УДК 517.5

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# ON THE MAXIMUM MODULUS POINTS OF ENTIRE AND MEROMORPHIC FUNCTIONS AND A PROBLEM OF ERDŐS

I. I. Marchenko. *On the maximum modulus points of entire and meromorphic functions and a problem of Erdős*, Mat. Stud. **38** (2012), 212–215.

Two conjectures concerning the number of maximum modulus points of an entire and a meromorphic function on the circle $\{z\colon |z|=r\}$ are presented and discussed.

И. И. Марченко. *Точки максимума модуля целых и мероморфных функций и проблема Ердеша* // Мат. Студiї. – 2012. – Т.38, №2. – C.212–215.

Обсуждаются две гипотезы о точках максимума модуля на окружности $\{z\colon |z|=r\}$ целых и мероморфных функций.

Let $\nu(r)$ be the number of maximum modulus points of $f(z)$ on the circle $\{z\colon |z|=r\}$. In 1964 P. Erdős (see [1]) posed the following **question:** *can we find an entire function* $f(z)\neq cz^p$ *such that* $\nu(r)\to\infty$ $(r\to\infty)$*?* In 1968 F. Herzog and G. Piranian in [8] constructed an example of such a function. In their example the entire function is of infinite order. For entire functions of finite order the question remains open. It should be mentioned here that in [1] this question was called Clunie's problem, however, as was communicated to us by G. Piranian, this problem was first stated by P. Erdős.

In 1995 we introduced the notion of separated maximum modulus points for meromorphic functions. Let $p(r,\infty,f)$ be the number of components of the set $\{|\xi|=r\colon |f(\xi)|>1\}$, which contain at least one point of maximum modulus of $f(z)$. We set

$$p(\infty,f)=\liminf_{r\to\infty} p(r,\infty,f).$$

It is interesting to observe the influence of this quality on the magnitude of growth and the value distribution of a meromorphic function.

Let $\mathcal{L}(r,\infty,f):=\max_{|z|=r}\ln^+|f(z)|$, $T(r,f)$ be Nevanlinna's characteristic of a meromorphic function $f(z)$, $T'_-(r,f)$ the left derivative of $T(r,f)$,

$$\beta(\infty,f)=\liminf_{r\to\infty}\frac{\mathcal{L}(r,\infty,f)}{T(r,f)}$$

the Petrenko's deviation of $f(z)$ from $\infty$ ([11]),

$$b(\infty,f)=\liminf_{r\to\infty}\frac{\mathcal{L}(r,\infty,f)}{rT'_-(r,f)}$$

2010 *Mathematics Subject Classification:* 30D30, 30D35.
*Keywords:* maximum modulus point, meromorphic function, Nevanlinna's characteristic, entire function, lower order.

the Eremenko's deviation ([5]).

**Theorem A** ([9])**.** *If $f(z)$ is a meromorphic function of finite lower order $\lambda$ then*

$$\beta(\infty,f)\le\begin{cases}\frac{\pi\lambda}{p(\infty,f)} & \text{if} \quad \frac{\lambda}{p(\infty,f)}\ge 0.5, \quad (1)\\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if} \quad p(\infty,f)=1 \text{ and } \lambda<0.5, \quad (2)\\ \frac{\pi\lambda}{p(\infty,f)}\sin\frac{\pi\lambda}{p(\infty,f)} & \text{if} \quad p(\infty,f)>1 \text{ and } \frac{\lambda}{p(\infty,f)}<0.5. \quad (3)\end{cases}$$

**Corollary.** *If $f(z)$ is a meromorphic function of finite lower order $\lambda$ then*

$$p(\infty,f)\le\max\left(\left[\frac{\pi\lambda}{\beta(\infty,f)}\right],1\right),$$

*where $[x]$ is the integer part of $x$.*

If $f(z)$ is an entire function then $\beta(\infty,f)\ge 1$, therefore, for an entire function of finite lower order $\lambda$ we have $p(\infty,f)\le\max([\pi\lambda],1)$.

The estimates in Theorem A are sharp. To see this, for each positive $\lambda$ and a natural number $n$ such that $\frac{\lambda}{n}\ge 0.5$, let us consider the function

$$f_1(z)=E_{\frac{\lambda}{n}}(z^n),$$

where $E_\rho(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\frac{n}{\rho})}$ is Mittag-Leffler's entire function of order $\rho$. It is clear that $f_1(z)$ is an entire function of lower order $\lambda$. From the asymptotics of the function $E_\rho(z)$ it follows easily that $p(\infty,f_1)=n$ and $\beta(\infty,f_1)=\frac{\pi\lambda}{n}$. Thus, estimate (1) is attained for the function $f_1(z)$. Estimate (2) is attained for the function $E_\lambda(z)$. To prove the sharpness of estimate (3), we consider the meromorphic function

$$f_2(z)=\frac{1}{2}f_{\frac{\lambda}{n}}(z^n),$$

where $f_\lambda(z)$ is the function from Example 3 in [6], p. 282, $\lambda>0$ and $n$ satisfy the inequality $\frac{\lambda}{n}<\frac{1}{2}$. From the asymptotics of $f_\lambda(z)$ it follows easily that the lower order of $f_2(z)$ is $\lambda$ and also $p(\infty,f_2)=n$ and $\beta(\infty,f_2)=\frac{\pi\lambda}{n}\sin\frac{\pi\lambda}{n}$. Consequently, estimate (3) is also attained.

In 1998 we considered the influence of the value $p(\infty,f)$ on the quantity $b(\infty,f)$. As a result, the following relationship was obtained.

