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HYPERCYCLIC OPERATORS ON LIPSCHITZ SPACES

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We consider hypercyclic operators on free Banach spaces and little Lipschitz spaces which are some kind of generalizations of shift operators and composition operators respectively.

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Рассматриваются гиперциклические операторы на свободных банаховых пространствах и малых липшицевых пространствах, которые, в некотором смысле, обобщают операторы сдвига и операторы композиции соответственно.

1. Introduction. Let X be a nonempty metric space and we fix a point $\theta_X \in X$. The pair (X, θ_X) is called a *pointed* space.

Definition 1. Let X and Y be metric spaces. A map $f\colon X \to Y$ is *Lipschitz* if there exists a constant $L_f \geq 0$ such that

$$\rho(f(p), f(q)) \leq L_f \rho(p, q)$$

for all $p, q \in X$.

The least such L_f is called the *Lipschitz constant* of f. We denote by $\mathrm{Lip}_0(X, Y)$ the space of all Lipschitz maps between pointed metric spaces (X, θ_X) and (Y, θ_Y) which map θ_X into θ_Y. In the case, when Y is a linear space we suppose that $\theta_Y = 0$. It is known (see e.g. [4, 5]) that for an arbitrary metric pointed space (X, θ_X) there is a unique (up to isometrical isomorphism) Banach space $B(X)$ and a Lipschitz embedding $\nu\colon X \to B(X)$ such that for every normed space E and any map $f(x) \in \mathrm{Lip}_0(X, E)$ there is a linear operator $\widetilde{f}(x)\colon B(X) \to E$ with $\widetilde{f}(\nu(x)) = f(x)$, $x \in X$ and $\|\widetilde{f}\| = L_f$. We denote by $\mathrm{span}X$ the linear span of $\nu(X)$ in $B(X)$ and elements by $\underline{x} = \nu(x)$. By the construction, elements $\sum_{k=1}^n \lambda_k \underline{x}_k$ are dense in $B(X)$. The space $B(X)$ is called a *free Banach space*.

A map F from metric space X to X is called *topologically transitive* if there is a vector $x \in X$ such that the orbit $\mathrm{Orb}(F, x) = \{F^n(x) = \underbrace{F \circ \cdots \circ F}_{n}(x)\colon n \in \mathbb{N}\}$ is dense in X. In the case when (X, θ_X) is a metric pointed space we require that $\mathrm{Orb}(F, x)$ is dense in $X \setminus \theta$. Let E be a Fréchet space. A linear continuous operator $T\colon E \to E$ is called *hypercyclic* if T is topologically transitive. An element $x \in E$ is called a *hypercyclic vector* for T if $\mathrm{Orb}(T, x)$

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is dense in E. A vector $x \in E$ is *cyclic* for T provided the linear span of orbit $\mathrm{Orb}(T, x)$ is dense in E.

The study of hypercyclic operators started after Birkhoff's result ([3]) that the operator of composition with translation $x \mapsto x + a$, $a \neq 0$, $T_a \colon f(x) \mapsto f(x + a)$ is hypercyclic in the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$. R. Aron and J. Bès in [1] proved that the operator of composition with translation T_a is hypercyclic in the space of weakly continuous analytic functions on all bounded subsets of a separable Banach space X which are bounded on bounded subsets. A detailed survey of hypercyclic operators is given in [2].

2. Hypercyclic operators on $B(X)$.

Theorem 1. *Let (X, θ) be a complete metric pointed space and $F \colon X \to X$ be a topologically transitive map with $F(\theta) = \theta$. Then the linear operator $\widehat{F} \colon B(X) \to B(X)$ is cyclic.*

Proof. Let $x \in X$ be a hypercyclic vector of F. Then $\mathrm{Orb}(\widehat{F}, \underline{x})$ is dense in $\nu(X) = \underline{X}$. By the definition of $B(X)$, the space $\mathrm{span}X$ is dense in $B(X)$. Then the space $\mathrm{span}(\mathrm{Orb}(\widehat{F}, \underline{x}))$ is dense in $B(X)$ as well. □

We remark that in the general case, hypercyclicity of operator $\widehat{F}$ does not follow from the topological transitivity of F.

