A QUESTION ON THE SPECTRA OF ALGEBRAS OF SYMMETRIC FUNCTIONS ON $L_\infty$ RELATED TO THE MOMENT PROBLEM


Problem 1. Describe the set of all sequences of complex numbers $\{a_n\}_{n=1}^\infty$ such that $R_n(x) = \int_{[0, 1]} (x(t))^n dt = a_n$ for some $x \in L_\infty[0, 1]$.

2. Partial solutions and relations to the moment problem. In [5] it is proved that, for any finite sequence $\{a_n\}_{n=1}^m$ there is a function $x(\cdot) \in L_\infty[0, 1]$ such that $R_n(x) = a_n$ for $n \in \{1, 2, \ldots, m\}$ and $R_n(x) = 0$ for $n > m$.

Let us recall that a nondecreasing function $\sigma(\theta)$ is a solution of the trigonometric moment problem for a given sequence $\{c_k\}_{k=-\infty}^\infty$, $c_{-k} = \overline{c_k}$, $c_0 \in \mathbb{R}$ if

$$c_k = \frac{1}{2\pi} \int_{[-\pi, \pi]} \exp(ik\theta) d\sigma(\theta), \quad k \in \mathbb{Z}.$$
It is known (see e.g. [1, Theorem 5.1.2]) that the trigonometric moment problem has a solution if and only if the Hermitian forms

$$\omega_n(\xi) = \sum_{\alpha, \beta=0}^{n} c_{\alpha-\beta} \xi_\alpha \overline{\xi_\beta},$$

$$\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$$

are nonnegative functions defined for all \( n \in \mathbb{N} \). Making some simple calculations we can get the following result.

**Theorem 1.** Let \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{C} \). If \( \sigma(\theta) \) is a strictly monotone solution of the trigonometric moment problem for \( \{c_n\}_{n=-\infty}^{\infty} \), \( c_n = a_n \) for \( n > 0 \), \( c_0 = 1 \) and \( c_{-n} = \overline{a_n} \), then \( x_\sigma = \exp(i\sigma^{-1}(2\pi t - \pi)) \) satisfies \( R_n(x_\sigma) = a_n \) \((n \in \mathbb{N})\).

**Corollary 1.** If \( \{b_n\}_{n=1}^{\infty} \subset \mathbb{C} \) is such that for a given fixed number \( z \in \mathbb{C}, \{a_n\}_{n=1}^{\infty} = \{b_n/z^n\}_{n=1}^{\infty} \) satisfies the condition of Theorem 1, then \( x(t) = z \exp(i\sigma^{-1}(2\pi t - \pi)) \) satisfies \( R_n(x) = b_n \) \((n \in \mathbb{N})\).

It is not difficult to construct a function \( x(t) \in L_\infty[0, 1] \) such that \( \{b_n\}_{n=1}^{\infty} = \{R_n(x)\}_{n=1}^{\infty} \) does not satisfy the conditions of Corollary 1.

A discrete analogue of Problem 1, namely, the problem to describe all sequences \( \{a_n\}_{n=1}^{\infty} \) such that \( F_n(x) = a_n \), where \( x = (x_1, \ldots, x_n, \ldots) \in \ell_1 \) and \( F_n(x) = \sum_{k=1}^{\infty} x_k^n \), was investigated in [2, 3] and is related to the problem of description of the set of all characters on the algebra \( H_{bs}(\ell_1) \) of symmetric analytic functions of bounded type on \( \ell_1 \). Note that the discrete case is quite different from the continuous one. In particular, in [2] it is proved that, if for some \( m > 0 \), \( F_n(x) = 0 \) for all \( n > m \), then \( x = 0 \), while there is a character \( \psi \) on \( H_{bs}(\ell_1) \) such that \( \psi(F_1) = 1 \) and \( \psi(F_n) = 0 \) for \( n > 1 \). From results of [3] it follows that for every \( x \in \ell_1 \), \( \sum_{n=0}^{\infty} G_n(z) z^n \) is a function of exponential type with zeros at \( z_n = -1/x_n \) for all \( x_n \neq 0 \), where \( G_0(z) = 1, G_1(z) = F_1(z), nG_n(z) = F_1(z)G_{n-1}(z) - F_2(z)G_{n-2}(z) + \ldots + (-1)^{n-1}F_n(z), n > 1 \). However, the problem about description of all characters on \( H_{bs}(\ell_1) \) is still open.

**REFERENCES**


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