p'} d\tau\Big)^{\frac{1}{p'}} \le$$

$$\le \frac{C}{c_q^\alpha}\Big(\frac{(\frac{r}{c_q}-\frac{2}{3})^{1-\alpha p}}{1-\alpha p}\Big)^{\frac{1}{p}} \Big(\int_{\frac{2}{3}}^{1} \Big(\ln\frac{2}{1-\tau}\Big)^{p'} d\tau\Big)^{\frac{1}{p'}} \le \frac{C(\alpha,p)}{r^\alpha}.$$

It remains to consider the case $\frac{r}{2} < c_q \le r$. Similarly to the previous cases we obtain

$$J_{q2} \le C\int_{2c_q/3}^{c_q} \frac{(r-t)^{-\alpha}}{c_q}\ln\frac{2}{1-\frac{t}{c_q}}dt + C\int_{c_q}^{r}\frac{(r-t)^{-\alpha}}{t}\ln\frac{2}{1-\frac{c_q}{t}}dt =$$

$$= \frac{C}{c_q^\alpha}\int_{2/3}^{1}(\frac{r}{c_q}-\tau)^{-\alpha}\ln\frac{2}{1-\tau}d\tau + \frac{C}{c_q^\alpha}\int_{1}^{r/c_q}(\frac{r}{c_q}-\tau)^{-\alpha}\ln\frac{2\tau}{\tau-1}\frac{d\tau}{\tau} \le$$

$$\le \frac{C}{c_q^\alpha}\int_{2/3}^{1}(1-\tau)^{-\alpha}\ln\frac{2}{1-\tau}d\tau + \frac{C}{c_q^\alpha}\Big(\int_{1}^{r/c_q}(\frac{r}{c_q}-\tau)^{-\alpha p}d\tau\Big)^{\frac{1}{p}}\Big(\int_{1}^{\frac{r}{c_q}}\Big(\frac{\ln\frac{2\tau}{\tau-1}}{\tau}\Big)^{p'}d\tau\Big)^{\frac{1}{p'}} \le$$

$$\le \frac{C(\alpha)}{c_q^\alpha} + \frac{C}{c_q^\alpha}\Big(\frac{(\frac{r}{c_q}-1)^{1-\alpha p}}{1-\alpha p}\Big)^{\frac{1}{p}}\Big(\int_{1}^{2}\Big(\frac{\ln\frac{2\tau}{\tau-1}}{\tau}\Big)^{p'}d\tau\Big)^{\frac{1}{p'}} \le \frac{C(\alpha,p)}{r^\alpha}, \quad \alpha p < 1 < p.$$

We see that

$$J_{q2} = O(r^{-\alpha}), \quad r\to+\infty,\ \frac{r}{2}\le c_q \le kr. \tag{11}$$

We now able to give an upper estimate for $\mathcal{I}_\alpha[f](r)$. It follows from (4), (7), (10), and (11) that

$$I_\alpha[f](r) = \int_0^{2\pi} D^{\alpha-1}\frac{|f'(re^{i\varphi})|}{|f(re^{i\varphi})|}d\varphi \le C\Big(\frac{T(kr,f)}{r^\alpha} + \sum_{c_q\le\frac{r}{2}}\frac{r^{1-\alpha}}{c_q} + \sum_{\frac{r}{2}<c_q<kr}\frac{1}{r^\alpha}\Big) \le$$

$$\le C\Big(\frac{T(kr,f)+n(kr,0,\infty,f)}{r^\alpha}) + \frac{1}{r^{\alpha-1}}\int_0^{r/2}\frac{dn(t,0,\infty,f)}{t}\Big) \le$$

$$\le C\Big(\frac{T(kr,f)+n(kr,0,\infty,f)}{r^\alpha}) + \frac{1}{r^{\alpha-1}}\int_0^{r/2}\frac{n(t,0,\infty,f)}{t^2}dt\Big) \le C\Big(\frac{T(kr,f)+\frac{N(k^2r,0,\infty,f)}{\ln k}}{r^\alpha} +$$

$$+\frac{1}{r^{\alpha-1}}\int_0^{r/2}\frac{n(t,0,\infty,f)dt}{t^2}\Big) \le C(\alpha,k)\Big(\frac{T(k^2r,f)}{r^\alpha} + \frac{1}{r^{\alpha-1}}\int_0^{r/2}\frac{n(t,0,\infty,f)dt}{t^2}\Big).$$