Example 1. Let $X = S^1 \bigcup \theta$ be the space with natural metric and $\theta = (0, 0) \in \mathbb{R}^2$, where S^1 is the unit sphere in $\mathbb{R}^2$. We define a map of rotation F on X by an irrational angle α. It is known that F is topologically transitive and for every $x \in S^1$, $\mathrm{Orb}(F, x)$ is dense in S^1. So $\widehat{F}$ is a cyclic operator and for every $x \in S^1$, $\underline{x}$ is a cyclic vector. But the norm ofa hypercyclic operator must be strictly greater than 1 and we have $\|\widehat{F}\| = L_F = 1$.

We will use the Hypercyclicity Criterion (see [2]) to establish conditions of hypercyclicity of operators on a free Banach space.

Theorem 2. (Hypercyclicity Criterion) *Let E be a separable Fréchet space. An operator $T \colon E \to E$ satisfies the Hypercyclicity Criterion provided there exist $X_0 \subset E$, $Y_0 \subset E$ dense subsets of E and maps $S_n \colon Y \to E$, $n \in \mathbb{N}$, such that:*

(i) $T_n x \to 0$, $n \to \infty$ *for all* $x \in X_0$,

(ii) $S_n y \to 0$, $n \to \infty$ *for all* $y \in Y_0$,

(iii) $(T_n \circ S_n) y \to y$, $n \to \infty$ *for all* $y \in Y$.

Theorem 3. *Let (X, θ) be a separable complete metric space and $F \colon X \to X$, $F(\theta) = \theta$ be a 1–Lipschitz map. Suppose that X can be represented as a countable union of nonempty, pairwise disjoint sets*

$$X = \bigcup_{n=0}^{\infty} A_n,$$

where $A_0 = \theta$, $F(A_n) = A_{n-1}$ for any $n > 0$ and the restriction of F to A_n is injective for every $n > 1$. Then $T = \lambda \widehat{F}$ is a hypercyclic operator on $B(X)$ for any λ, $|\lambda| > 1$.

Proof. We define $Y_0 = X_0 = \mathrm{span}(X \setminus \theta)$. It is a dense subset in $B(X)$ for every $z = \sum a_i \underline{x}_i \in X_0$, $S(z) = \sum a_i \frac{\widehat{F}^{-1}}{\lambda}(\underline{x}_i)$, $S_n(z) = S^n(z)$.

Since span$(X\backslash\theta)$ consists of formal finite sums, for every $z \in \text{span}(X \setminus \theta)$, $T^m(z) = 0$ starting with some number m. Therefore condition 1) is fulfilled. Thus if $|\lambda| > 1$ then

$$S^n(z) \leq \frac{1}{|\lambda|^n}\|z\| \to 0$$

as $n \to \infty$. Therefore condition 2) is fulfilled. Further $(T^n \circ S^n) = Id$ is the identical operator and therefore condition 3) is also fulfilled. □

Example 2. Let $X = \mathbb{N}\bigcup 0$ be the space with discrete metric and fixed point $\theta = 0$. It is known that $B(X) = \ell_1(\mathbb{N})$. We define $F\colon \mathbb{N} \to \mathbb{N}$, $F(n) = n - 1$ for $n \neq 0$ and $F(0) = 0$. Let $A_n = \{n\}$, then F satisfies the conditions of Theorem 3. Observe, that $\lambda\widehat{F}(a_1, \dots, a_n, \dots) = \lambda(a_2, \dots, a_n, \dots)$ is the weighted left shift. It is well known that $\lambda\widehat{F}$ is hypercyclic for $|\lambda| > 0$.