**Theorem B** ([10])**.** *For a meromorphic function $f(z)$ of lower order $0<\lambda\le\infty$, we have*

$$b(\infty,f)\le\begin{cases}\frac{\pi}{p(\infty,f)} & \text{if} \quad \frac{\lambda}{p(\infty,f)}\ge 0.5 \text{ or } \lambda=\infty, \quad (4)\\ \frac{\pi}{\sin\pi\lambda} & \text{if} \quad p(\infty,f)=1 \text{ and } \lambda<0.5, \quad (5)\\ \frac{\pi}{p(\infty,f)}\sin\frac{\pi}{p(\infty,f)} & \text{if} \quad p(\infty,f)>1 \text{ and } \frac{\lambda}{p(\infty,f)}<0.5. \quad (6)\end{cases}$$

**Corollary.** *If $f(z)$ is a meromorphic function then we have*

$$p(\infty,f)\le\max\left(1,\left[\frac{\pi}{b(\infty,f)}\right]\right).$$

The estimates in Theorem B are sharp. In the case of $\lambda = \infty$, it is attained for the function $E_0(z^n)$, where $n$ is any given natural number and $E_0(z)$ is an entire function from [7], § 4.1. For this function one has $p(\infty, f) = n$ and $b(\infty, f) = \frac{\pi}{n}$. If $\lambda < \infty$ and $\frac{\lambda}{n} \geq \frac{1}{2}$ estimate (4) is attained for the function $f_1(z)$ defined above. For this function the lower order is $\lambda$, $p(\infty, f_1) = n$ and $b(\infty, f_1) = \frac{\pi}{n}$. Estimate (5) is attained by Mittag-Leffler's function $E_\lambda(z)$. Finally, estimate (6) is attained for the function $f_2(z)$ defined above. This is a function of lower order $\lambda$, $p(\infty, f_2) = n$ and $b(\infty, f_2) = \frac{\pi}{n} \sin \frac{\pi\lambda}{n}$.

Together with Ciechanowicz we managed to generalize Theorem A. Let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r > 0$, such that $\phi(r) = o(T(r, f))$, $r \to \infty$. We shall denote by $\overline{p}_\phi(r, \infty, f)$ the number of components of the set $\{|\xi| = r\colon \log |f(\xi)| > \phi(r)\}$ possessing at least one maximum modulus point of the function $f(z)$. Moreover, let us denote

$$\overline{p}_\phi(\infty, f) = \liminf_{r\to\infty} \overline{p}_\phi(r, \infty, f).$$

We set

$$\overline{p}(\infty, f) = \sup_{\{\phi\}} \overline{p}_\phi(\infty, f).$$

It is straightforward that $\overline{p}(\infty, f) \geq p(\infty, f)$. It was shown in [2] that Theorem A also holds if we replace $p(\infty, f)$ with $\overline{p}(\infty, f)$.

Let us mention here a few other results in the same direction obtained together with Ciechanowicz. For $0 < \eta \leq 1$ and $r > 0$ we denote by $\widetilde{p}_\eta(r, \infty, f)$ the number of component intervals of the set

$$\{|\xi| = r\colon \log |f(\xi)| > (1 - \eta)T(r, f)\}$$

possessing at least one maximum modulus point of the function $f(z)$. We set

$$\widetilde{p}_\eta(\infty, f) = \liminf_{r\to\infty} \widetilde{p}_\eta(r, \infty, f), \quad \widetilde{p}(\infty, f) = \sup_{\{\eta\}} \widetilde{p}_\eta(\infty, f).$$

As it is easy to notice: $\widetilde{p}(\infty, f) \geq \overline{p}(\infty, f)$. In [3] we obtained the estimate of $\widetilde{p}(\infty, f)$ via the value of deviation.

**Theorem C** ([3])**.** *For a meromorphic function $f(z)$ of finite lower order $\lambda$ the following inequality is true*

$$\widetilde{p}(\infty, f) \leq \max\left(\left[\frac{2\pi\lambda}{\beta(\infty, f)}\right], 1\right),$$

*where $[x]$ is the integer part of $x$.*

For an entire function $f$ we have $\beta(\infty, f) \geq 1$, that leads to the following conclusion.

**Corollary.** *For an entire function $g(z)$ of finite lower order $\lambda$ we have*

$$\widetilde{p}(\infty, g) \leq \max([2\pi\lambda], 1).$$

The estimate of $\widetilde{p}(\infty, f)$ via $b(\infty, f)$ was established in [4].

**Theorem D.** *For a meromorphic function $f$ of lower order $\lambda$, $0 < \lambda \leq \infty$ we have*

$$\widetilde{p}(\infty, f) \leq \max\left(1, \left[\frac{2\pi}{b(\infty, f)}\right]\right).$$

We did not manage to decrease the intervals to such extent that they would contain only one maximum modulus point of $f(z)$. Probably the results are true not only for separated maximum modulus points but also for maximum modulus points.

Let $\nu(r)$ be the number of maximum modulus points of $f(z)$ on the circle $\{z\colon |z| = r\}$.

**Conjecture 1.** *Let $f(z) \neq cz^p$ be an entire function of finite lower order. Then*

$$\liminf_{r\to\infty} \nu(r) < \infty.$$

**Conjecture 2.** *Let $f(z) \neq cz^p$ be a meromorphic function and $b(\infty, f) > 0$. Then*

$$\liminf_{r\to\infty} \nu(r) < \infty.$$

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*Received 16.07.2012*