Here we used inequality ([7]) $n(kr,0,\infty,f) \le \frac{N(k^2r,0,\infty,f)}{\ln k}$. Choosing $k=\sqrt{\beta}$, we arrive to the assertion of the theorem when $\beta\in(1,\frac{9}{4})$. The general case follows from the monotonicity of the Nevanlinna characteristic. □

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REFERENCES

1. W. Bergweiler, Ph.J. Rippon, G.M. Stallard, *Dynamics of meromorphic functions with direct or logarithmic singullarities*, Proc. London Math. Soc., (3) **97** (2008), 368–400.
2. I. Chyzhykov, G.G. Gundersen, J. Heittokangas, *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc., (3) **86** (2003), 735–754.
3. I. Chyzhykov, J. Heittokangas, J. Rättyä, *On the finiteness of φ-order of solutions of linear differential equations in the unit disc*, J. d'Analyse Math., **109** (2009), V.1, 163–196.
4. I. Chyzhykov, J. Heittokangas, J. Rättyä, *Sharp logarithmic derivative estimates with applications to ODE's in the unit disc*, J. Austr. Math. Soc., **88** (2010), 145–167.
5. S.D. Eidelman, S.D. Ivasyshen, A.N. Kochubei, Analitic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type, Birkhäuser, Basel, 2004.
6. P.C. Fenton, M.M. Strumia, *Wiman-Valiron theory in the disc*, Journal of the London Mathematical Society, **79** (2009), 478–496.
7. A.A. Gol'dberg, I.V. Ostrovskii, Value distribution of meromorphic functions, Transl. Math. Monographs, V.236, Amer. Math. Soc., 2008.
8. A.A. Gol'dberg, N.N. Strochik, *Asymptotic behavior of meromorphic functions of completely regular growth and of their logarithmic derivatives*, Siberian Math. Journal, **26** (1985), №6, 802–809.
9. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., (2) **37** (1988), 88–104.
10. H. Hedenmaln, B. Korenblum, K. Zhu, Theory of Bergman spaces. – Springer Verlag, 2001.
11. A.A. Kilbas, M. Rivero, L. Rodriguez-Germa, J.J. Trujilo, *α-Analitic solutions of some linear fractional differential equations with variable coefficients*, Appl. Math. Comput., **187** (2007), 239–249.
12. A.A. Kilbas, H.M. Srivastava, J.J. Trujilo, Theory and Applications of Fractional Differential Equations. – Elsevier, Amsterdam, 2006.
13. A.N. Kochubei, *Fractional differential equations: α-entire solutions, regular and irregular singularities*, Fract. Calc. Appl. Anal., **12** (2009), №2, 135–158.
14. I. Laine, Nevanlinna Theory and Complex Differential Equations. – Walter de Gruyter, Berlin, 1993.
15. T.M. Salo, O.B. Skaskiv, Ya.Z. Stasyuk, *On a central exponent of entire Dirichlet series,* Mat. Stud. **19** (2003), №1, 61–72.
16. S.G. Samko, A.A. Kilbas, O.I. Marichev, Integrals and derivatives of fractional order and some of their applications, Nauka i tekhnika, Minsk, 1987. (in Russian)
17. O. Skaskiv, *On the logarithmic derivative and central exponent of Dirichlet series*, Math. Bull. of Shevchenko Scientific Soc., **2** (2005), 213–230. (in Ukrainian)
18. O.B. Skaskiv, Ya.Z. Stasyuk, *On derivatives of Dirichlet series*, Visnyk of Lviv Univ., **64** (2005), 258–265. (in Ukrainian)
19. Sh. Strelitz, Asymptotic properties of analytic solutions of differential equations. – Izdat Mintis, Vilnius, 1972, 468 p. (in Russian)
20. H. Wittich, Neuere Untersuchungen über eindeutige analytische Funktionen, 2nd edn. – Springer, Berlin, 1968.

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