Theorem 4. *Let E be a separable Frechet space. If $T\colon E \to E$ is a hypercyclic operator satisfying the Hypercyclicity Criterion, then $\widehat{T}\colon B(E) \to B(E)$ is also a hypercyclic operator and satisfies the Hypercyclicity Criterion.*

Proof. Since T satisfies the Hypercyclicity Criterion, there are apropriated spaces X_0, Y_0 and sequence of maps S_n. The spaces X_0, Y_0 are dense in E, then spanX_0 and spanY_0 are dense sets in $B(E)$. We define $\widehat{S}_n(z) = \sum a_k S_n(x_k)$ for every $z = \sum a_k \underline{x}_k \in \text{span}Y_0$. It is easy to see that for $\widehat{T}$, $\widehat{S}_n$, spanX_0 and spanY_0 the conditions of the criterion are fulfilled. Therefore $\widehat{T}$ satisfies the Hypercyclicity Criterion. □

3. Hypercyclic operators on little Lipschitz space lip(X). We know that every Lip$_0(X)$ is a dual space. Like with a first predual, for a larger class of examples we actually have a nice, explicit description of a double predual. It is the subspace of Lip$_0(X)$ consisting of precisely those Lipschitz functions with a certain local flatness property ([5, p. 73]). This space is the "little" Lipschitz space lip$_0(X)$. The norm topology is the primary topology on lip$_0(X)$, while it plays the same role that the weak* topology in the case of Lip$_0(X)$. Also, the theory of lip$_0(X)$ breaks down when X is not compact ([5]). We denote $\mathcal{M}_0^c$ the class of compact pointed metric spaces.

Definition 2. Let $X \in \mathcal{M}_0^c$ and Y be a metric space, and let $f \in \text{Lip}(X, Y)$. Then f is a *little Lipschitz function* if for every $\epsilon > 0$ the exists $\delta > 0$ such that

$$\rho(p, q) \leq \delta \quad \Rightarrow \quad \rho(f(p), f(q)) \leq \epsilon\rho(p, q).$$

The little Lipschitz space of real valued functions lip(X) is the subset of Lip(X). Similarly, lip$_0(X)$ is the subset of the space of Lipschitz functions Lip$_0(X)$. The space lip$_0(X)$ is a Banach space. It is known that lip$_0(X)^{**} \cong$ Lip$_0(X)$ (see [5]).

Let $T \in \text{lip}_0(X, X)$ for a compact metric space X. Our purpose is to study the composition operator on lip$_0(X)$.

Theorem 5. *Let X be a compact metric space and $T \in \text{lip}_0(X, X)$ be a surjective map. Suppose that X can be represented as a countable union of nonempty, pairwise disjoint sets*

$$X = \bigcup_{n=0}^{\infty} A_n,$$

where $T(A_n) = A_{n+1}$ *for any* $n > 0$ *and* $A_0 = \theta$. *Then for every* λ, $|\lambda| > 1$, *the composition operator*

$$C_{\lambda T}\colon \operatorname{lip}_0(X) \to \operatorname{lip}_0(X), \quad \lambda f \mapsto f \circ T$$

is a hypercyclic operator on $\operatorname{lip}_0(X)$.

Proof. Let us consider an operator $\lambda\widehat{T}\colon B(X) \to B(X)$, where $B(X)$ is a free Banach space. We know that $\operatorname{lip}_0^{**}(X) = B^*(X)$. We will assert that $\widehat{T}^*\colon B^*(X) \to B^*(X)$ satisfies the Hypercyclic Criterion. Let us define $Y_0 = X_0 = \operatorname{span}(X)$. It is a dense subset in $B^*(X)$. Since $T(A_n) = A_{n+1}$, $T^*(\operatorname{span} A_{n+1}) = \operatorname{span} A_n$. For every $z = \sum a_i \underline{x}_i \in X_0$, $S(z) = \sum a_i \frac{\widehat{T^*}^{-1}}{\lambda}(\underline{x}_i)$ for any λ, $|\lambda| > 1$, $S_n(z) = S^n(z)$. It is easy to see that $(\lambda\widehat{T}^*)^m = 0$ starting with some number m, $S^n(z) \to 0$ and $(\lambda T^n \circ S^n) = Id$. So $\lambda\widehat{T}^*$ satisfies the Hypercyclicity Criterion. Hence, the restriction of $\lambda\widehat{T}^*$ to $\operatorname{lip}_0(X) \subset B^*(X)$, $C_{\lambda T}$ satisfies the Hypercyclicity Criterion. Therefore the composition operator $C_{\lambda T}$ is hypercyclic on $\operatorname{lip}_0(X)$. □

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